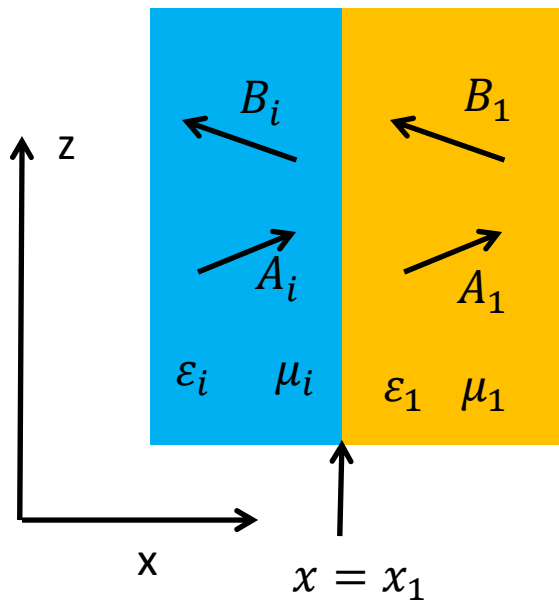


# Boundary Condition: TE



Incidence layer:

$$E_{iy} = e^{j(\omega t - \beta z)} \{ A_i e^{-jk_{ix}(x-x_1)} + B_i e^{jk_{ix}(x-x_1)} \}$$

$$H_{iz} = \frac{1}{\omega \mu_i} e^{j(\omega t - \beta z)} \{ k_{ix} A_i e^{-jk_{ix}(x-x_1)} - k_{ix} B_i e^{jk_{ix}(x-x_1)} \}$$

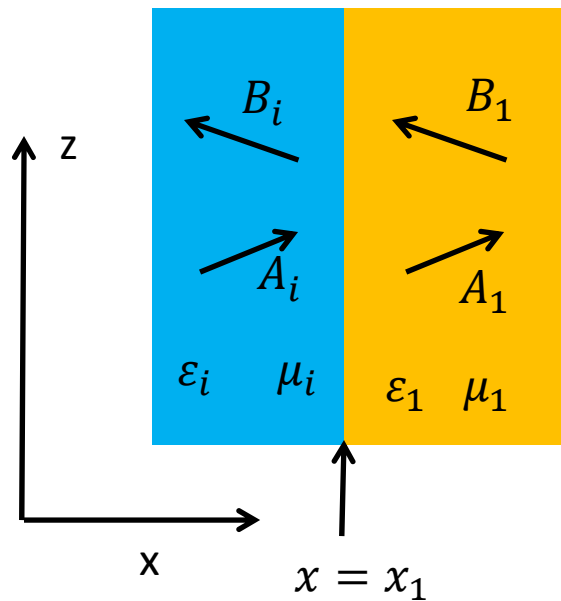
1st layer:

$$E_{1y} = e^{j(\omega t - \beta z)} \{ A_1 e^{-jk_{1x}(x-x_1)} + B_1 e^{jk_{1x}(x-x_1)} \}$$

$$H_{1z} = \frac{1}{\omega \mu_1} e^{j(\omega t - \beta z)} \{ k_{1x} A_1 e^{-jk_{1x}(x-x_1)} - k_{1x} B_1 e^{jk_{1x}(x-x_1)} \}$$

Assign as HW, calculate two H field components and check boundary conditions.

# Boundary Condition: TE



$x = x_1$  Incidence layer:

$$E_{iy} = e^{j(\omega t - \beta z)} \{A_i + B_i\}$$

$$H_{iz} = \frac{1}{\omega \mu_i} e^{j(\omega t - \beta z)} \{k_{ix} A_i - k_{ix} B_i\}$$

1st layer:

$$E_{1y} = e^{j(\omega t - \beta z)} \{A_1 + B_1\}$$

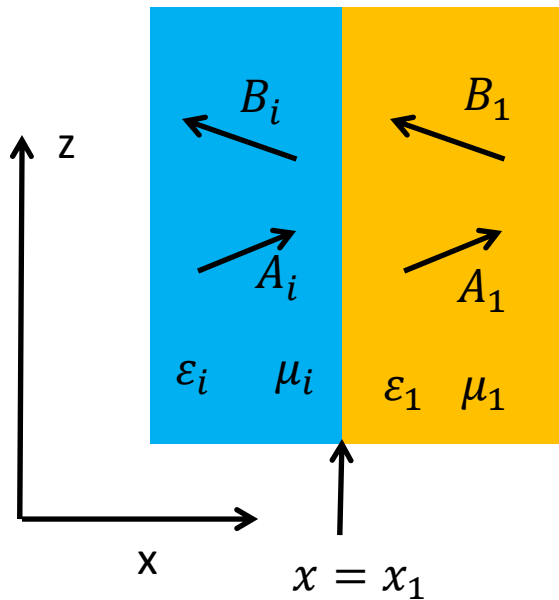
$$H_{1z} = \frac{1}{\omega \mu_1} e^{j(\omega t - \beta z)} \{k_{1x} A_1 - k_{1x} B_1\}$$

$$A_i + B_i = A_1 + B_1$$

$$\frac{1}{\mu_i} \{k_{ix} A_i - k_{ix} B_i\} = \frac{1}{\mu_1} \{k_{1x} A_1 - k_{1x} B_1\}$$

$$\begin{bmatrix} 1 & 1 \\ \frac{k_{ix}}{\mu_i} & -\frac{k_{ix}}{\mu_i} \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{k_{1x}}{\mu_1} & -\frac{k_{1x}}{\mu_1} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

# Boundary Condition: TE

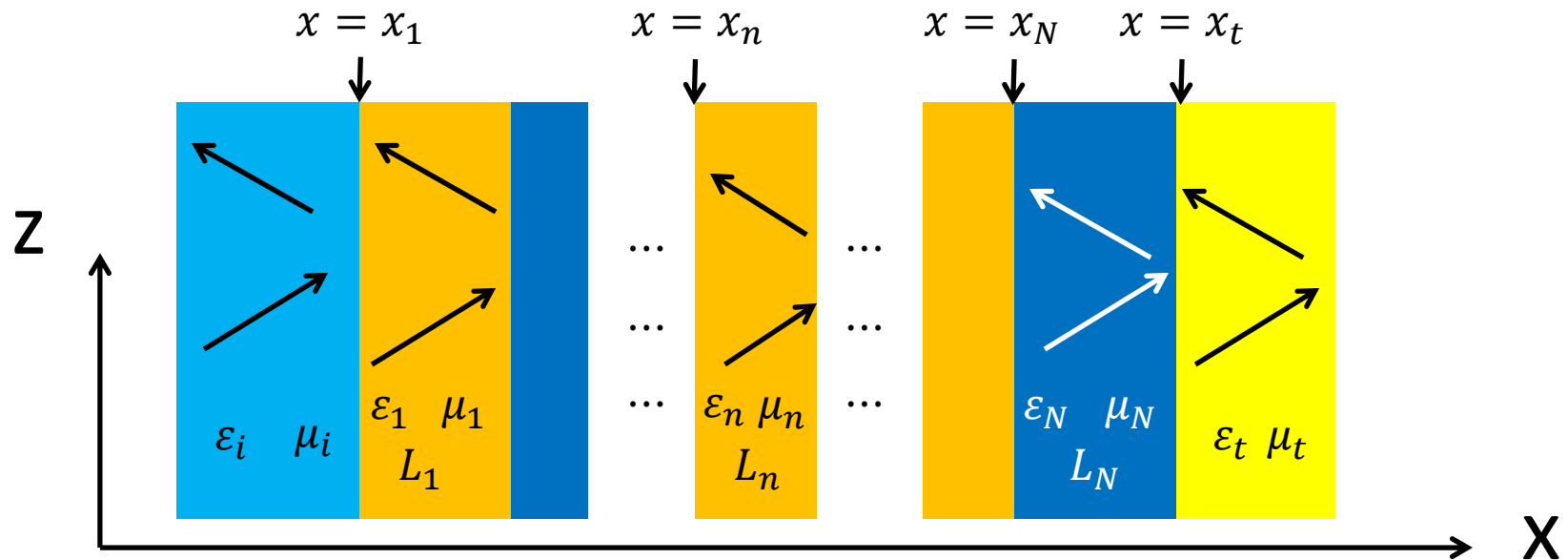


$$\begin{bmatrix} 1 & 1 \\ \frac{k_{ix}}{\mu_i} & -\frac{k_{ix}}{\mu_i} \end{bmatrix} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{k_{1x}}{\mu_1} & -\frac{k_{1x}}{\mu_1} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$D_i^{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix} = D_1^{TE} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$D_n^{TE} = \begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\mu_n} & -\frac{k_{nx}}{\mu_n} \end{bmatrix}$$

