

Lagrangian Dynamics 2008/09

Lecture 15: The Symmetric Top – Precession; the Tennis Racquet Theorem

Technical Note: Euler's equations apply in a frame rotating with the body. So why isn't $\underline{\omega}$ identically zero – surely there is no motion to describe, if we're working in this frame? This is a good question. The physical answer is that since the body frame is *non-inertial*, an observer in this frame is able to sense that she is rotating, and indeed could perform experiments which would reveal exactly the value of $\underline{\omega}$ (any suggestions?). Mathematically, $\underline{\omega}$ was defined as the rotation velocity of the body frame *relative to an inertial one*; so this is what Euler's equations involve.

The Symmetric Top: If the body has an axis of symmetry (call this axis 3), then $I_1 = I_2$. Assume (for definiteness later) that I_3 is smaller than the other two; then Euler's equations become:

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_1 - I_3) \omega_2 \omega_3 \\ I_1 \dot{\omega}_2 &= (I_3 - I_1) \omega_3 \omega_1 \\ I_3 \dot{\omega}_3 &= 0 \end{aligned}$$

Hence ω_3 is a constant, and we can write

$$\dot{\omega}_1 = \frac{I_1 - I_3}{I_1} \omega_3 \omega_2 \equiv \Omega \omega_2 \quad (1)$$

$$\dot{\omega}_2 = -\Omega \omega_1 \quad (2)$$

Differentiating wrt t again, we easily obtain

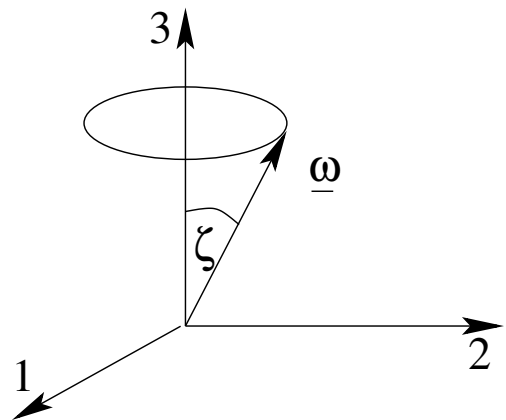
$$\begin{aligned} \ddot{\omega}_1 &= -\Omega^2 \omega_1 \\ \ddot{\omega}_2 &= -\Omega^2 \omega_2 \end{aligned}$$

Both of these are equations for SHM at angular frequency Ω . Since $\dot{\omega}_2 = -\Omega \omega_1$, there is a $\pi/2$ phase difference between ω_1 and ω_2 (eg, if $\omega_2 = a \cos(\Omega t + \alpha)$, then $\omega_1 = a \sin(\Omega t + \alpha)$).

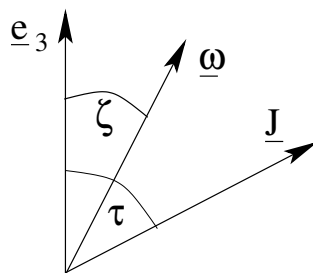
This corresponds to *circular motion* of $\underline{\omega}$ in the (1,2) plane, with a constant ω_3 .

Hence $\underline{\omega}$ **precesses** about the 3 axis with precession frequency $\Omega = (I_1 - I_3)\omega_3/I_1$. This precession of $\underline{\omega}$ in the body frame traces out the *body cone*.

Since ω_3 is constant, and so is $\omega_1^2 + \omega_2^2$ (circular motion in the 1, 2 plane) we have $|\underline{\omega}| = \text{constant}$. Since $\underline{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, and $I_1 = I_2$, $|\underline{J}|$ is constant for the same reason, and so is $\underline{J} \cdot \underline{\omega}$.

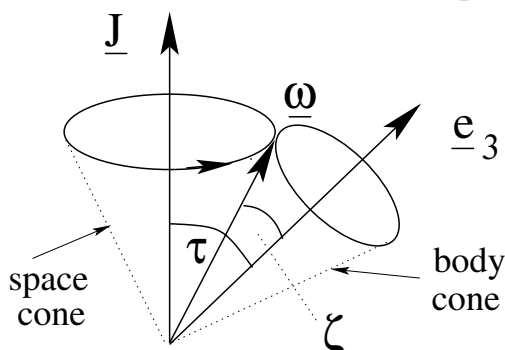


The fact that $\underline{J} \cdot \underline{\omega}$, $\underline{J} \cdot \underline{e}_3$ and $\underline{\omega} \cdot \underline{e}_3$ are *all* constants means that \underline{J} , $\underline{\omega}$ and \underline{e}_3 make constant angles with each other, and are coplanar because $I_1 = I_2$. Hence $\underline{\omega}$ and \underline{J} precess *together* at frequency Ω about the 3 axis at inclination angles $\zeta = \cos^{-1}(\omega_3/|\underline{\omega}|)$, $\tau = \cos^{-1}(I_3\omega_3/|\underline{J}|)$ which are fixed by the IC's. Here, $\tau > \zeta$ because $I_3 < I_1 = I_2$.



