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LQG for (and by) the Bewildered

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ABSTRACT: We present a pedagogical introduction to the notions underlying the connection formulation of General Relativity - Loop Quantum Gravity (LQG) - with an emphasis on the physical aspects of the framework. We explain, in a concise and clear manner, the steps which lead from the Einstein-Hilbert action for gravity to the construction of the quantum states of geometry, known as *spin-networks*, which provide the basis for the kinematical Hilbert space of quantum general relativity. Along the way we introduce the various associated concepts of *tetrads*, *spin-connection* and *holonomies* which are a pre-requisite for understanding the LQG formalism. Having provided a minimal introduction to the LQG framework, we discuss its applications to the problems of black hole entropy and of quantum cosmology. A list of the most common criticisms of LQG is presented, which are then tackled one by one in order to convince the reader of the physical viability of the theory.

An extensive set of appendices provide accessible introductions to several key notions such as the *Peter-Weyl theorem*, *duality* of differential forms and *Regge calculus*, among others. The presentation is aimed at graduate students and researchers who are familiar with the tools of quantum mechanics and field theory but are intimidated by the seeming technical prowess required to browse through the existing LQG literature. Our hope is to make the formalism appear a little less bewildering to the un-initiated and to help lower the barrier for entry into the field.

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*Todo list

1 Introduction

The goal of Loop Quantum Gravity (LQG) is to take two extremely well-developed and successful theories, General Relativity and Quantum Field Theory, at "face value" and attempt to combine them into a single theory with a minimum of assumptions and deviations from established physics. Our goal, as authors of this paper, is to provide a succinct but clear description of LQG - the main body of concepts in the current formulation of LQG, some of the historical basis underlying these concepts, and a few simple yet interesting results - aimed at the reader who has more curiosity than familiarity with the underlying concepts, and hence desires a broad, pedagogical overview before attempting to read more technical discussions. As the title suggests, this paper derives from the desire on our part to clarify our own understanding of the material by attempting to explain it to others. There are several other reviews of this subject [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], which the reader may refer to in order to gain a broader understanding of the subject and in order to sample the various points of view held by researchers in the field.

We will begin with a brief review of the history of the field of quantum gravity in the remainder of this section. Following this we review some topics in General Relativity in Section 2 and Quantum Field Theory in section 3, which hopefully fall into the "Goldilocks zone", providing all the necessary basis for LQG, and nothing more. We may occasionally introduce concepts in greater detail than the reader considers necessary, but we feel that when introducing concepts to a (hopefully) wide audience who find them unfamiliar, insufficient detail is more harmful than excessive detail. We will then sketch a conceptual outline of the broad program of quantization of the gravitational field in section 4, before moving on to our main discussion of the Loop Quantum Gravity approach in 5. In section 6 we cover applications of the ideas and methods of LQG to the counting of microstates of black holes and to the problem of quantum cosmology. Section 7 briefly describes some of the more recent developments. We conclude with criticisms of LQG and rebuttals thereof in 8 along with a discussion of its present status and future prospects.

It is assumed that the reader has a minimal familiarity with the tools and concepts of differential geometry, quantum field theory and general relativity, though we aim to remind the reader of any relevant technical details as necessary.

Before we begin, it would be helpful to give the reader a historical perspective of the developments in theoretical physics which have led us to the present stage.

We are all familiar with classical geometry consisting of points, lines and surfaces. The framework of Euclidean geometry provided the mathematical foundation for Newton's work on inertia and the laws of motion. In the 19^{th} century Gauss, Riemann and Lobachevsky, among others, developed notions of *curved* geometries in which one or more of Euclid's postulates were loosened. The resulting structures allowed Einstein and Hilbert to formulate the theory of General Relativity which describes the motion of matter through spacetime as a consequence of the curvature of the background geometry. This curvature in turn is induced by the matter content as encoded in Einstein's equations (2.9). Just as the parallel postulate was the unstated assumption of Newtonian mechanics, whose rejection led to Riemmanian geometry, the unstated assumption underlying the framework of general relativity is that of the smoothness and continuity of spacetime on all scales.