# TE Transfer Matrix



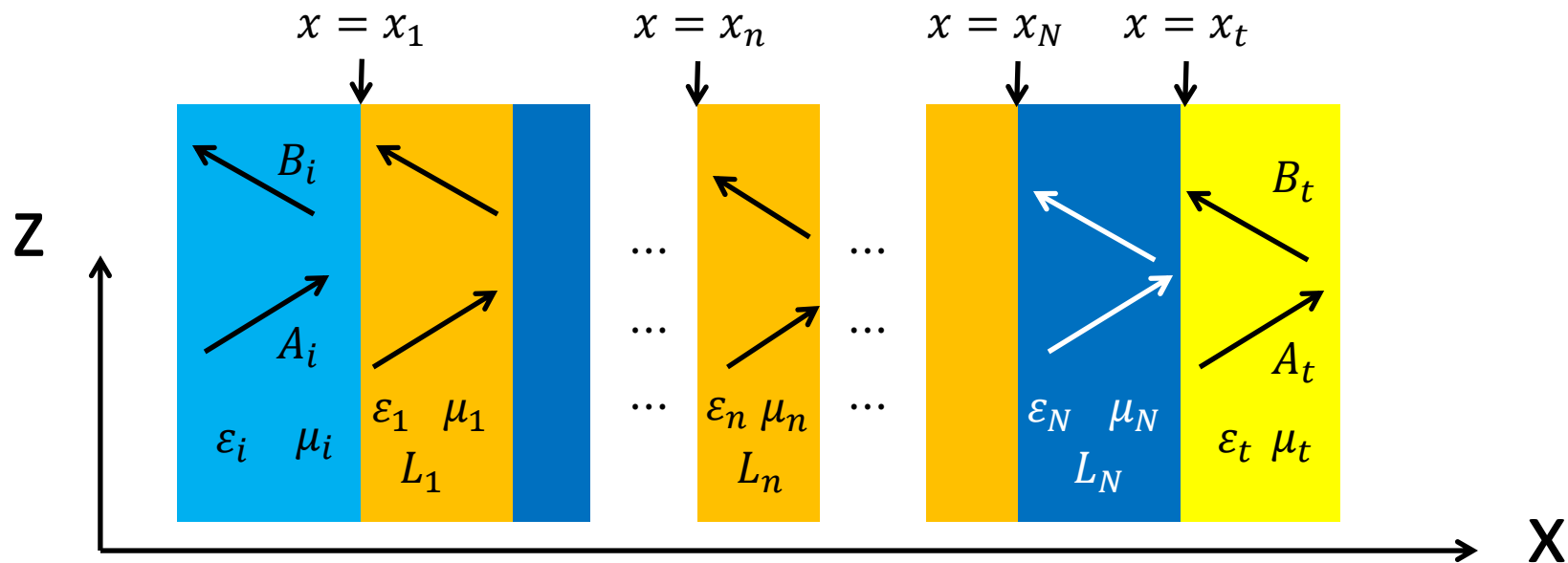
$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = [(D_t^{TE})^{-1} D_N^{TE} P_N^{TE}] T_{N-1}^{TE} \cdots T_n^{TE} \cdots T_2^{TE} T_1^{TE} [(D_1^{TE})^{-1} D_i^{TE}] \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = T_{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix} \quad T_{TE} = [(D_t^{TE})^{-1} D_N^{TE} P_N^{TE}] T_{N-1}^{TE} \cdots T_n^{TE} \cdots T_2^{TE} T_1^{TE} [(D_1^{TE})^{-1} D_i^{TE}]$$

$$T_n = [D_{n+1}^{TE}]^{-1} D_n^{TE} P_n^{TE}$$

Assign as HW: determine the transfer matrix between the Nth and the transmission layer.

# Reflection and Transmission: TE



$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = T_{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

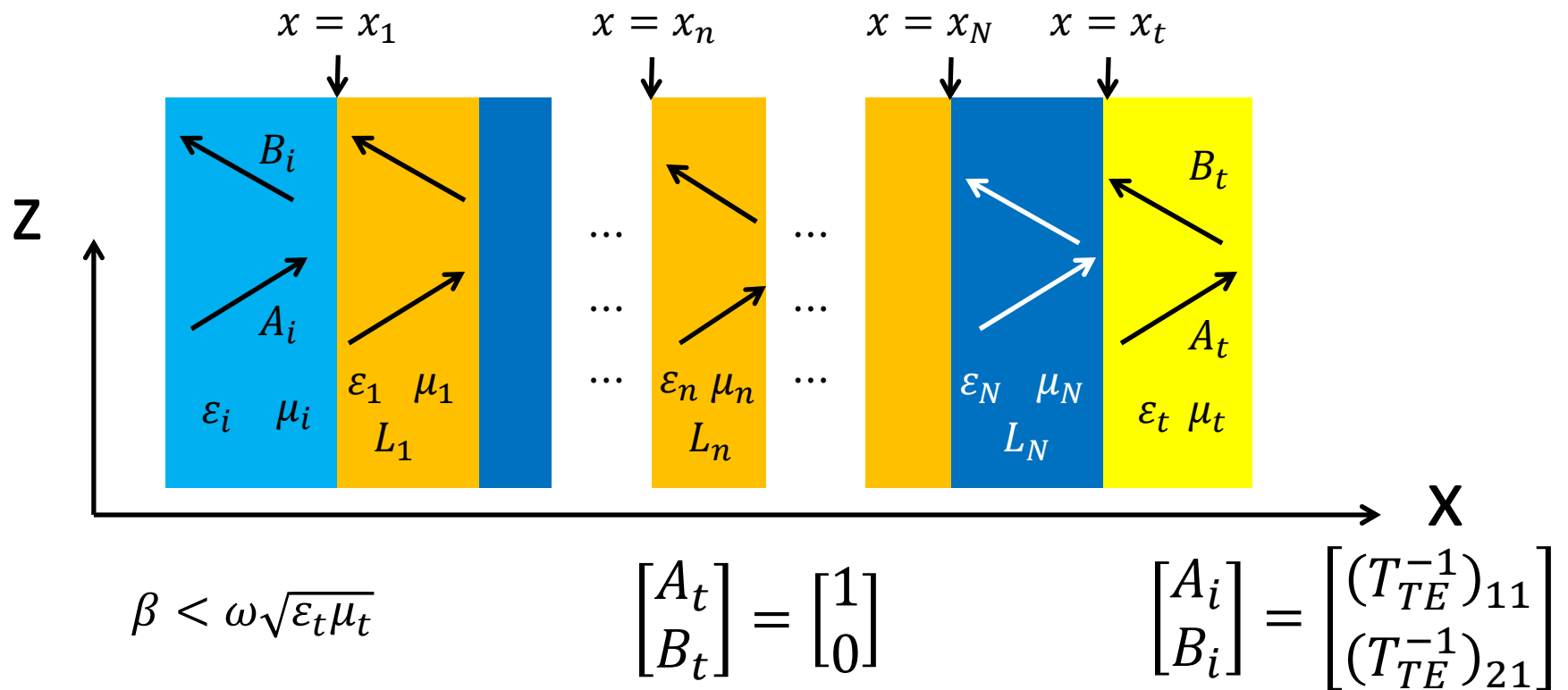
Boundary Condition:  
Only outgoing uniform EM wave  
exist in the transmission layer

$$k_{nx} = \sqrt{\omega^2 \epsilon_t \mu_t - \beta^2} \quad \beta < \omega \sqrt{\epsilon_t \mu_t}$$

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} (T_{TE}^{-1})_{11} & (T_{TE}^{-1})_{12} \\ (T_{TE}^{-1})_{21} & (T_{TE}^{-1})_{22} \end{bmatrix} \begin{bmatrix} A_t \\ B_t \end{bmatrix}$$

$$\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} (T_{TE}^{-1})_{11} \\ (T_{TE}^{-1})_{21} \end{bmatrix}$$

# Reflection: TE



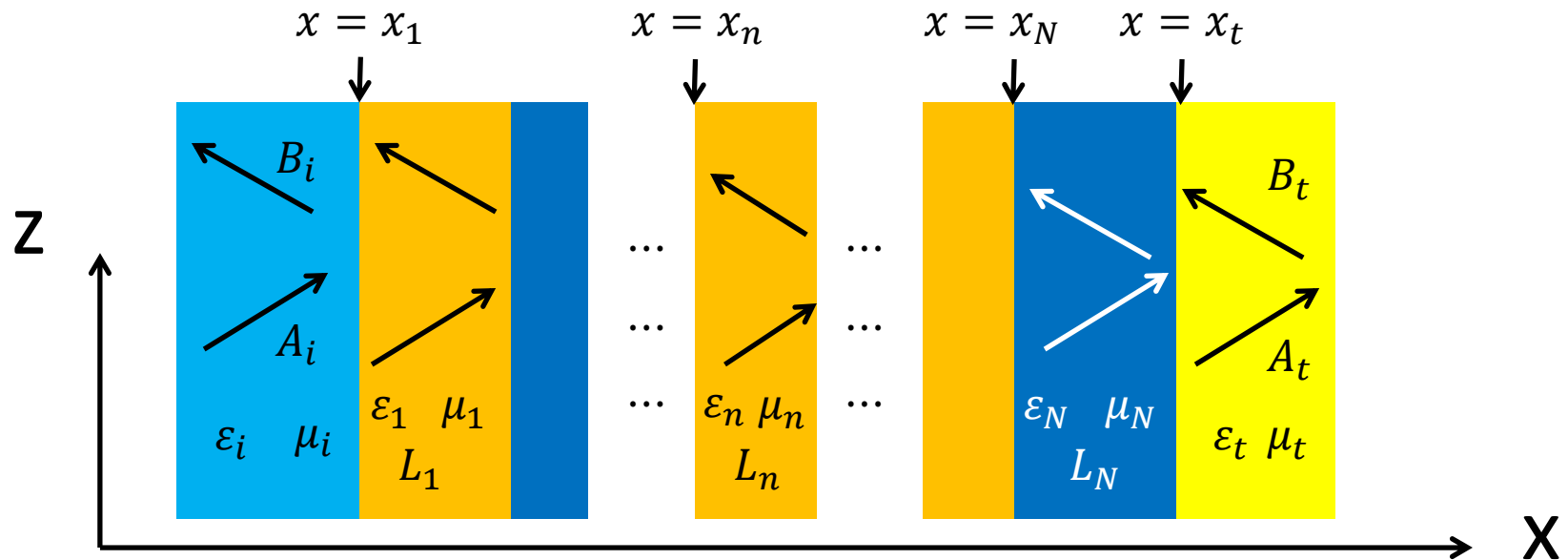
$S_{rx}$ : Poynting vector associated with the reflected wave.