Note: If all three principal moments are equal (isotropic body), Euler's equations reduce to $\dot{\omega} = 0$. This is trivially true because \underline{J} and $\underline{\omega}$ are parallel for this case.

Symmetric Top in the Lab Frame: We now have to translate these results into the (inertial) lab frame. We assumed $\underline{G} = 0$ and therefore, in the lab, \underline{J} is constant. Clearly, \underline{J} , $\underline{\omega}$ and \underline{e}_3 remain coplanar in this frame. Hence we have the picture:



So, $\underline{\omega}$ traces out the surface of a cone (the *space cone*) about \underline{J} . Since $\underline{\omega}$ simultaneously executes a cone about \underline{e}_3 (see above), we find that *the body cone rolls without slipping around the outside of the space cone* – a nice picture if you like geometry! If $I_3 > I_1 = I_2$, the body cone rolls inside the space cone.

An example is the free precession of the earth (wobble): $\omega_3 = 2\pi$ per day; the asphericity $(I_1 - I_3)/I_3 \simeq 0.3\%$ giving a precession period of about 300 days. The observed value is 430 days because the earth is not quite a rigid body (it has a molten core), and the amplitude is about $10m$. (Note: this phenomena is not precession of the equinoxes.)

Asymmetric Top: the Tennis Racquet theorem: When all three principal moments of inertia are different, the general solution of Euler's equations for torque-free motion involves elliptic integrals and is not illuminating.

We instead ask the following: Can one have motions where $\underline{\omega}$ is constant in the body frame? Inspection of Euler's equations shows that this is only possible if two of the components of $\underline{\omega}$ (say, ω_1 and ω_2) are zero. Hence

An asymmetric body can only rotate with constant angular velocity if the rotation axis is one of the principal axes.

Is the resulting motion stable? To investigate, write $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_2$ and $\omega_3 = \omega + \epsilon_3$ where the ϵ 's are small. Expanding Euler's equations to first order in ϵ 's gives

$$\begin{aligned} I_1 \dot{\epsilon}_1 &= (I_2 - I_3)\omega \epsilon_2 \\ I_2 \dot{\epsilon}_2 &= (I_3 - I_1)\omega \epsilon_1 \end{aligned}$$

$$I_3 \dot{\epsilon}_3 = 0$$

Eliminating ϵ_2 (say) gives the second order equation

$$I_1 I_2 \ddot{\epsilon}_1 = (I_2 - I_3)(I_3 - I_1) \omega^2 \epsilon_1$$

The same equation holds for $\ddot{\epsilon}_2$. The solution is

$$\ddot{\epsilon}_1 = A e^{i\Omega t} + B e^{-i\Omega t}$$

with

$$\Omega^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega^2$$

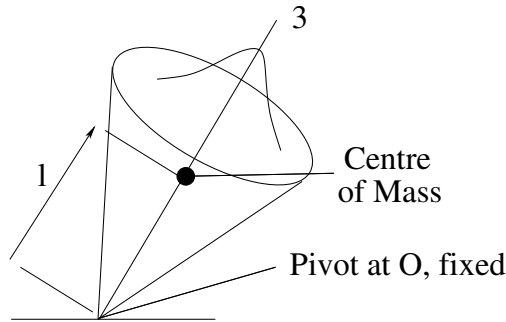
We conclude that if I_3 is *either the smallest or the largest* moment of inertia, then the RHS is positive and Ω is real. Hence perturbations to steady motion about this axis oscillate but do not grow. In contrast, if I_3 is the middle axis, Ω is imaginary and perturbations grow exponentially. This is the tennis racquet theorem (easily demonstrated with said object or similar):

Steady rotation about the intermediate axis is unstable; about the major or minor axis, it is stable.

Lagrangian Dynamics of Rigid Bodies

We now turn to the Lagrangian approach, which is suitable for systems in which there is an external torque acting.