Loop quantum gravity and related approaches invite us to consider that our notion of spacetime as a smooth continuum must give way to an atomistic description of geometry in which the classical spacetime we observe around us emerges from the interactions of countless (truly indivisible) *atoms* of spacetime. This idea is grounded in mathematically rigorous results, but is also a natural continuation of the trend that began when 19^{th} century attempts to reconcile classical thermodynamics with the physics of radiation encountered fatal difficulties - such as James Jeans' "ultraviolet catastrophe". These difficulties were resolved only when work by Planck, Einstein and others in the early 20^{th} century provided an atomistic description of electromagnetic radiation in terms of particles or "quanta" of light known as *photons*. This development spawned quantum mechanics, and in turn quantum field theory, while around the same time the special and general theories of relativity were being developed.

In the latter part of the 20^{th} century physicists attempted, without much success, to unify the two great frameworks of quantum mechanics and general relativity . For the most part it was assumed that gravity was a phenomenon whose ultimate description was to be found in the form of a quantum field theory as had been so dramatically and successfully accomplished for the electromagnetic, weak and strong forces in the framework known as the Standard Model. These three forces could be understood as arising due to interactions between elementary particles mediated by gauge bosons whose symmetries were encoded in the groups $U(1)$, $SU(2)$ and $SU(3)$ for the electromagnetic, weak and strong forces, respectively. The universal presumption was that the final missing piece of this "grand unified" picture, gravity, would eventually be found as the QFT of some suitable gauge group. This was the motivation for the various grand unified theories (GUTs) developed by Glashow, Pati-Salam, Weinberg and others where the hope was that it would be possible to embed the gravitational interaction along with the Standard Model in some larger group (such $SO(5)$, $SO(10)$ or E_8 depending on the particular scheme). Such schemes could be said to be in conflict with Occam's dictum of simplicity and Einstein and Dirac's notions of beauty and elegance. *More importantly all these models assumed implicitly that spacetime remains continuous at all scales.* As we shall see this assumption lies at the heart of the difficulties encountered in unifying gravity with quantum mechanics.

Rejecting the notion that systems could absorb or transmit energy in arbitrarily small amounts led to the photonic picture of electromagnetic radiation and the discovery of quantum mechanics. Likewise, rejecting the notion that spacetime is arbitrarily smooth at all scales - and replacing it with the idea that geometry at the Planck scale must have a discrete character - leads us to a possible resolution of the ultraviolet infinities encountered in quantum field theory and to a theory of "quantum gravity".

Bekenstein's observation [12, 13, 14] of the relationship between the entropy of a black hole and the area of its horizon combined with Hawking's work on black hole thermodynamics led to the realization that there were profound connections between thermodynamics, information theory and black hole physics. These can be succinctly summarized by the famous *area law* relating the entropy of a *macroscopic* black hole S_{BH} to its surface area

A:

$$S_{BH} = \gamma A \tag{1.1}$$

where γ is a universal constant and $A \gg A_{pl}$, with $A_{pl} \propto l_p^2$ being the Planck area. While a more detailed discussion will wait until 6.1, we note here that if geometrical observables such as area are quantized, (1.1) can be seen as arising from the number of ways that one can join together $\mathcal{N}$ quanta of area to form a horizon. In LQG the quantization of geometry arises naturally - though not all theorists are convinced that geometry should be quantized or that LQG is the right way to do s.

With this historical overview in mind, it is now worth summarising the basic notions of General Relativity and QFT before we attempt to see how these two disciplines may be unified in a single framework.

