SUPERNATURALS AND THEIR PROPERTIES

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For reference, let $\mathbb{N}^\infty$ denote the set of all ∞-tuples of the form $(n_1, n_2, \dots)$ with $n_i \in \mathbb{N}$, $\forall i$ and a finite amount of non-zero entries. Also let $\mathbb{N}^* = \{1, 2, 3, \dots\}$. Let $n \in \mathbb{N}^*$. Then by the Fundamental Theorem of Arithmetic, we can represent n as a unique factorization of primes. That is

$$n = p_1^{e_1} p_2^{e_2} \cdots = \prod_{k=1}^{\infty} p_k^{e_k}.$$

Note that for all n there exists an m such that $n < p_m$.

Definition. Define a function $\phi : \mathbb{N}^* \to \mathbb{N}^\infty$ such that $\phi(n) = (e_1, e_2, \dots) = (e_k)_{k=1}^\infty$. Further, define a binary operation $\oplus : \mathbb{N}^\infty \times \mathbb{N}^\infty \to \mathbb{N}^\infty$ such that

$$\oplus((a_k)_{k=1}^\infty, (b_k)_{k=1}^\infty) = (a_k + b_k)_{k=1}^\infty.$$

Theorem 1. *$(\mathbb{N}^\infty, \oplus)$ is a commutative monoid.*

Proof. Let $u, v \in \mathbb{N}^\infty$ with $u = (a_k)_{k=1}^\infty$ and $v = (b_k)_{k=1}^\infty$. Then we have:

$$u \oplus v = (a_k + b_k)_{k=1}^\infty = (b_k + a_k)_{k=1}^\infty = v \oplus u.$$

Since $a_k + b_k = b_k + a_k \in \mathbb{N}$, $\forall k$, it follows that $u \oplus v = v \oplus u \in \mathbb{N}^\infty$. So $(\mathbb{N}^\infty, \oplus)$ is both closed and commutative. Now let $w \in \mathbb{N}^\infty$ with $w = (c_k)_{k=1}^\infty$. Then it follows

$$(u \oplus v) \oplus w = ((a_k + b_k) + c_k)_{k=1}^\infty = (a_k + (b_k + c_k))_{k=1}^\infty = u \oplus (v \oplus w).$$

Thus $(\mathbb{N}^\infty, \oplus)$ is associative. And lastly, choose $1_{\mathbb{N}^\infty} \in \mathbb{N}^\infty$ such that $1_{\mathbb{N}^\infty} = (0)_{k=1}^\infty$. Then for any $u = (a_k)_{k=1}^\infty \in \mathbb{N}^\infty$, we have

$$1_{\mathbb{N}^\infty} \oplus u = (0 + a_k)_{k=1}^\infty = (a_k)_{k=1}^\infty = (a_k + 0)_{k=1}^\infty = u \oplus 1_{\mathbb{N}^\infty}.$$

Thus, $1_{\mathbb{N}^\infty}$ is the identity element of $(\mathbb{N}^\infty, \oplus)$. $\square$

Theorem 2. *ϕ is a commutative monoid homomorphism.*

Proof. Let $a, b \in \mathbb{N}^*$. Then by the Fundamental Theorem of Arithmetic, both a and b have a unique factorization of primes. That is $a = \prod_{k=1}^\infty p_k^{e_k}$ and $b = \prod_{k=1}^\infty p_k^{f_k}$. Then $ab = \prod_{k=1}^\infty p_k^{e_k+f_k}$. So we have

$$\phi(ab) = \phi(\prod_{k=1}^{\infty} p_k^{e_k+f_k}) = (e_k + f_k)_{k=1}^\infty = (e_k)_{k=1}^\infty \oplus (f_k)_{k=1}^\infty = \phi(a) \oplus \phi(b),$$

as required. $\square$

Definition. Define a function $\psi : \mathbb{N}^\infty \to \mathbb{P}[X]^\infty$, where $\mathbb{P}[X]^\infty$ is the set of all polynomials in one variable, such that $\psi((a_k)_{k=1}^\infty) = \sum_{k=1}^\infty a_k x^k$. Also, define the operations $\otimes' : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ and $\otimes : \mathbb{N}^\infty \times \mathbb{N}^\infty \to \mathbb{N}^\infty$ such that for all $a, b \in \mathbb{N}^*$ and $u, v \in \mathbb{N}^\infty$ we have $\otimes(u, v) = \psi(u)\psi(v)$ and $\otimes'(a, b) = \phi(a) \otimes \phi(b)$.

It can be easily shown that ψ and ϕ are isomorphisms, and many interesting results arise from these definitions. One example is:

$$6 \otimes' 8 \overset{\phi}{=} (1,1,0,\dots) \otimes (3,0,0,\dots) \overset{\psi}{=} (1+x)(3) = 3+3x \overset{\psi^{-1}}{=} (3,3,0,\dots) \overset{\phi^{-1}}{=} 2^3 3^3 = 216$$

Exercise. Prove that if a and b are prime, then $(a \otimes' b)$ is also prime.

Proposition 3. *Fix p prime. Any $0 < a \leq p$ can be factored uniquely into primes by the Fundamental Theorem of Arithmetic. That is, if $0 < p_1 < p_2 < \cdots < p_n = p$ are all primes less than or equal to p, then*

$$a = \prod_{k=1}^{n} p_k^{e_k}.$$

This allows us to define another function $\phi : \mathbb{Z}_p^* \to \mathbb{Z}_p^n$ which is an isomorphism. Note that $\mathbb{Z}_p^n$ is an abelian group and that ϕ is a abelian group isomorphism.