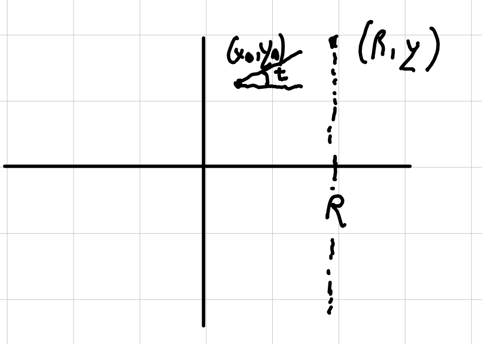
Suppose you have a point source $(x_0, y_0)$ that emits radiation in uniform random directions $t \sim U[-\pi, \pi)$. What is the density $p(y)$ of photons hitting a vertical detector at $x = R$?




The governing equations for a photon travelling distance $r$ at angle $t$ and hitting the vertical detector is then

$$R = x = x_0 + r\cos t$$
$$y = y_0 + r\sin t$$

since

$$r = \frac{R - x_0}{\cos t}$$

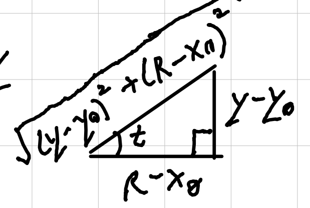
then

$$y = y_0 + (R - x_0)\tan t$$

we can then transform the uniform distribution of angles with the substitution

$$y = y_0 + (R - x_0)\tan t$$
$$dy = (R - x_0)\sec^2 t \, dt$$
$$\frac{\cos^2 t}{R - x_0}\, dy = dt$$

Using the substitution $\tan t = \frac{y - y_0}{R - x_0}$ we can draw a triangle to put $t$ in terms of $y$




$$\cos t = \frac{R - x_0}{\sqrt{(y - y_0)^2 + (R - x_0)^2}}$$

hence

$$\cos^2 t = \frac{(R - x_0)^2}{(y - y_0)^2 + (R - x_0)^2}$$

There are 2 periods of $\tan t$ in $t \in [-\frac{\pi}{2}, \frac{3\pi}{2})$ so we have a multiplier of 2 and the bounds $y \in (-\infty, \infty)$

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} dt = \frac{\cancel{2}}{\cancel{2}\pi}\int_{-\infty}^{\infty} \frac{|R - x_0|}{(y - y_0)^2 + (R - x_0)^2}\, dy$$

and so

$$p(y) = \frac{1}{\pi}\frac{|R - x_0|}{(y - y_0)^2 + (R - x_0)^2}$$

This is the Cauchy distribution

with a little rewriting, we can make its CDF more apparent

$$p(y) = \frac{1}{\pi|R - x_0|}\frac{1}{1 + \left(\frac{y - y_0}{R - x_0}\right)^2}$$

$$\Phi(y) = \frac{1}{\pi|R - x_0|}\int_{-\infty}^{y} \frac{1}{1 + \left(\frac{s - y_0}{R - x_0}\right)^2}\, ds$$

$$v = \frac{s - y_0}{|R - x_0|}$$
$$|R - x_0|\, dv = ds$$

$$= \frac{1}{\pi\cancel{|R - x_0|}}\int_{-\infty}^{\frac{y - y_0}{|R - x_0|}} \frac{\cancel{|R - x_0|}}{1 + v^2}\, dv$$

$$= \frac{1}{\pi}\tan^{-1}(v)\Big|_{-\infty}^{\frac{y - y_0}{|R - x_0|}}$$

$$= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{y - y_0}{|R - x_0|}\right) - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{y - y_0}{|R - x_0|}\right) + \frac{1}{2}$$

So the normalized distribution

$$1 = \frac{1}{Z}\frac{1}{\pi}\int_a^b \frac{|R - x_0|}{(R - x_0)^2 + (y - y_0)^2}\, dy$$

$$Z = \Phi(b) - \Phi(a)$$