Physics Forum Post on Legendre Polynomial Orthogonality

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There is something slightly messy about the proof of orthogonality given by maverick6664. Technically, one would have to check that the boundary terms after each integration by parts vanish. For example, the first boundary term would show up as follows: let $V = x^2 - 1$, then

$$\int_{-1}^{1} \frac{d^n V^n}{dx^n} \frac{d^k V^k}{dx^k}\, dx = \left(\frac{d^n V^n}{dx^n} \frac{d^{k-1} V^k}{dx^{k-1}} \right)\Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n+1} V^n}{dx^{n+1}} \frac{d^{k-1} V^k}{dx^{k-1}}\, dx$$

Now, it's true that the boundary term vanishes, but I think it certainly needs to be considered rigorously. Rewrite $V = x^2 - 1$ as $V = (x+1)(x-1)$. Then, it follows from the Liebniz rule that

$$\frac{d^j V^k}{dx^j} = \sum_{m=0}^{j} \binom{j}{m} \left(\frac{d^{j-m}}{dx^{j-m}} (x+1)^k \right) \left(\frac{d^m}{dx^m} (x-1)^k \right)$$

Thus, it's clear that if $j < k$, all of the terms of $d^j V^k / dx^j$ have a power of $(x-1)$ and $(x+1)$ in them and so they all vanish at 1 and -1. In the case at hand, we care about $j = k-1$ which is indeed less than k.

A cleaner way to prove orthogonality is as follows. We know that $P_n(x)$ and $P_m(x)$ satisfy the Legendre differential equation by definition:

$$\frac{d}{dx}\left((1-x^2) \frac{dP_n}{dx} \right) + n(n+1) P_n = 0 \qquad \text{similarly for } P_m$$

I will rewrite this as $((1-x^2)P_n')' + n(n+1)P_n = 0$ and similarly for P_m. Multiply the P_n equation by P_m and vice versa and subtract the two. You get

$$\left(n(n+1) - m(m+1) \right) P_n P_m = ((1-x^2)P_n')' P_m - ((1-x^2)P_m')' P_n$$

Let $n\&m = n(n+1) - m(m+1)$. Now we can use integration by parts once on each term on the right:

$$n\&m \int_{-1}^{1} P_n P_m \, dx = (1-x^2)P_n' P_m \big|_{-1}^{1} - (1-x^2)P_m' P_n \big|_{-1}^{1} - \int_{-1}^{1} (1-x^2) P_n' P_m' \, dx + \int_{-1}^{1} (1-x^2) P_m' P_n' \, dx$$

Now, it's very clear that the right hand side vanishes. The last two integrals are the exact same integral with opposite signs, so they cancel. And, since Legendre polynomials have finite degree (maximum power of x), they and their derivatives are always finite. Thus, the boundary terms vanish because of the $(1 - x^2)$ term, which is zero both at 1 and -1. Since $n\&m \neq 0$ if $n \neq m$, we have proven that

$$\int_{-1}^{1} P_n P_m \, dx = 0 \qquad \text{if } n \neq m$$