

# The Logistic Map

## 1 Introduction

Chaotic dynamics was made popular by the computer experiments of Robert May and Mitchell Feigenbaum on a mapping known as the logistic map. The remarkable feature of the logistic map is in the simplicity of its form (quadratic) and the complexity of its dynamics. It is the simplest model that shows chaos.

The logistic map is the simplest model in population dynamics that incorporates the effects of both birth and death rates. It is given by the formula:

$$x_{n+1} = f(x_n) = \lambda x_n (1 - x_n) . \quad (1)$$

where the function  $f(x) = \lambda x(1 - x)$  is called the logistic mapping and the parameter  $\lambda$  models the *effective growth rate*. The population size,  $x_n$  at the  $n^{\text{th}}$  year, is defined relative to the maximum population size the ecosystem can sustain and is therefore a number between 0 and 1. The parameter  $\lambda$  is restricted between 0 and 4 to keep the system bounded and therefore the model to make physical sense.

The logistic equation gives the rule for determining the relative population  $x_{n+1}$  at the  $(n+1)^{\text{th}}$  year in terms of the population in the  $n^{\text{th}}$  year. To get a physical understanding of the terms in the the logistic equation, we can think of the  $\lambda x_n$  term as a positive feedback term in the sense that as  $x_n$  increases so does the value of  $\lambda x_n$ . This is same as saying that the population size in the next year ( $x_{n+1}$ ) is determined by the product of the previous population size  $x_n$  and the rate ( $\lambda$ ) at which the population grows. Similarly, the term  $(1 - x_n)$  can be thought of as a negative feedback, since increasing  $x_n$  will decrease  $(1 - x_n)$  and therefore  $(1 - x_n)$  can be thought of as population decline due to over population and scarce resources.

## 2 Fixed Points

It is clear that for values of  $\lambda$  between  $[0, 1]$ , if we start iterating the equation with any value  $x_0$  the value of  $x_n$  will settle down to 0. This can be understood from the fact that the the logistic equation is a product of three numbers, namely,  $\lambda$ ,  $x_n$  and  $(1 - x_n)$  which are all between  $[0, 1]$ , the population at the next time step must always be smaller than what is at the current time step. Thus, no matter what initial value we choose for  $x_0$ , if  $\lambda \leq 1.0$ , the population is doomed to extinction. The point zero is called a **fixed point**. This is an example of an **attracting** fixed point as the iterates converge towards that point.

For values of  $\lambda$  between  $(1, 3)$  the iterates instead of being attracted to zero, get attracted to a different fixed point. How did the fixed point change ? What happened to the fixed point located at  $x = 0$ . Without going into the details now, we state that the fixed point  $x = 0$  is unstable for  $\lambda$  between  $(1, 3)$  and a new fixed point exists for  $\lambda \geq 1$ .

To determine the fixed point of a map, we do the following. By definition, a fixed point is a point which when fed back into the map gives back the same point. Mathematically, this expressed by the condition  $x = f(x)$ . So for the Logistic map the fixed points are given by

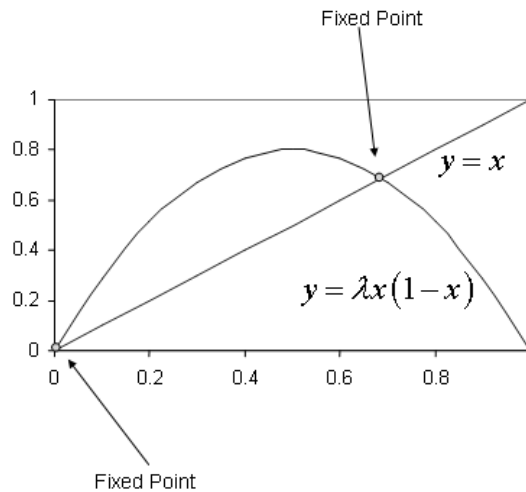


Figure 1: Fixed Points of the Logistic Map

solving

$$x = \lambda x(1 - x) . \quad (2)$$

Solving this equation gives two values,  $x^* = 0$  and  $x^* = 1 - 1/\lambda$ . The second fixed point changes its position according to the value of  $\lambda$  and doesn't exist for  $\lambda < 1$ , because  $x^*$  becomes negative and that is not allowed domain for the logistic equation. The fixed point can also be determined graphically. All fixed points, regardless of the value of  $\lambda$ , can be found by seeing where the parabola ( intersects the identity line  $y = x$ . For values of  $\lambda \geq 1$ , the parabola will always intersect at two points. The fixed point at zero is unstable for  $b \geq 1$  and the second fixed point is stable for  $\lambda$  between  $(1, 3)$ .

### 3 Limit Cycles

We begin by asking, what happens to the long term behavior of the logistic equation when  $\lambda \geq 3$ ? As long as the choice of the initial value of the iterate,  $x_0$ , is not 0 or  $1 - 1/\lambda$ , the logistic equation will **never** converge to any fixed point. Figure (3) shows the **time series** when  $\lambda = 3.2$  and  $x_0 = 0.4$ . In the jargon of dynamicists, you say, the system settles down to a **period 2 limit cycle**. That is, the iterates of the logistic equation **oscillate** between **two values**. The fixed points 0 and  $1 - 1/\lambda$  still exist but they are unstable.

