Derivation: Lorentz Transformations of Motion

To reconcile the solutions of the Maxwellian equations of electromagnetism with the known laws of physics, the physicist Hendrik Lorentz derived a new set of equations for motion based upon the two following postulates:

- Postulate one - The laws of physics hold true for every inertial frame of reference.
- Postulate two - The velocity of light is magnitude (c) in a vacuum independent of the motion of any frame of reference.

Let us begin by imagining two observers, one with relative motion to the other of velocity (v) along the x axis. Let us track events by defining the time and spatial coordinates: for observer one, the time elapsed and distance traversed can be defined as (x,t). In a similar manner, for observer two, the time elapsed and distance traversed can be of definition: (x′,t′). Let the origin of the observer's coordinates coincide, $x = t = x' = t' = 0$

Let us now track the time and spatial coordinates of a beam of light relative to the two observers. As the beam propagates along the positive x axis, the coordinates are $(x - ct) = 0$, and similarly, $(x' - ct') = 0$

Along the negative x axis is $(x + ct) = 0$, and $(x' + ct') = 0$

It is of importance to recognize that for the concision of the origins to remain, a number of change gamma (γ) must be applied to the equations:

$(x' - ct') = \gamma(x - ct)$ and along the negative axis, in a similar manner $(x + ct) = \gamma(x' + ct')$

The easiest way to conceptualize the placement of the number gamma in the equations is to understand that the sign of propagation along the x axis is relative to which observer you are thinking about, and recognizing that the velocity relative to one observer is of the same magnitude to that of the other.

Solving for gamma in one equation, and substituting it into the other produces:

$$(x' - ct') = \frac{(x-ct)(x+ct)}{(x'+ct')}, \text{ or}$$

$$x'^2 - c^2t'^2 = x^2 - c^2t^2$$

It is the equation of the form above that allows for us to solve for how the coordinates are relative to each other. Understanding that velocity $v = \frac{x'}{t'} = \frac{x}{t}$ and solving for t′ yields:

$$t'^2 = t^2\left(1-\frac{v^2}{c^2}\right) + x'^2$$

Allowing for x′ to go to zero, we obtain:

Eq 1) $t'\gamma = t = \frac{t'}{\sqrt{1-\beta^2}}$, where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ (time dilation)

In a similar manner, solving for x yields:

$$x'\gamma = x = \frac{x'}{\sqrt{1-\beta^2}} \text{ (length contraction)}$$

Let us now track the position of an observer relative to the other. Taking a general case of an object in motion, Eq 2) $x = \gamma(x' + vt')$. Now assuming that $\frac{x'}{t'} = k$, the velocity of a third body moving relative to the second, we search for an equation of the velocity of the third relative to the first.($w = \frac{x}{t}$). Working with Eq 1), we obtain:

$$t' = \gamma(t - \frac{vx}{c^2})$$

Understanding what happens when we exchange places with the prime observer, we obtain:

$$\text{Eq 3) } t = \gamma(t' + \frac{vx}{c^2})$$

Dividing Eq 2) by Eq 3), we arrive at a useful velocity addition equation:

$$w = \frac{x}{t} = \frac{k+v}{1+(vk)/c^2}$$

Derivation: Special Theory of Relativity

Were Lorentz is credited with the transformations of motion, Einstein takes the transformations and applies them. We begin by noting the momentum of a particle in motion, p = mv. As consequence of the Lorentz transformations, the particle, and its attributes will likely change as it has relative motion to an observer. Let us note that the particle's momentum at time (t) is at rest, and (t′) represents the particle's motion relative to an observer.

Since $t = \frac{t'}{\gamma}$, and $p = mv = m\frac{dx}{dt}$, we obtain:

$$p = m\frac{dx}{dt'}(\gamma) = mv\gamma$$

Recall that Force dotted with displacement is of equivalence to work,

$$dw = Fdx = m\frac{dv}{dt}dx = (mv)\ dv = (v)\ dp$$

We have concluded that an infinitesimal amount of work being done is equivalent to the object's velocity multiplied into the infinitesimal momentum being changed.

Differentiating p with respect to v yields:

$$dp = (m\gamma + \frac{mv^2}{c^2(1-\beta^2)^{\frac{3}{2}}})dv$$

Multiplying though by v, and applying the work equation:

$$dw = vdp = (mv\gamma)dv + (\frac{mv^3}{c^2(1-\beta^2)^{\frac{3}{2}}})dv$$

By summing the infinitesimal work being applied from zero velocity to the observed velocity, we get the energy it takes for the particle to reach velocity v, or the kinetic energy of the particle:

$$\int_0^{KE} dw = \int_0^v (\frac{mv}{(1-\beta^2)^{\frac{1}{2}}})dv + \int_0^v (\frac{mv^3}{c^2(1-\beta^2)^{\frac{3}{2}}})dv$$

Taking m to be a constant scalar multiple from any frame of reference, recalling that $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, and applying integration by parts to the integral furthest right, we obtain:

$$KE = -mc^2 + mc^2\sqrt{1 - \beta^2} + mv^2\gamma$$

Algebraic steps yield:

$$KE = mc^2(\gamma-1) = mc^2(\frac{1}{\sqrt{1-v^2/c^2}} - 1)$$

Allowing for the relative motion to fall to zero, we get the energy of potential to be:

$PE = mc^2$, using the knowledge that the total energy of an object is PE + KE,

$$E = mc^2\gamma$$

We have now arrived at the famous mass-energy equivalence equations. Let us now take a further step in an attempt to relate momentum to energy. Recall in classical mechanics that

KE = $\frac{mv^2}{2}$, and p = mv, where the common relation between momentum and energy is 2KE = $\frac{p^2}{m}$, now understanding the new relativistic momentum and energy equations, we know that v = $\frac{pc^2}{E}$, where this substitution into E = $\frac{mc^2}{\sqrt{1-v^2/c^2}}$ yields:

$$E^2 = (pc)^2 + (mc^2)^2$$

We have thus achieved the relativistic equations of energy and momentum. One step further shows to us that the classical equations which predict the energy and momentum of particles are in accordance with the relativistic equations of motion.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Using the Taylor expansion form of gamma, we obtain the $\gamma = 1 + \frac{V^2}{2c^2} + \frac{3v^4}{8c^4} \ldots$

Understanding the equation E = $mc^2\gamma$, we see that when $v \ll c$, $E \approx mc^2 + \frac{mv^2}{2}$, where the energy of gain under application of a force is $\frac{mv^2}{2}$ which is consistent with the classical equations of motion.