$S_{tx}$ : Poynting vector associated with the transmitted wave.

$$R = \frac{S_{rx}}{S_{ix}} = \left| \frac{B_i}{A_i} \right|^2 = \left| \frac{(T_{TE}^{-1})_{21}}{(T_{TE}^{-1})_{11}} \right|^2$$

Reflection coefficient  $R$  represents the fraction of EM wave reflected back by the planar structure.

# Transmission: TE



$$\beta < \omega\sqrt{\epsilon_t\mu_t}$$

$$E_{iy} = e^{j(\omega t - \beta z)} \{ A_i e^{-jk_{ix}(x-x_1)} + B_i e^{jk_{ix}(x-x_1)} \}$$

$$E_{ty} = e^{j(\omega t - \beta z)} \{ A_t e^{-jk_{tx}(x-x_t)} + B_t e^{jk_{tx}(x-x_t)} \}$$

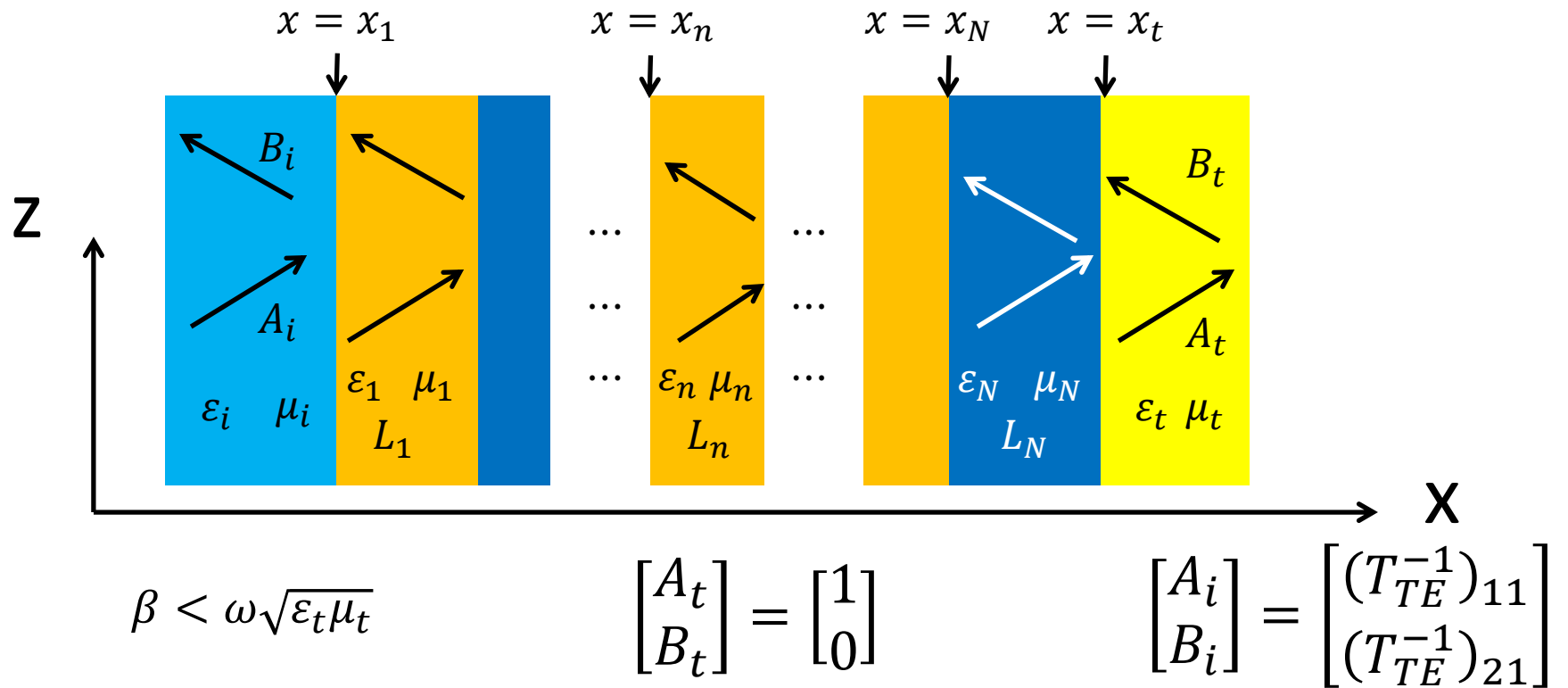
$$\vec{E}(\vec{r}, t) = E_0 e^{j\omega t} e^{-jk_0 z} \vec{e}_x$$

$$\vec{S}(\vec{r}, t) = \frac{1}{2} \frac{k_0}{\omega \mu} |E_0|^2 \vec{e}_z$$

$$S_{ix} = \frac{k_{ix}}{2\omega\mu_i} |A_i|^2$$

$$S_{tx} = \frac{k_{tx}}{2\omega\mu_t} |A_t|^2$$

# Transmission: TE



$$T = \frac{S_{tx}}{S_{ix}} = \frac{\frac{k_{tx}}{2\omega\mu_t} |A_t|^2}{\frac{k_{ix}}{2\omega\mu_i} |A_i|^2} = \frac{\mu_i k_{tx}}{\mu_t k_{ix}} \frac{1}{|(T_{TE}^{-1})_{11}|^2}$$



# Reflection and Transmission: TE

- EM fields take the form of uniform plane wave in the transmission layer:  $\beta < \omega\sqrt{\epsilon_t\mu_t}$
- Calculate the transmission matrix of the entire layered planar structure:  $\begin{bmatrix} A_t \\ B_t \end{bmatrix} = T_{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$
- Calculate reflection and transmission coefficient:  $\begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} (T_{TE}^{-1})_{11} \\ (T_{TE}^{-1})_{21} \end{bmatrix}$
- Calculate the (amplitude and power) reflection and transmission coefficients:

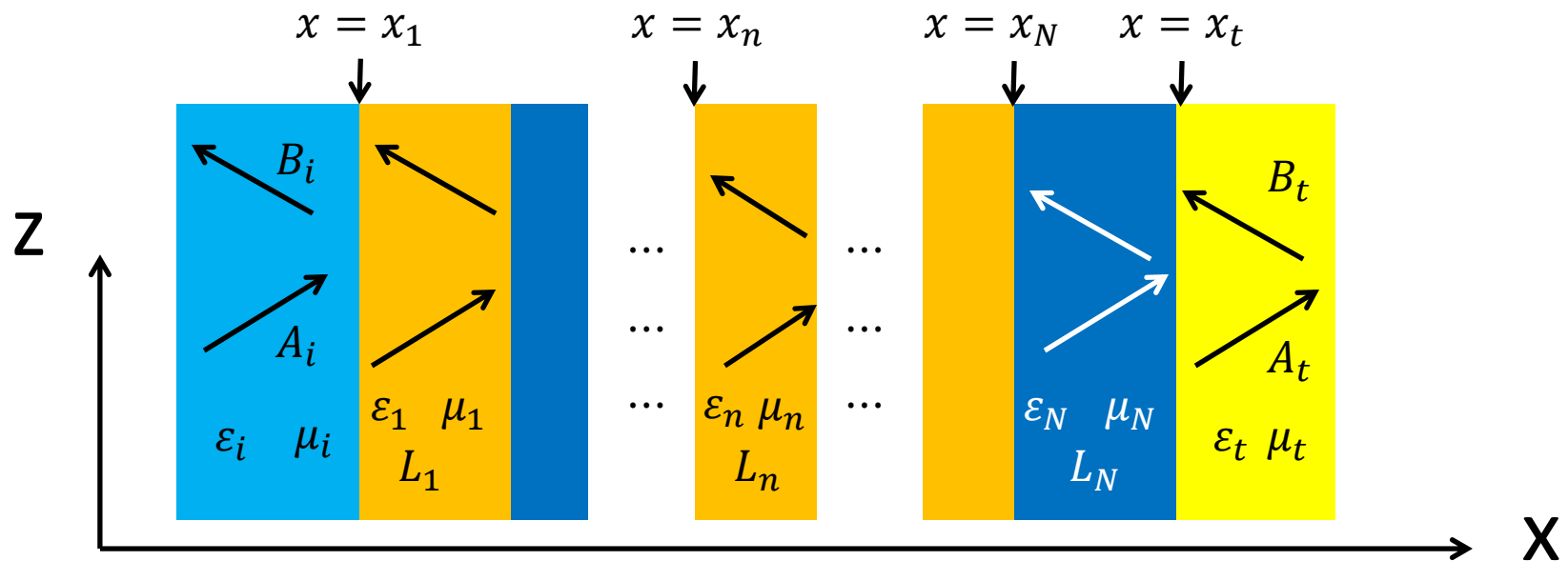
$$r = \frac{B_i}{A_i} = \frac{(T_{TE}^{-1})_{21}}{(T_{TE}^{-1})_{11}}$$

$$R = \frac{S_{rx}}{S_{ix}} = \left| \frac{(T_{TE}^{-1})_{21}}{(T_{TE}^{-1})_{11}} \right|^2$$

$$t = \frac{A_t}{A_i} = \frac{1}{(T_{TE}^{-1})_{11}}$$

$$T = \frac{S_{tx}}{S_{ix}} = \frac{\mu_i k_{tx}}{\mu_t k_{ix}} \frac{1}{|(T_{TE}^{-1})_{11}|^2}$$

# TM Solution



Contains  $H_y$ ,  $E_x$ , and  $E_z$  components. All other components are zero.

$$\vec{H}_n = e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \} \vec{e}_y$$

