

EE5323 Signals and Systems

Laplace Transform

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1 Introduction To Laplace Transform

Consider a LTI system with input $x(t) = e^{st}$ and impulse response $h(t)$. The output of the system

$$y(t) = H(s)e^{st} \quad (1)$$

where

$$H(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \quad (2)$$

the function $H(s)$ is referred to as Laplace transform of $h(t)$.

Definition 1.1 *Let $x(t)$ is a general C.T. signal. The Bilateral (two sided) Laplace transform of $x(t)$ is defined as*

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \quad (3)$$

where the variable s is generally complex-valued

$$s = \sigma + j\omega$$

Definition 1.2 *The Unilateral (one sided) Laplace transform of the C.T. signal $x(t)$*

$$X_{\ell}(s) = \int_0^{\infty} x(t)e^{-st} dt \quad (4)$$

Symbolically, we can write Laplace transform pair of the signal $x(t)$ as

$$X(s) = \mathcal{L}\{x(t)\}$$

or

$$\boxed{x(t) \xleftrightarrow{\mathcal{L}} X(s)}$$

Definition 1.3 The values of s for which the Laplace transform converges is called the Region of Convergence (ROC).

Example 1.4 Consider the signal

$$x(t) = e^{-at}u(t)$$

the Laplace transform of $x(t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st} dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt \\ &= \int_0^{\infty} e^{-at}e^{-st} dt \\ &= \int_0^{\infty} e^{-t(a+s)} dt \\ &= \frac{-1}{s+a} \left[e^{-t(s+a)} \right]_0^{\infty} \\ &= \frac{1}{s+a} \end{aligned}$$

The integral in example 1.4 converges if $e^{-t(s+a)}$ is finite, the integral is from 0 to ∞ , i.e., the time, $t > 0$, which means, $s+a > 0$ in order for the exponential function to decay

$$a + s > 0$$

$$a + \sigma + j\omega > 0$$

$$a + \sigma > 0$$

$$\sigma > -a$$

$$\text{ROC: } \Re\{s\} > -a$$

The ROC in example 1.4 represents a line in the complex plane as shown in Figure 1.

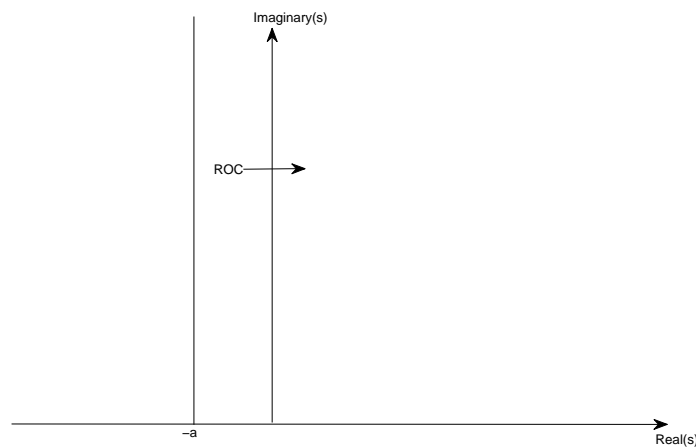


Figure 1: ROC: $\Re\{s\} > -a$

Example 1.5 Consider the signal

$$x(t) = -e^{-at}u(-t)$$

The Laplace transform of $x(t)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= - \int_{-\infty}^0 e^{-t(s+a)}dt \\ &= \frac{1}{s+a} \end{aligned}$$

This result is the same as the one obtained in example 1.4. We conclude that the ROC is very necessary here to distinguish signals from each other. The ROC for this example

$$\text{ROC: } \Re\{s\} < -a$$

so the only difference is the region of convergence!

2 Poles and Zeros of $X(s)$

Usually $X(s)$ is a rational function

$$\begin{aligned} X(s) &= \frac{a_0 s^m + a_1 s^{m-1} + \dots + a_m}{b_0 s^n + b_1 s^{n-1} + \dots + b_n} \\ &= \frac{a_0 (s - z_1) \dots (s - z_m)}{b_0 (s - p_1) \dots (s - p_n)} \end{aligned}$$

Important Notes:

- a_k and b_k are real constants.
- n and m are positive integers.
- If $n > m$, $X(s)$ is Proper.
- If $n \leq m$, $X(s)$ is Improper.
- The roots of the numerator polynomial are called the zeros z_k of $X(s)$.
- The roots of the denominator polynomial are called the poles p_k of $X(s)$.
- When $s = p_k$ for $k = 1, \dots, n$, $X(s)$ does not converge. Hence, the ROC can not contain any poles.

Example 2.1 Let

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3}$$

$X(s)$ can be written as

$$X(s) = 2 \frac{s + 2}{(s + 1)(s + 3)}$$

$$z_1 = -2, p_1 = -1, p_2 = -3.$$

3 Properties Of The ROC

P1 ROC does not contain any poles.

P2 If $x(t)$ is a finite duration signal, $x(t) \neq 0$, $t_1 < t < t_2$ and is absolutely integrable, the ROC is the entire s plane.

P3 If $x(t)$ is a right sided signal, $x(t) = 0$, $t < t_0$, the ROC is of the form

$$\Re\{s\} > \max\{\Re\{p_k\}\}$$

i.e., ROC is to the right of the rightmost pole, in this case the system is causal.

P4 If $x(t)$ is a left sided signal $x(t) = 0$, $t > t_0$, the ROC is of the form

$$\Re\{s\} < \min\{\Re\{p_k\}\}$$

i.e., ROC is to the left of the leftmost pole, in this case the system is anti-causal.