To determine the conditions for a 2 cycle to occur we note that the 2 period limit cycle of the logistic equation can be thought of as the fixed point of the two composition of the logistic equation. As a matter of fact, an **m period limit cycle** of the logistic equation can be thought of as the fixed point of the m composition of the logistic equation. The 2 period limit cycle  $(x_a, x_b)$  implies that  $x_a = f(x_b)$  and  $x_b = f(x_a)$ . Substituting the second equation in the first gives  $x_a = f(f(x_a))$ . So to find the 2 period limit cycle of the logistic equation for a particular value of  $\lambda$  one has to solve the equation:

$$x = f(f(x)) . \quad (3)$$

This quite often written as

$$x = f^2(x) . \quad (4)$$

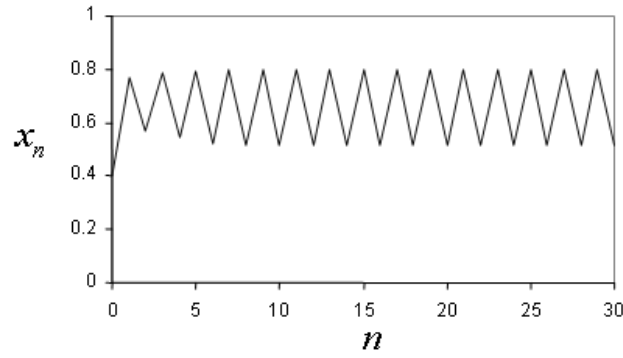


Figure 2: Time Series of Limit Cycle of Period 2

Generalizing, the above argument for an  $m$  period limit cycle, we say, that the existence of  $m$  period limit cycle is established by real solutions to the equation  $x = f^m(x)$ , provided the solutions lie between 0 and 1.

## 4 Stability - Instability of Fixed Points and Limit Cycles

For one-dimensional maps like the logistic map, there is a very simple method to determine if a fixed point or limit cycle is stable or unstable. A point is stable if a small perturbation in the initial condition leads to a same behavior by the iteration. A point is unstable if a small perturbation in the initial condition leads to a different behavior by the iteration. The idea behind the technique is to examine the local behavior of the map in the vicinity of the fixed point or the limit cycle. The first derivative of the logistic equation is given by  $f'(x) = \lambda(1 - 2x)$  which is the slope of the parabola at point  $x$ . To find the stability of the fixed point, we evaluate the slope at the fixed point and the following conditions characterize the behavior of the fixed point:

$$|f'(x)| < 1 \quad \text{stable and attracting} \quad (5)$$

$$f'(x) = 0 \quad \text{super stable} \quad (6)$$

$$|f'(x)| > 1 \quad \text{unstable and repelling} \quad (7)$$

$$|f'(x)| = 1 \quad \text{neutral} \quad (8)$$

The extension of the above analysis to determine the stability of a limit cycle is nearly identical except for the fact that a limit cycle will oscillate between, say,  $m$  points in an orbit:  $x_1, x_2, x_3, \dots, x_m$ , where we have labeled the orbit in the order they are visited. As mentioned earlier, an  $m$  period limit cycle of the logistic equation is the fixed point of the  $m$  composition of the logistic equation, therefore to determine the stability of  $m$  period limit cycle, we evaluate the slope of  $f^m(x)$  at any one of the  $m$  points of the limit cycle and the stability of the limit cycle is determined by checking which one of the above four conditions is satisfied. Actually, we can do a little better than this. Using the chain rule for differentiation, the above condition can be reduced to the evaluation of the expression

$$|f'(x_1) \times f'(x_2) \times f'(x_3) \dots f'(x_m)| \quad (9)$$

and then checking to see which one of the above four conditions is satisfied, to determine the stability of the limit cycle.

If we increase the value of  $\lambda$  even more, we see 4 period limit cycle bifurcating to 8 period limit cycle, then a 16 period limit cycle, and so on. At a very special critical value, the logistic system falls into what is essentially an infinite-period limit cycle. This is **chaos**.

## 5 Cobweb Diagram

One can plot the results of this sequence with a "Cobweb" diagram. If we plot the logistic function  $y = f(x)$ , we will have a curve in the  $xy$  plane whose position relative to the  $x = y$  line is determined by the control parameter  $\lambda$ . A Cobweb diagram traces out the sequence created by the Logistic map as follows: we begin at the initial condition along the  $x$  axis. This is used to seed the mapping and produce our next number, which is on the  $y$ -axis and is located by the logistic function curve. Thus we draw a vertical line from the  $x$ -axis to the logistic curve. In our next iteration, this number acts as our  $x$ -axis number so we jump to the  $x = y$  line and back up to the logistic curve identifying the  $y$  result, repeating the process. The cobweb diagram for the 2 period cycle with  $\lambda = 3.2$  is displayed in the figure below.

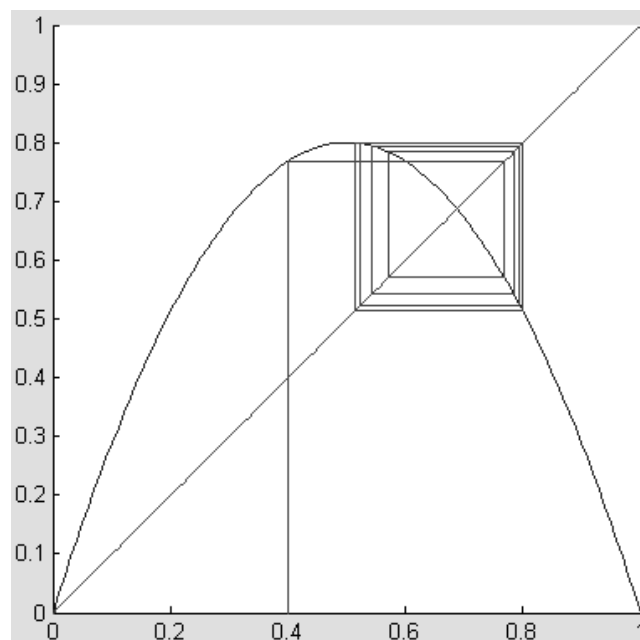


Figure 3: Cob web diagram for a Limit Cycle of Period 2