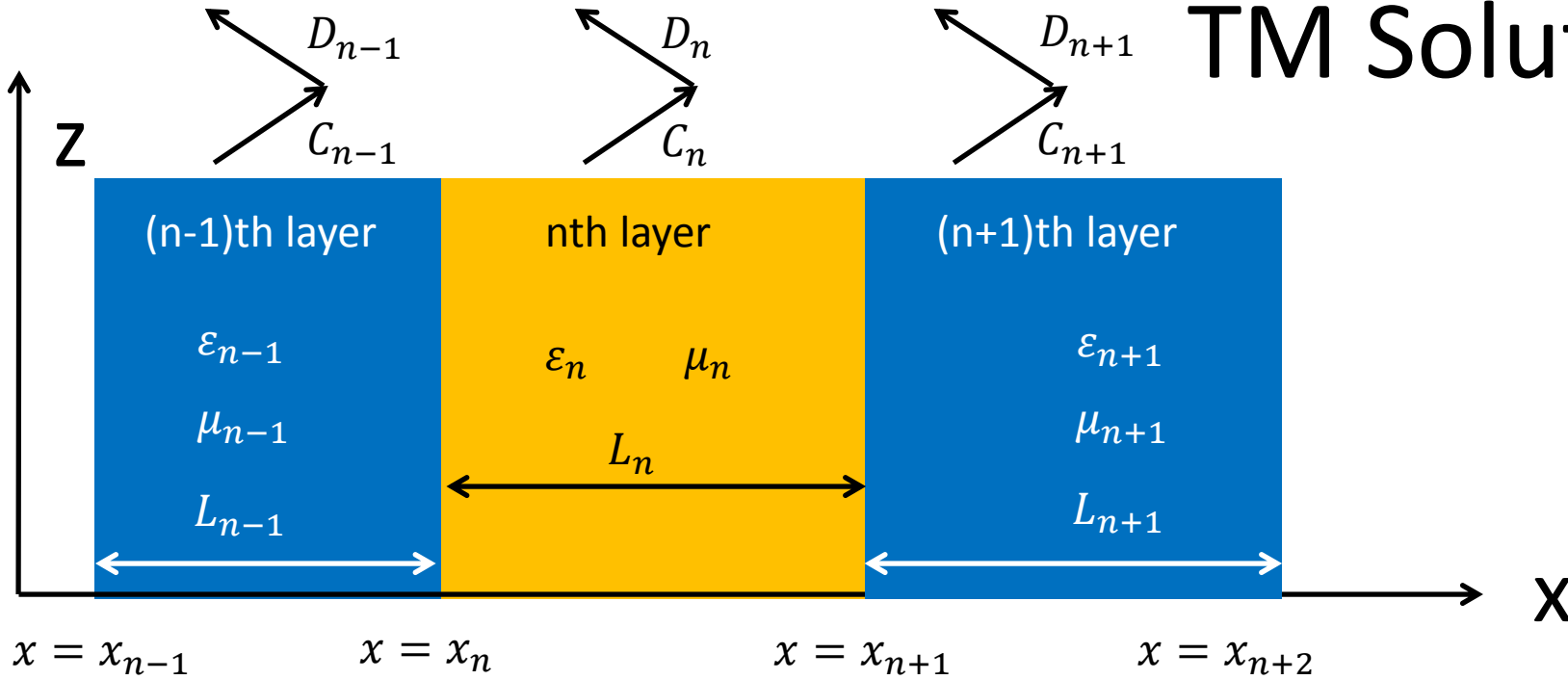
$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} = \frac{\nabla \times \vec{H}}{j\epsilon\omega}$$

$$E_{nx} = \frac{\beta}{\epsilon_n \omega} e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \}$$

$$E_{nz} = \frac{1}{j\epsilon_n \omega} e^{j(\omega t - \beta z)} \{ -jk_{nx} C_n e^{-jk_{nx}(x-x_n)} + jk_{nx} D_n e^{jk_{nx}(x-x_n)} \}$$

# TM Solution



nth layer

$$H_{ny} = e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}L_n} + D_n e^{jk_{nx}L_n} \}$$

$$E_{nz} = \frac{1}{j\epsilon_n\omega} e^{j(\omega t - \beta z)} \{ -jk_{nx}C_n e^{-jk_{nx}L_n} + jk_{nx}D_n e^{jk_{nx}L_n} \}$$

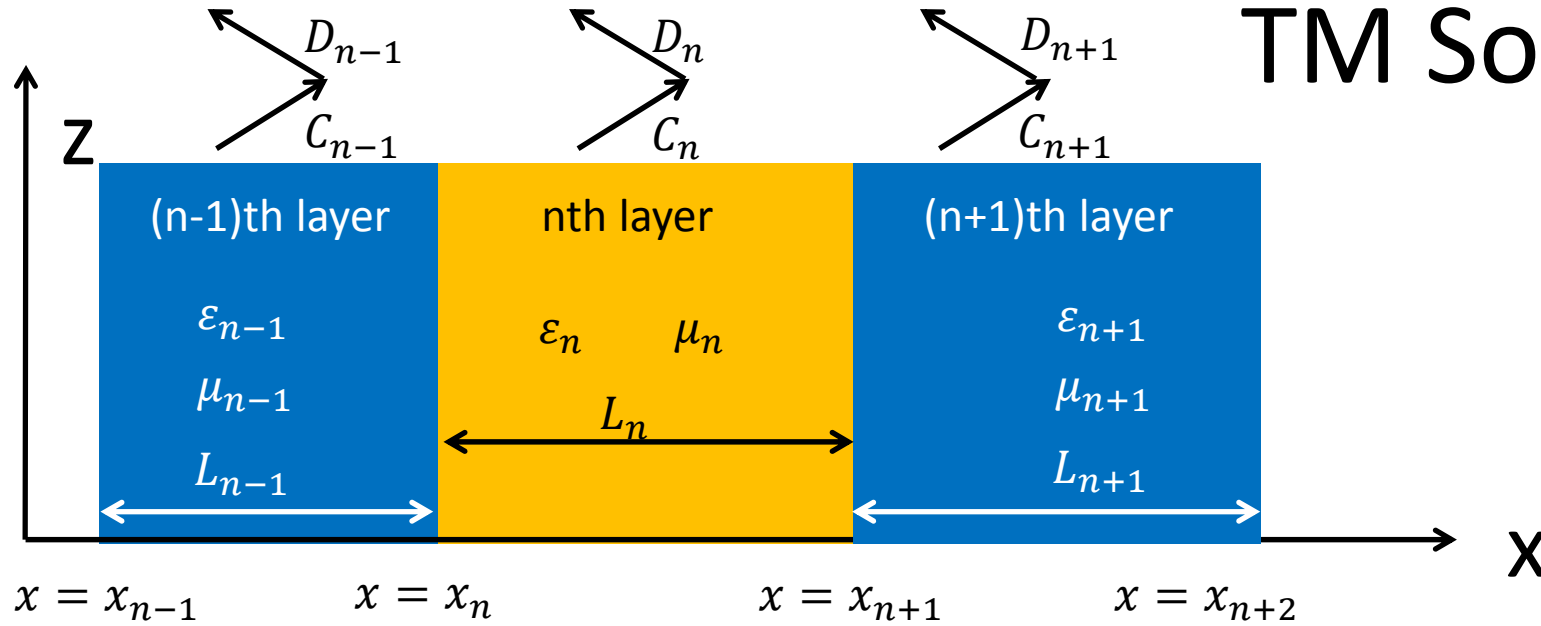
@  $x = x_{n+1}$

(n+1)th layer

$$H_{n+1,y} = e^{j(\omega t - \beta z)} \{ C_{n+1} + D_{n+1} \}$$

$$E_{n+1,z} = \frac{1}{j\epsilon_{n+1}\omega} e^{j(\omega t - \beta z)} \{ -jk_{n+1,x}C_{n+1} + jk_{n+1,x}D_{n+1} \}$$

