Lagrangian Dynamics 2008/09

Lecture 14: Rigid Body Motion – Introduction & Euler's Equations

Section VII: Rigid Body Motion

Nomenclature: the motion of rigid bodies was studied in depth by Euler and Lagrange. Their names are attached to equations (*eg* Euler's equations of motion for a top) which are unrelated to those met previously (Euler's equation for stationarity of an integral). This is confusing, but not as confusing as trying to give these equations new names.

Recap: A rigid body is viewed as a system of N particles, $a = 1, \cdots, N$, with constraints

$$|\underline{r}_a - \underline{r}_b| = \rho_{ab} = \text{constant}$$

for each pair (a, b). These are not all linearly independent, however. In three dimensions, we end up with 6 degrees of freedom (see below).

For such a system, we showed

$$M\ddot{\underline{R}} = \underline{F}^{\text{ext}}$$

(centre of mass motion), and

$$\dot{\underline{L}} = \underline{G}^{\text{ext}}$$

We also showed, for angular momentum $\underline{L}$ about some stated axis and for kinetic energy T

$$\begin{aligned} \underline{L} &= \underline{J} + \underline{R} \times \underline{P} & (1) \\ T &= T_{\text{c-of-mom}} + \frac{1}{2}MV^2 & (2) \end{aligned}$$

ie, the value in a general (inertial) frame is the value *in* the centre of momentum (c-of-mom) frame plus a contribution from the motion *of* the centre of mass, treated as though it were a point particle.

NB Remember that in the c-of-mom frame, the intrinsic angular momentum $\underline{J}$ does not depend on the origin chosen (though, because of the second term in equation (1) this will not be true for $\underline{L}$ in a general frame). Recall that the c-of-mom frame is an inertial frame (*not a rotating frame*) in which the centre of mass is stationary, but not *necessarily* at the origin. In such a frame, at least instantaneously, the motion of a rigid body is purely rotational.

Euler's Theorem: *Any displacement of a rigid body with one point fixed in space can be described as a rotation about some single axis*

This is surely obvious(!), and the formal proof is not illuminating. To specify a finite rotation requires an axis (unit vector, two degrees of freedom) and a magnitude of the rotation (one dof). Hence we will need 3 generalised coordinates to specify the centre of mass motion and three for the angular motion. In practice, there are many ways to choose three angular coordinates: we do not make an explicit choice at this stage.

The velocity of the system is usually specified by $\dot{\underline{R}}(t)$ (the centre of mass motion) and $\underline{\omega}(t)$ (the angular velocity about the centre of mass), *ie*, also six independent components.

Technical Note: In choosing angular coordinates, there is some subtlety since finite rotations about different axes do not commute. Hence $\underline{\Omega}(t) = \int_0^t \underline{\omega}(t')\,dt'$ is not a valid generalised coordinate; the same value of this quantity can be obtained for different sequences of rotations and hence different actual orientations of the system. Hence the need for "Euler angles" introduced later.

The Inertia Tensor: Working in the c-of-mom frame and choosing our origin at the centre of mass, the instantaneous velocity of the a^{th} particle is

$$\dot{\underline{r}}_a = \underline{\omega}(t) \times \underline{r}_a$$

The intrinsic angular momentum is therefore

$$\begin{aligned} \underline{J} &= \sum_a \underline{r}_a \times \underline{p}_a = \sum_a m_a(\underline{r}_a \times \dot{\underline{r}}_a) \\ &= \sum_a m_a(\underline{r}_a \times (\underline{\omega} \times \underline{r}_a)) \;=\; \sum_a m_a r_a^2\,\underline{\omega} - \sum_a m_a \underline{r}_a(\underline{r}_a \cdot \underline{\omega}) \end{aligned} \tag{3}$$

(using $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$). The first term is a vector parallel to $\underline{\omega}$, the second term is a vector that is, in general, not parallel to $\underline{\omega}$.