Choice of Origin: We will be concerned almost exclusively with spinning objects (tops) with a point O held fixed:



The fixed point is usually not the centre of mass. From now on we denote by I the inertia tensor about O (before this was called I'). The angular momentum (about O) of such a body is

$$\underline{L} = I \underline{\omega}$$

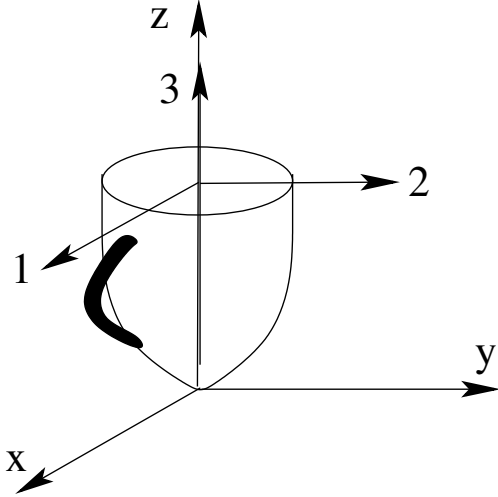
and its kinetic energy (*including* centre of mass motion – remember Euler's theorem) is

$$T = \frac{1}{2} \underline{\omega} \cdot I \underline{\omega} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

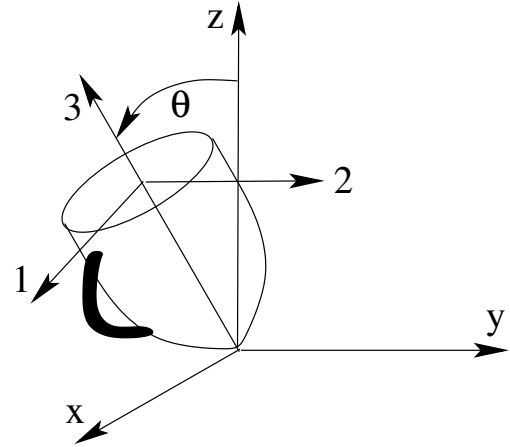
where $I_{1,2,3}$ are the eigenvalues of I . We take these as arbitrary.

Choice of Coordinates – Euler Angles: We need a set of three generalised coordinates to specify the orientation of the body. (According to Euler’s theorem, since O is fixed, there are only three degrees of freedom left.)

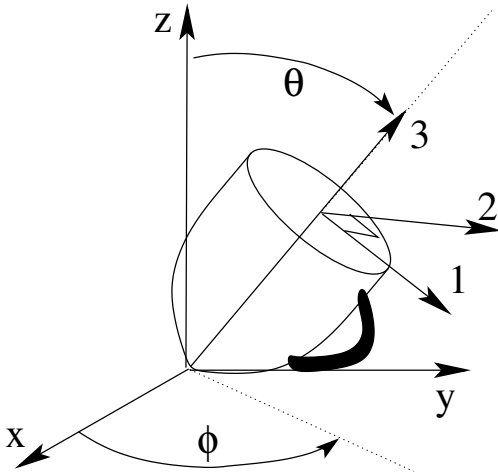
Three *Euler Angles* (θ, ϕ, ψ) are defined by a sequence of rotations. We shall use one of numerous possible choices for three angular coordinates. (Even among those normally called “Euler Angles” there are several conventions, of which one used here is called the *y* convention – see Goldstein if interested.) The *body axes* and fixed *space axes* share a common origin at the pivot point. In the figures below, the body axes have been displaced for clarity.



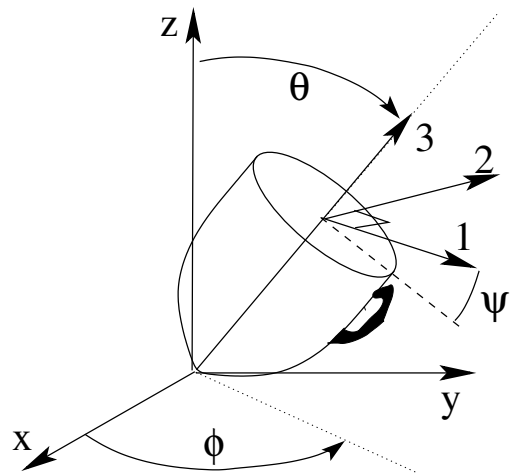
Reference State: The *body axes* 1, 2 & 3 are aligned with the fixed *space axes* x , y & z . For reference, we introduce a “handle” which lies in the (z, x) plane.



The First Rotation is through angle θ in the (z, x) plane. The handle remains in the (z, x) plane.



The Second Rotation is through angle ϕ about the z axis. The 3 axis is now specified by (θ, ϕ) as in spherical polars. The handle still lies in the vertical plane,



The Third Rotation is through angle ψ about the 3 axis which is fixed to the body. The handle is rotated by ψ away from the vertical.

This sequence defines (θ, ϕ, ψ) the three Euler angles specifying the orientation of an arbitrary rigid body. The main advantage of the Euler angles over other choices is that the various conservation laws emerge more tidily.