# TM Solution

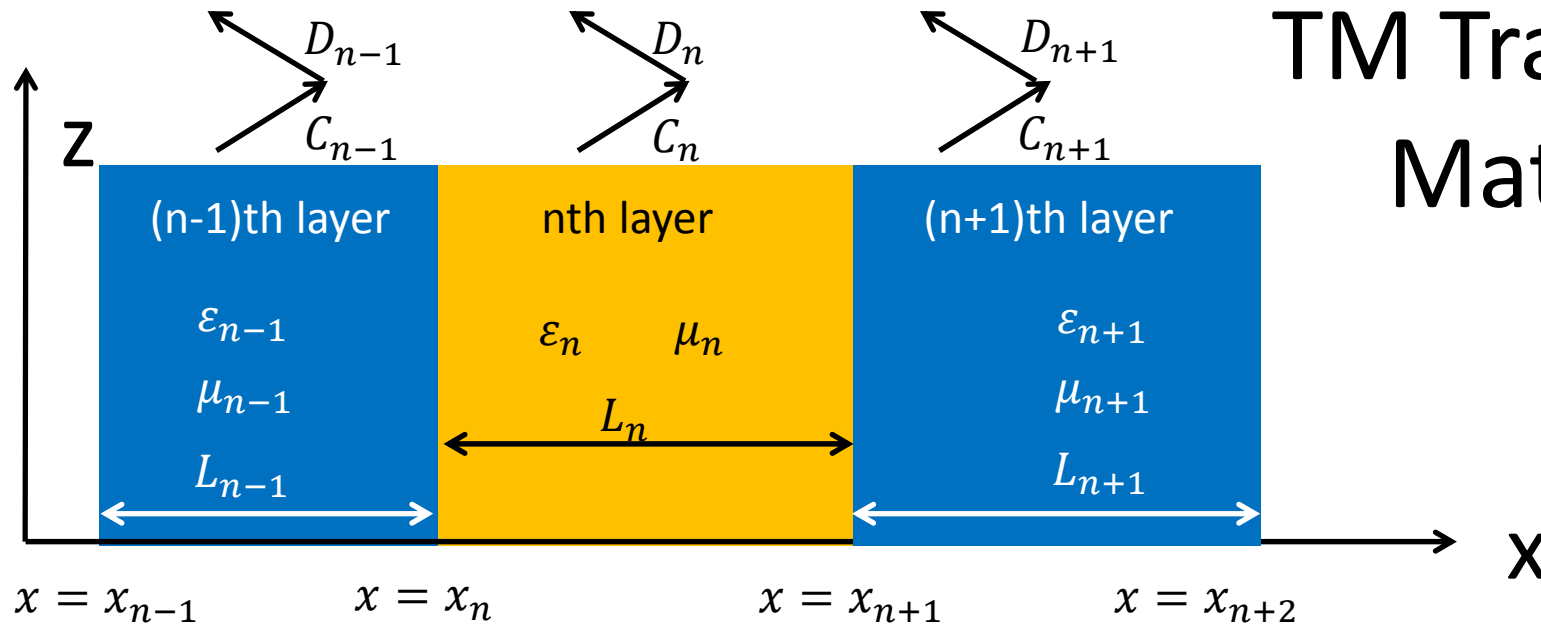


$$C_n e^{-jk_{nx}L_n} + D_n e^{jk_{nx}L_n} = C_{n+1} + D_{n+1}$$

$$\frac{k_{nx}}{\epsilon_n} C_n e^{-jk_{nx}L_n} - \frac{k_{nx}}{\epsilon_n} D_n e^{jk_{nx}L_n} = \frac{k_{n+1,x}}{\epsilon_{n+1}} C_{n+1} - \frac{k_{n+1,x}}{\epsilon_{n+1}} D_{n+1}$$

$$\begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\epsilon_n} & -\frac{k_{nx}}{\epsilon_n} \end{bmatrix} \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{k_{n+1,x}}{\epsilon_{n+1}} & -\frac{k_{n+1,x}}{\epsilon_{n+1}} \end{bmatrix} \begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix}$$

# TM Transfer Matrix



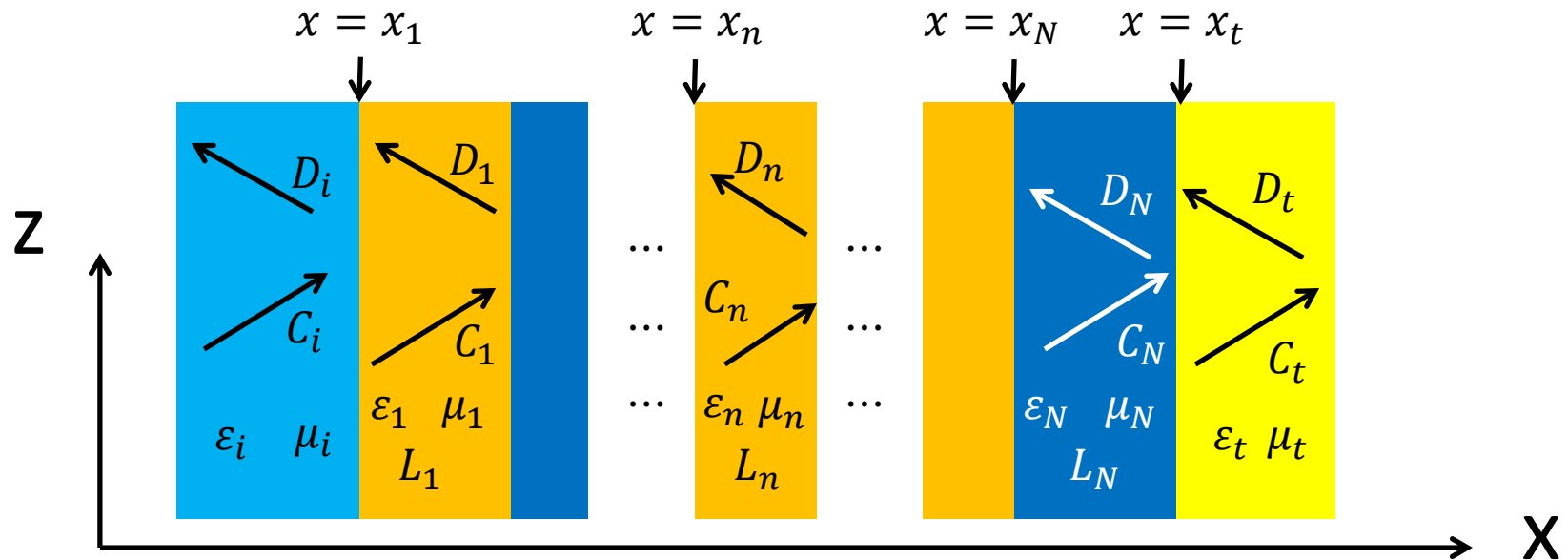
$$D_n^{TM} P_n^{TM} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = D_{n+1}^{TM} \begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix}$$

$$D_n^{TM} = \begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\epsilon_n} & -\frac{k_{nx}}{\epsilon_n} \end{bmatrix}$$

$$P_n^{TM} = \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix}$$

$$D_{n+1}^{TM} = \begin{bmatrix} 1 & 1 \\ \frac{k_{n+1,x}}{\epsilon_{n+1}} & -\frac{k_{n+1,x}}{\epsilon_{n+1}} \end{bmatrix}$$

# TM Transfer Matrix

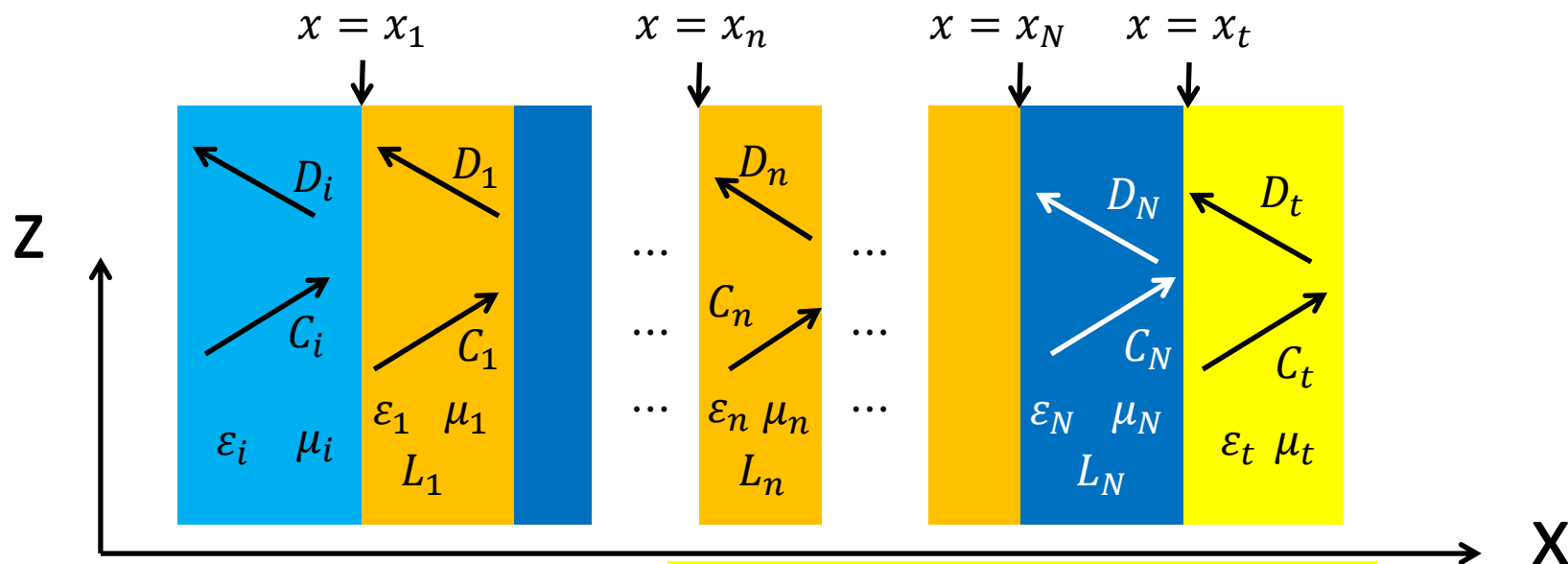


$$\vec{H}_n = e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \} \vec{e}_y$$

$$\begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix} = T_n^{TM} \begin{bmatrix} C_n \\ D_n \end{bmatrix} \quad T_n^{TM} = [D_{n+1}^{TM}]^{-1} D_n^{TM} P_n^{TM}$$

$$\begin{bmatrix} C_N \\ D_N \end{bmatrix} = T_{N-1}^{TM} \cdots T_n^{TM} \cdots T_2^{TM} T_1^{TM} \begin{bmatrix} C_1 \\ D_1 \end{bmatrix}$$