The intrinsic angular momentum $\underline{J}$ is thus a vector which depends linearly on $\underline{\omega}$ (for example, $\underline{J}(2\underline{\omega}) = 2\underline{J}(\underline{\omega})$) but need not be parallel to it. So there must be a relationship

$$\underline{J} = I\underline{\omega} \tag{4}$$

where I is a rank-2 tensor (which we represent in cartesian coordinates by a 3×3 matrix), called the inertia tensor. In Cartesian coordinates, equation (4) becomes $J_i = I_{ij}\,\omega_j$. From equations (3) and (4)), it is a straightforward exercise to show that

$$I_{ij} \;=\; \sum_a m_a \left\{ r_a^2 \delta_{ij} - x_{a,i} x_{a,j} \right\} \tag{5}$$

or, in terms of explicit cartesians $\underline{r}_a = (x_a, y_a, z_a)$, we have

$$I = \sum_a m_a \begin{pmatrix} (y_a^2 + z_a^2) & -x_a y_a & -x_a z_a \\ -x_a y_a & (x_a^2 + z_a^2) & -y_a z_a \\ -x_a z_a & -y_a z_a & (x_a^2 + y_a^2) \end{pmatrix} \tag{6}$$

The expression for I explicitly involves the particle coordinates $\underline{r}_a$, which are measured relative to the centre of mass. For a **rigid** body, these coordinates can be taken as constants, though this applies only in a frame of reference in which the body is completely stationary (*ie*, $\dot{\underline{r}}_a = 0$, $\forall a$). For a body that is actually rotating, this is a noninertial frame, the *body frame* which rotates and translates with the body. Despite this inconvenience, one *defines* the inertia tensor I to be evaluated in such a frame, that is, with respect to a set of x, y, z axes *fixed in the body.* The inertia tensor is therefore a *time independent* characteristic of how the mass is distributed in the body.

Kinetic Energy: In the centre of momentum frame, the KE is

$$T \;=\; \sum_a \frac{1}{2} m_a (\underline{\omega} \times \underline{r}_a) \cdot (\underline{\omega} \times \underline{r}_a)$$

and (using $(\underline{a} \times \underline{b}) \cdot \underline{c} = \underline{a} \cdot (\underline{b} \times \underline{c})$)

$$T \;=\; \frac{1}{2}\underline{\omega} \cdot \sum_a m_a \underline{r}_a \times (\underline{\omega} \times \underline{r}_a) \;=\; \frac{1}{2}\underline{\omega} \cdot \underline{J} \;=\; \frac{1}{2}\underline{\omega} \cdot I\underline{\omega}$$

Principal Axes: In this course, we will generally not need the explicit form of I found above; we take I as given. But it is important that (however irregular the body) the tensor I is *symmetric*. Hence there exist three real eigenvalues, I_1, I_2, I_3, and three mutually perpendicular eigenvectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$.

Choosing these to define new axes, the *Principal Axes* (PA), fixed in the body, we have the diagonal form

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Where (I_1, I_2, I_3) are the *Principal Moments of Inertia.* Moreover, in this coordinate system

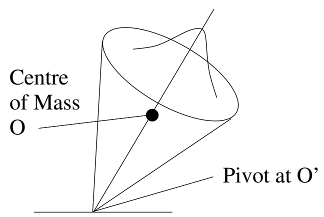
$$\underline{J} = (I_1\omega_1,\ I_2\omega_2,\ I_3\omega_3)$$

and

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

Here lies the central problem: These "nice" axes are **fixed** in the body. But the body is rotating. Hence in lab coordinates (or any other inertial frame), the principal axes have time-dependent orientations. This is why rigid body motion is not a straightforward application of linear algebra.

Shift of Origin: We have placed the origin of our coordinate system for the $\underline{r}$'s at the centre of mass O of the body. This is the best choice for a body undergoing arbitrary motion, or whose centre of mass is constrained to be fixed in space. In systems where some other point O' is constrained to be fixed instead (for a spinning top, this is usually the point of contact with the table) one can use the usual formula $\underline{L} = \underline{J} + M\underline{R} \times \underline{P}$ to calculate the angular momentum $\underline{L}$ with respect to O'.