2 Classical GR

General Relativity (GR) is an extension of Einstein's Special Theory of Relativity (SR), which was required in order to include observers in non-trivial gravitational backgrounds. SR applies in the absence of gravity, and in essence it describes the behavior of vector quantities in a four-dimensional Galilean space, with the Minkowski metric[1]:

$$\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1), \tag{2.1}$$

leading to a 4D line-element

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{2.2}$$

The speed of a light signal, measured by any inertial observer, is a constant, denoted c. If we denote the components of a vector in four-dimensional spacetime with Greek indices (e.g. v^μ) the Minkowski metric[2] divides vectors into three categories; *timelike* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu < 0$), *null* or *light-like* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu = 0$), and *spacelike* (those vectors for which $\eta_{\mu\nu} v^\mu v^\nu > 0$). Any point, with coordinates (ct, x, y, z), is referred to as an *event*, and the set of all null vectors having their origin at any event define the future light-cone and past light-cone of that event. Events having time-like or null displacement from a given event E_0 (i.e. lying inside or on E_0's lightcones) are causally connected to E_0. Those in/on the past light-cone can influence E_0, those in/on the future lightcone can be influenced by E_0.

General Relativity extends these concepts to non-Euclidean spacetime. The metric of this (possibly curved) spacetime is denoted $g_{\mu\nu}$. Around each event it is possible to consider a sufficiently small region that the curvature of spacetime within this region is negligible, and hence the central concepts of Special Relativity apply locally. Rather than developing

[1] Of course the choice diag$(+1, -1, -1, -1)$ is equally valid but we will have occassion later to restrict our attention to the spacial part of the metric, in which case a positive (spatial) line-element is cleaner to work with.

[2] Strictly speaking it is a pseudo-metric, as the distance it measures between two distinct points can be zero.

the idea that the curvature of spacetime gives rise to gravitational effects, we shall treat this as assumed knowledge, and discuss how the curvature of spacetime may be investigated. Since spacetime is not assumed to be flat (we'll define "flat" and "curved" rigorously below) and Euclidean, in general one cannot usefully extend the coordinate system from the region of one point in spacetime (one event) to the region of another arbitrary point. This can be seen from the fact that a Cartesian coordinate system which defined "up" to be the z-axis at one point on the surface of the Earth, would have to define "up" not to be parallel to the z-axis at most other points. In short, a freely-falling reference frame cannot be extended to each point in the vicinity of the surface of the Earth - or any other gravitating body. We are thus forced to work with local coordinate systems which vary from region to region. We shall refer to the basis vectors of these local coordinate systems by the symbols e_i. A set of four such basis vectors at any point is called a *tetrad* or *vierbein*. As these basis vectors are not necessarily orthonormal, we also may define a set of dual basis vectors e^i, where $e^i \cdot e_j = \delta^i_j$.

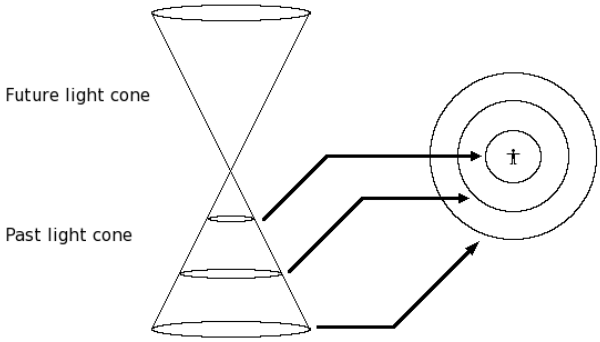


Figure 1: The future-pointing and past-pointing null vectors at a point define the future and past light cones of that point. Slices (at constant time) through the past light cone of an observer are two-spheres centred on the observer, and hence map directly to that observer's celestial sphere .

2.1 Parallel Transport and Curvature

Given the basis vectors e_i of a local coordinate system, an arbitrary vector is written in terms of its components v^i as $\vec{V} = v^i e_i$. The metric is used to switch between components referred to the basis or dual basis, e.g. $v_j = g_{ij} v^i$. When we differentiate a vector along a curve parametrised by the coordinate u^k we must apply the chain rule, as the vector itself can change direction and length, and the local basis will in general also change along the curve, hence

$$\frac{\mathrm{d}\vec{V}}{\mathrm{d}u^k} = \frac{\partial v^j}{\partial u^k} e_j + v^j \frac{\partial e_j}{\partial u^k}. \tag{2.3}$$