# Reflection and Transmission: TM



Assign as HW, derive this transfer matrix

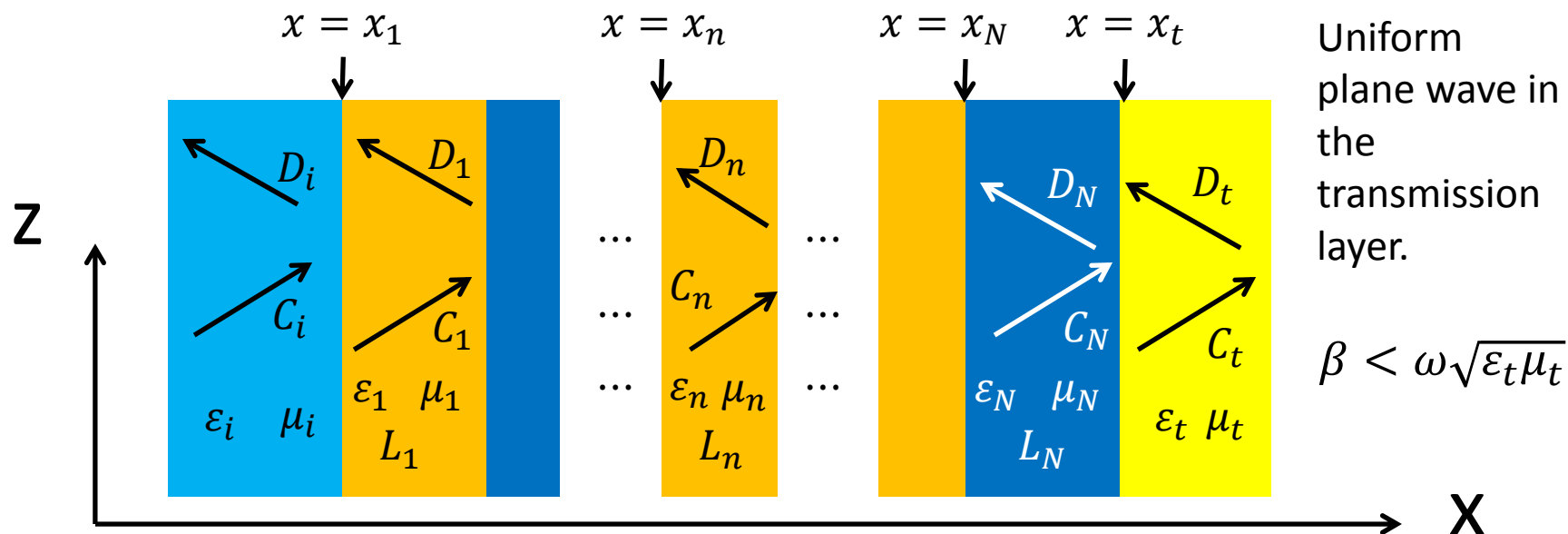
$$\begin{bmatrix} C_t \\ D_t \end{bmatrix} = [(D_t^{TM})^{-1} D_N^{TM} P_N^{TM}] T_{N-1}^{TM} \cdots T_n^{TM} \cdots T_2^{TM} T_1^{TM} \left[ (D_1^{TM})^{-1} D_i^{TM} \right] \begin{bmatrix} C_i \\ D_i \end{bmatrix}$$

$$\begin{bmatrix} C_t \\ D_t \end{bmatrix} = T_{TM} \begin{bmatrix} C_i \\ D_i \end{bmatrix}$$

$$T_n^{TM} = [D_{n+1}^{TM}]^{-1} D_n^{TM} P_n^{TM}$$

$$T_{TM} = [(D_t^{TM})^{-1} D_N^{TM} P_N^{TM}] T_{N-1}^{TM} \cdots T_n^{TM} \cdots T_2^{TM} T_1^{TM} [(D_1^{TM})^{-1} D_i^{TM}]$$

# Reflection and Transmission: TM



$$\vec{H}_n = e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \} \vec{e}_y$$

Uniform Plane Wave  $Z = \frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu}{\epsilon}}$

$$|\vec{S}| = \frac{1}{2} \text{Re}[\vec{E} \times \vec{H}^*] = \frac{1}{2Z} |\vec{E}|^2 = \frac{Z}{2} |\vec{H}|^2$$

$$|\vec{S}| = \frac{1}{2} \frac{k}{\omega\mu} |\vec{E}|^2 = \frac{1}{2} \frac{k}{\omega\mu} \frac{\mu}{\epsilon} |\vec{H}|^2 = \frac{1}{2} \frac{k}{\omega\epsilon} |\vec{H}|^2$$

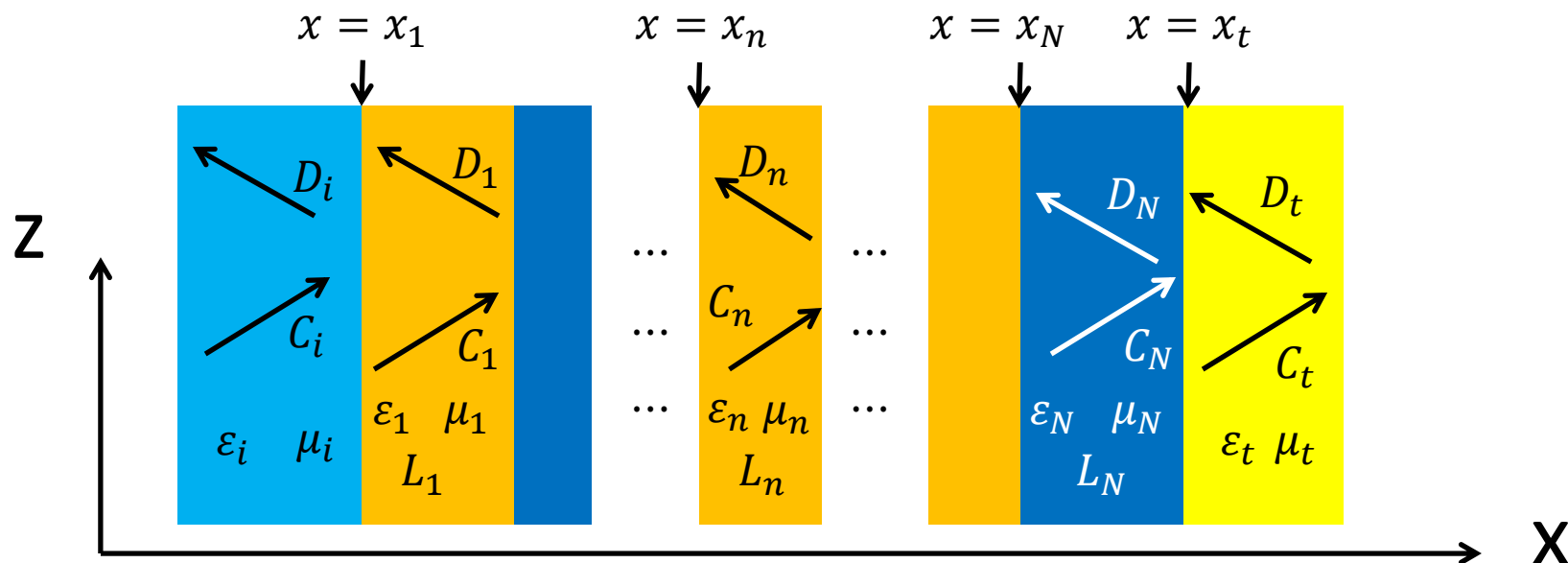
$$S_{ix} = \frac{1}{2} \frac{k_{ix}}{\omega\epsilon_i} |C_i|^2$$

$$S_{rx} = \frac{1}{2} \frac{k_{ix}}{\omega\epsilon_i} |D_i|^2$$

$$S_{tx} = \frac{1}{2} \frac{k_{tx}}{\omega\epsilon_t} |C_t|^2$$



# Reflection and Transmission: TM



Uniform plane wave in the transmission layer.

$$\beta < \omega \sqrt{\epsilon_t \mu_t}$$

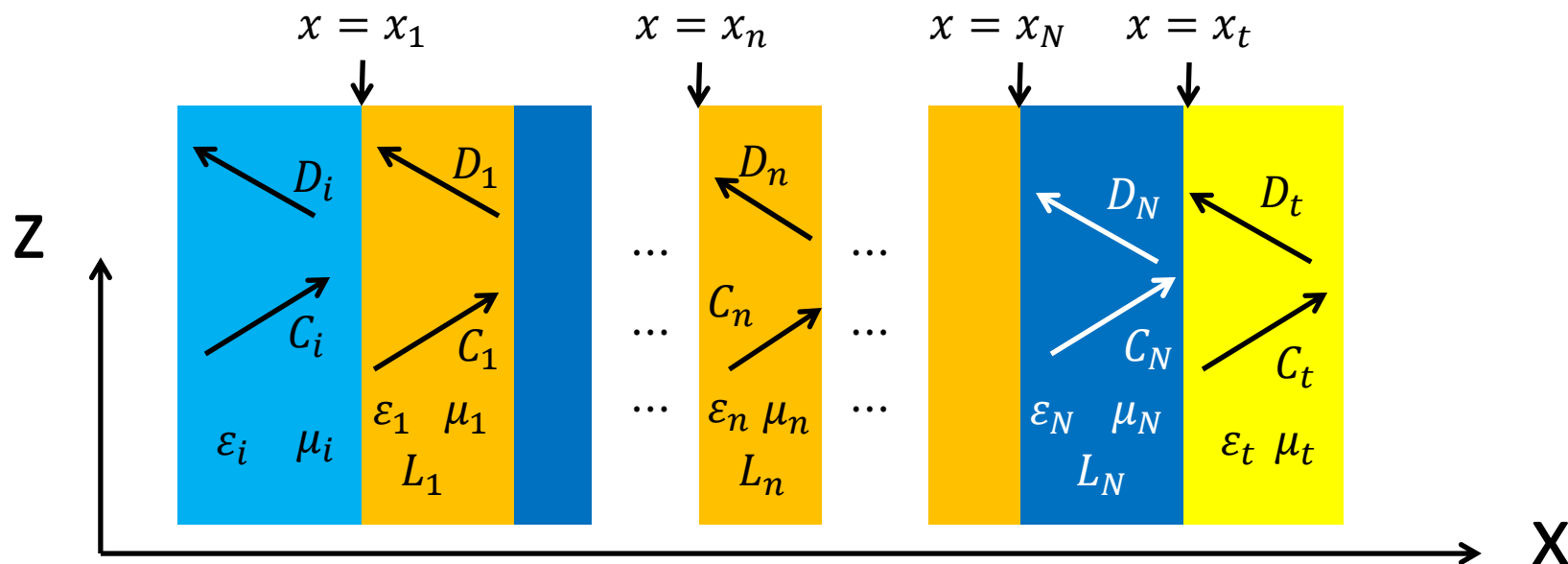
$$\begin{bmatrix} C_t \\ D_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_i \\ D_i \end{bmatrix} = \begin{bmatrix} (T_{TM}^{-1})_{11} \\ (T_{TM}^{-1})_{21} \end{bmatrix}$$

$$\begin{bmatrix} C_t \\ D_t \end{bmatrix} = T_{TM} \begin{bmatrix} C_i \\ D_i \end{bmatrix}$$

$$R = \frac{S_{rx}}{S_{ix}} = \left| \frac{D_i}{C_i} \right|^2 = \left| \frac{(T_{TM}^{-1})_{21}}{(T_{TM}^{-1})_{11}} \right|^2$$