P5 If $x(t)$ is a double sided signal, the ROC is of the form

$$p_1 < \Re\{s\} < p_2$$

P6 If the ROC includes the $j\omega$ axis, Fourier transform exists and the system is stable.

Example 3.1 Let

$$X(s) = \frac{s+3}{(s+1)(s-2)}$$

Poles at $s = -1$, $s = 2$. Three ROC's

- i. $\Re\{s\} < -1$, P4 applies, the signal is left sided, the ROC doesn't include $j\omega$ axis, hence, Fourier transform doesn't exist.
- ii. $-1 < \Re\{s\} < 2$, P5 applies, the signal is double sided, the ROC includes the $j\omega$ axis, Fourier transform exists.
- iii. $\Re\{s\} > 2$, P3 applies, right sided signal, the ROC doesn't include $j\omega$ axis, hence, Fourier transform doesn't exist.

Usually, Laplace transform pairs are obtained using Laplace transform table. Table of the most common Laplace transform pairs is shown in Figure 2.

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\text{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -\text{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -\text{Re}(a)$
$te^{-at} u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -\text{Re}(a)$
$-te^{-at} u(-t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) < -\text{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}(s) > 0$
$e^{-at} \cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$
$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$

Figure 2: Laplace Transform Table

4 Unilateral Laplace Transform Properties

Bilateral Laplace transform reduces to unilateral Laplace transform for causal systems. In this section we will discuss the unilateral Laplace transform properties.

P1 Linearity

If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$$

and

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$$

then

$$\boxed{\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{\mathcal{L}} \alpha X_1(s) + \beta X_2(s)}$$

$$ROC = ROC_1 \cap ROC_2$$

Example 4.1 Let

$$x(t) = (A + Be^{-bt})u(t)$$

Using Laplace transform table, we have

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b}$$

$$X(s) = \frac{A}{s} + \frac{B}{s+b}$$

$$ROC_1 : \Re\{s\} > 0, \quad ROC_2 = \Re\{s\} > -b$$

$$ROC = \Re\{s\} > 0 \cap \Re\{s\} > -b = \max(-b, 0)$$

P2 Time Shifting

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

then

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} X(s)e^{-st_0}$$

for some positive fixed t_0 . The ROC will be the same.

Example 4.2 Consider the signal $x(t) = \text{rect}(\frac{t-a}{2a})$.

This signal can be written as

$$x(t) = u(t) - u(t - 2a)$$

Using linearity and time shifting

$$u(t) - u(t - 2a) \xleftrightarrow{\mathcal{L}} \frac{1}{s} - \frac{1}{s} e^{-2as}$$

$$ROC : \Re\{s\} > 0$$

P3 Shifting in the s Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

then

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0)$$

the ROC will be shifted by $\Re\{s_0\}$.

$$ROC = ROC_1 + \Re\{s_0\}$$

Example 4.3 Let $x(t) = Ae^{-at} \cos(\omega_0 t + \theta)u(t)$

$x(t)$ can be written as

$$x(t) = Ae^{-at} \cos \omega_0 t \cos \theta u(t) - Ae^{-at} \sin \omega_0 t \sin \theta u(t)$$

From the table, we have

$$\cos \omega_0 t u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_0^2}$$

and

$$\sin \omega_0 t u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2 + \omega_0^2}$$

Using frequency shifting and linearity

$$X(s) = A \frac{(s + a) \cos \theta - \omega_0 \sin \theta}{(s + a)^2 + \omega_0^2}$$

The ROC is

$$ROC = \Re\{s\} > -a, \text{ instead of } 0$$

P4 Time Scaling

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

the

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{a} X\left(\frac{s}{a}\right)$$

$$ROC = aROC_1$$

Example 4.4 Let $x(t) = u(at)$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \Re\{s\} > 0$$

$$u(at) \xleftrightarrow{\mathcal{L}} \frac{1}{a} \frac{1}{\frac{s}{a}} = \frac{1}{s}, \Re\{s\} > a \times 0$$

Note the $u(t) = u(at)$ and so, $\mathcal{L}\{u(t)\} = \mathcal{L}\{u(at)\}$

5 Inverse Laplace Transform

It is the process of finding $x(t)$ given $X(s)$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

There are two methods to obtain the inverse Laplace transform.

5.1 Inversion using Complex Line Integral

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

The values of c in this integral depends on the ROC. We will not be using this line integral, rather we will use the inversion using Laplace table.

5.2 Inversion Using Laplace Table

Laplace transform can be written as

$$\begin{aligned} X(s) &= \frac{NUM(s)}{DEN(s)} \\ &= K \frac{(s - z_1) \dots (s - z_m)}{(s - p_1) \dots (s - p_n)} \end{aligned}$$

We always make sure that $m < n$, in this case, Partial Fraction Expansion can be performed.

5.2.1 Partial Fraction expansion

A. All poles has multiplicity of 1

$$X(s) = \frac{c_1}{s - p_1} + \dots + \frac{c_n}{s - p_n}$$

where

$$c_k = (s - p_k)X(s)|_{s=p_k}$$

B. When one or more poles has multiplicity r

In this case $X(s)$ has the term $(s - p)^r$

$$X(s) = \frac{\lambda_1}{s - p} + \frac{\lambda_2}{(s - p)^2} + \dots + \frac{\lambda_r}{(s - p)^r}$$

The coefficients λ_k can be found as

$$\lambda_k = \left[\frac{1}{(r - k)!} \frac{d^{r-k}}{ds^{r-k}} ((s - p_k)^r X(s)) \right]_{s=p}$$