However, by Euler's Theorem, the result is the same as found by writing

$$\underline{L} = I'\,\underline{\omega}$$

where I' is found by replacing O by O' as the coordinate origin, in the explicit expressions for I given in equations (5) and (6) above. The kinetic energy, including that of the centre of mass motion, becomes, in the PA basis

$$T \;=\; \frac{1}{2}\underline{\omega}\cdot I'\,\underline{\omega} \;=\; \frac{1}{2}\,(I_1'\omega_1^2 + I_2'\omega_2^2 + I_3'\omega_3^2)$$

The tensor I' is called the inertia tensor about the point O', as opposed to the "inertia tensor about the centre of mass" which is, strictly speaking, the proper name for I. If O' lies a distance l from the centre of mass along (say) the 3-axis, then, again in the PA basis, one has

$$I_1' = I_1; \qquad I_2' = I_2; \qquad I_3' = I_3 + Ml^2$$

This is a simple example of the parallel-axes theorem proved in FoMP/T&F. In our discussion of spinning tops, we will just write I, meaning I' relative to the pivot where appropriate.

Euler versus Lagrange: At this point there are two divergent approaches:

Eulerian Approach: Study equations in a noninertial frame (the *body frame*) rotating with the body, so that principal axes are constant in time. Difficulty:

$$\dot{\underline{J}} = \underline{G}^{\text{ext}}$$

where $\underline{G}^{\text{ext}}$ is (usually) specified wrt axes *fixed in space* (*ie* in the lab). We can't translate $\underline{G}$ back into the body frame until *after* we have found the motion! In practice, the Eulerian approach is therefore useful only for *kinematics* (motion without external forces): $\underline{G}^{\text{ext}} = 0$.

Lagrangian Approach: Choose a convenient set of generalised coordinates (in practice, three angles specifying the orientation of the principal body axes with respect to another set, fixed in space). Then construct $\mathcal{L}$ and differentiate as usual to obtain equations of motion. This is more powerful but less intuitive; we follow Euler's approach first.

Euler's Equations Of Motion: These describe motion in the principal axes frame S rotating with the body – this is not an inertial frame. Consider S_0, an inertial frame instaneously coinciding with S. As we saw previously, if the instantaneous angular velocity of the body is $\underline{\omega}$, then for any vector $\underline{A}$

$$[\dot{\underline{A}}]_{S_0} = [\dot{\underline{A}}]_S + \underline{\omega} \times [\underline{A}]_S$$

Now, in the inertial frame S_0, we have the usual equation of motion

$$[\dot{\underline{L}}]_{S_0} = \underline{G}^{\text{ext}} \equiv \underline{G}$$

where $\underline{G}$ is the external torque on the system. (The label ext is implicit from now on.) Accordingly in the body frame S

$$\underline{G} = [\dot{\underline{L}}]_S + \underline{\omega} \times [\underline{L}]_S$$

Now, since S is the frame of the principal axes (see above)

$$[\underline{L}]_S = \underline{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

In this coordinate frame, the equation for $\underline{G}$ therefore reads

$$(G_1, G_2, G_3) = (I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

or, expanding this out

$$\begin{aligned} G_1 &= I_1\,\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 \\ G_2 &= I_2\,\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 \\ G_3 &= I_3\,\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \end{aligned}$$

These are **Euler's Equations of Motion**.

For the reasons stated above, the useful case is when $\underline{G} = 0$; then Euler's equations become

$$\begin{aligned} I_1\,\dot{\omega}_1 &= (I_2 - I_3)\,\omega_2\omega_3 \\ I_2\,\dot{\omega}_2 &= (I_3 - I_1)\,\omega_3\omega_1 \\ I_3\,\dot{\omega}_3 &= (I_1 - I_2)\,\omega_1\omega_2 \end{aligned}$$