# Reflection and Transmission: TM



$$\beta < \omega\sqrt{\epsilon_t\mu_t} \quad \begin{bmatrix} C_t \\ D_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} C_i \\ D_i \end{bmatrix} = \begin{bmatrix} (T_{TM}^{-1})_{11} \\ (T_{TM}^{-1})_{21} \end{bmatrix}$$

$$T = \frac{S_{tx}}{S_{ix}} = \frac{\frac{1}{2} \frac{k_{tx}}{\omega\epsilon_t} |C_t|^2}{\frac{1}{2} \frac{k_{ix}}{\omega\epsilon_i} |C_i|^2} = \frac{\epsilon_i k_{tx}}{\epsilon_t k_{ix}} \frac{1}{|(T_{TM}^{-1})_{11}|^2}$$

# Reflection and Transmission: TM

- EM fields take the form of uniform plane wave in the transmission layer:  $\beta < \omega\sqrt{\epsilon_t\mu_t}$
- Calculate the transmission matrix of the entire layered planar structure:  $\begin{bmatrix} C_t \\ D_t \end{bmatrix} = T_{TM} \begin{bmatrix} C_i \\ D_i \end{bmatrix}$
- Calculate reflection and transmission coefficient:  $\begin{bmatrix} C_t \\ D_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} C_i \\ D_i \end{bmatrix} = \begin{bmatrix} (T_{TM}^{-1})_{11} \\ (T_{TM}^{-1})_{21} \end{bmatrix}$
- Calculate the (amplitude and power) reflection and transmission coefficients:

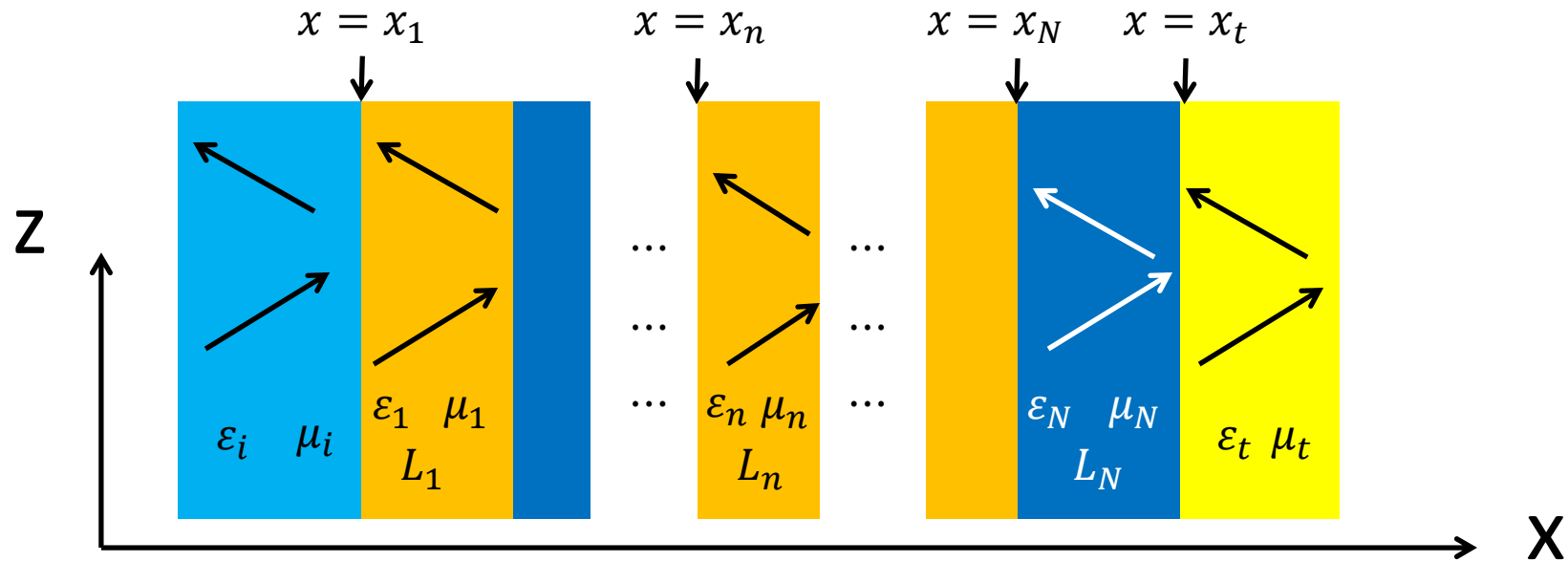
$$r = \frac{D_i}{C_i} = \frac{(T_{TM}^{-1})_{21}}{(T_{TM}^{-1})_{11}}$$

$$R = \frac{S_{rx}}{S_{ix}} = \left| \frac{(T_{TM}^{-1})_{21}}{(T_{TM}^{-1})_{11}} \right|^2$$

$$t = \frac{C_t}{C_i} = \frac{1}{(T_{TM}^{-1})_{11}}$$

$$T = \frac{S_{tx}}{S_{ix}} = \frac{\epsilon_i k_{tx}}{\epsilon_t k_{ix}} \frac{1}{|(T_{TM}^{-1})_{11}|^2}$$

# Comparison of TE and TM Cases



TE  $E_y$   $H_x$   $H_z$

$$\vec{E}_n = e^{j(\omega t - \beta z)} \{ A_n e^{-jk_{nx}(x-x_n)} + B_n e^{jk_{nx}(x-x_n)} \} \vec{e}_y$$

$$H_{nx} = -\frac{\beta}{\omega \mu_n} e^{j(\omega t - \beta z)} \{ A_n e^{-jk_{nx}(x-x_n)} + B_n e^{jk_{nx}(x-x_n)} \}$$

$$H_{nz} = \frac{1}{\omega \mu_n} e^{j(\omega t - \beta z)} \{ k_{nx} A_n e^{-jk_{nx}(x-x_n)} - k_{nx} B_n e^{jk_{nx}(x-x_n)} \}$$

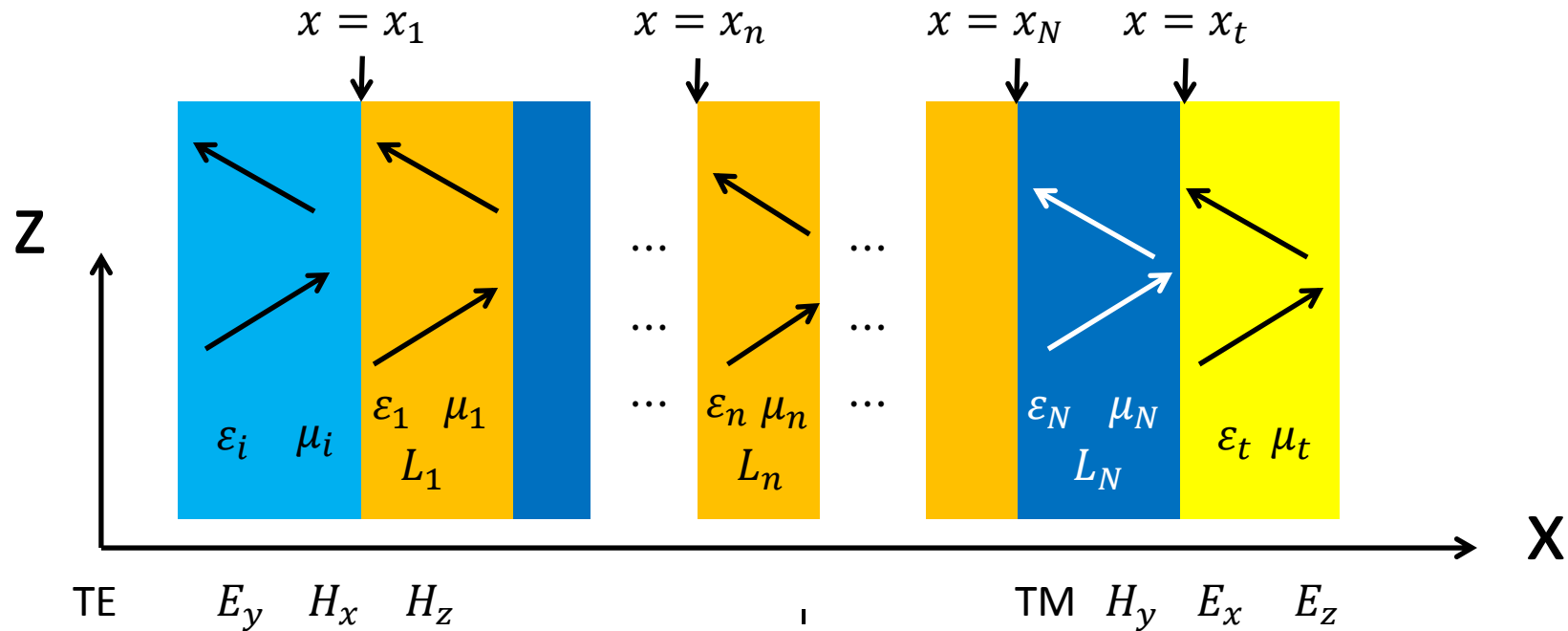
TM  $H_y$   $E_x$   $E_z$

$$\vec{H}_n = e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \} \vec{e}_y$$

$$E_{nx} = \frac{\beta}{\epsilon_n \omega} e^{j(\omega t - \beta z)} \{ C_n e^{-jk_{nx}(x-x_n)} + D_n e^{jk_{nx}(x-x_n)} \}$$

$$E_{nz} = \frac{1}{j \epsilon_n \omega} e^{j(\omega t - \beta z)} \{ -jk_{nx} C_n e^{-jk_{nx}(x-x_n)} + jk_{nx} D_n e^{jk_{nx}(x-x_n)} \}$$

# Comparison of TE and TM Cases



$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix} = T_n^{TE} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = [D_{n+1}^{TE}]^{-1} D_n^{TE} P_n^{TE} \begin{bmatrix} A_n \\ B_n \end{bmatrix}$$

$$D_n^{TE} = \begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\mu_n} & -\frac{k_{nx}}{\mu_n} \end{bmatrix}$$

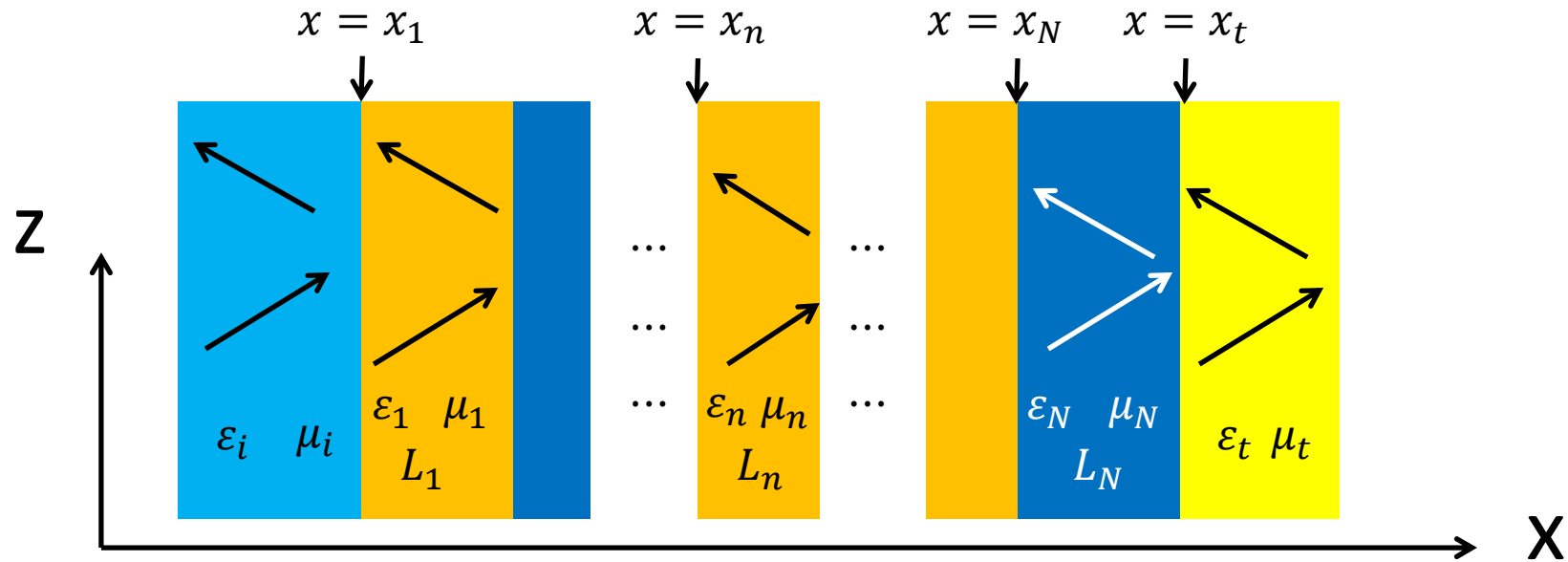
$$P_n^{TE} = \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix}$$

$$\begin{bmatrix} C_{n+1} \\ D_{n+1} \end{bmatrix} = T_n^{TM} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = [D_{n+1}^{TM}]^{-1} D_n^{TM} P_n^{TM} \begin{bmatrix} C_n \\ D_n \end{bmatrix}$$

$$D_n^{TM} = \begin{bmatrix} 1 & 1 \\ \frac{k_{nx}}{\epsilon_n} & -\frac{k_{nx}}{\epsilon_n} \end{bmatrix}$$

$$P_n^{TM} = \begin{bmatrix} e^{-jk_{nx}L_n} & 0 \\ 0 & e^{jk_{nx}L_n} \end{bmatrix}$$

# Comparison of TE and TM Cases



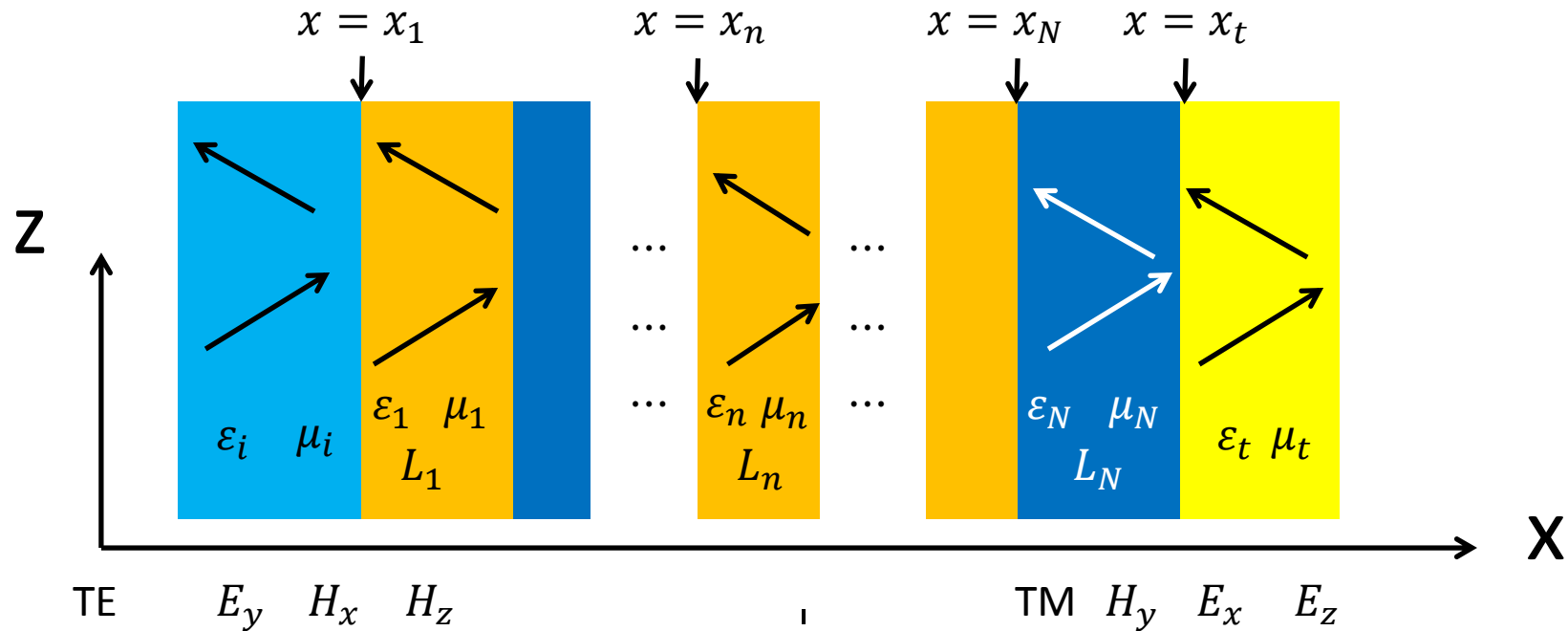
$$\text{TE} \quad \begin{bmatrix} A_t \\ B_t \end{bmatrix} = T_{TE} \begin{bmatrix} A_i \\ B_i \end{bmatrix}$$

$$T_{TE} = T_N^{TE} T_{N-1}^{TE} \cdots T_n^{TE} \cdots T_2^{TE} T_1^{TE} [(D_1^{TE})^{-1} D_i^{TE}]$$

$$\text{TM} \quad \begin{bmatrix} C_t \\ D_t \end{bmatrix} = T_{TM} \begin{bmatrix} C_i \\ D_i \end{bmatrix}$$

$$T_{TM} = T_N^{TM} T_{N-1}^{TM} \cdots T_n^{TM} \cdots T_2^{TM} T_1^{TM} [(D_1^{TM})^{-1} D_i^{TM}]$$

# Comparison of TE and TM Cases



$$r = \frac{(T_{TE}^{-1})_{21}}{(T_{TE}^{-1})_{11}}$$

$$R = \left| \frac{(T_{TE}^{-1})_{21}}{(T_{TE}^{-1})_{11}} \right|^2$$

$$t = \frac{1}{(T_{TE}^{-1})_{11}}$$

$$T = \frac{\mu_i k_{tx}}{\mu_t k_{ix}} \frac{1}{|(T_{TE}^{-1})_{11}|^2}$$

$$r = \frac{(T_{TM}^{-1})_{21}}{(T_{TM}^{-1})_{11}}$$

$$R = \left| \frac{(T_{TM}^{-1})_{21}}{(T_{TM}^{-1})_{11}} \right|^2$$

$$t = \frac{1}{(T_{TM}^{-1})_{11}}$$

$$T = \frac{\epsilon_i k_{tx}}{\epsilon_t k_{ix}} \frac{1}{|(T_{TM}^{-1})_{11}|^2}$$