We extract the i^{th} component by taking the dot product with the dual basis vector (basis one-form) e^i, since $e^i \cdot e_j = \delta^i_j$. Hence we obtain

$$\frac{\mathrm{d}v^i}{\mathrm{d}u^k} = \frac{\partial v^i}{\partial u^k} + v^j \frac{\partial e_j}{\partial u^k} \cdot e^i \tag{2.4}$$

which by a suitable choice of *notation* is usually rewritten in the form

$$\nabla_k v^i = \partial_k v^i + v^j \Gamma^i_{jk}. \tag{2.5}$$

The derivative written on the left-hand-side is termed the *covariant derivative*, and consists of a partial derivative due to changes in the vector, and a term Γ^i_{jk} called the *connection* due to changes in the local coordinate basis from one place to another. If a vector is parallel-transported along a path, its covariant derivative will be zero. In consequence any change in the components of the vector is due to (and hence equal and opposite to) the change in local basis, so that

$$\frac{\partial v^i}{\partial u^k} = -v^j \frac{\partial e_j}{\partial u^k} \cdot e^i \tag{2.6}$$

The transport of a vector along a single path between two distinct points does not reveal

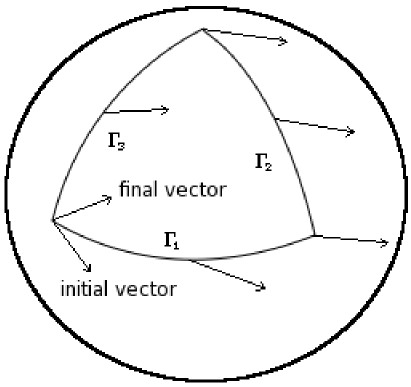


Figure 2: The parallel transport of a vector around a closed path tells us about the curvature of a region bounded by that path.

any curvature of the space (or spacetime) through which the vector is carried. To detect curvature it is necessary to carry a vector all the way around a closed path and back to its starting point, and compare its initial and final orientations. If they are the same, for an arbitrary path, the space (or spacetime) is *flat*. If they differ, the space is *curved*, and the amount by which the initial and final orientations of the vector differ provides a measure of how much curvature is enclosed within the path. Alternatively, one may transport two copies of a vector from the same starting point, A, along different paths, γ_1 and γ_2 to a common end-point, B. Comparing the orientations of the vectors after they have been transported along these two different paths reveals whether the space is flat or curved. It should be obvious that this is equivalent to following a closed path (moving along γ_1 from A to B, and then along γ_2 from B to A). The measure of how much this closed path (loop) differs from a loop in flat space (that is, how much the two transported vectors at B differ from each other) is called the *holonomy* of the loop, c.f. Figure 2

In light of the preceding discussion, suppose a vector $\vec{V}$ is transported from point A some distance in the μ-direction. The effect of this transport upon the components of $\vec{V}$ is given by the covariant derivative ∇_μ of $\vec{V}$. The vector is then transported in the ν-direction to arrive at point B. An identical copy of the vector is carried first from A in the ν-direction, and then in the μ-direction to B. The difference between the two resulting (transported) vectors, when they arrive at B is given by

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\vec{V}. \tag{2.7}$$

This commutator defines the Riemann curvature tensor,

$$R^\lambda{}_{\rho\mu\nu} v^\rho = [\nabla_\mu, \nabla_\nu] v^\lambda. \tag{2.8}$$

If and only if the space is flat, all the components of $R^\lambda{}_{\rho\mu\nu}$ will be zero, otherwise the space is curved.

2.2 Diffeomorphism Invariance

General relativity embodies a principle called *diffeomorphism invariance*. This principle states that the laws of physics should be invariant under a remapping of the coordinates - in fact coordinates have no physical meaning. The relationships between events are more important than the absolute locations of events. While the latter depend on the choice of the co-ordinate system, the former do not.

We require that any theory of quantum gravity should also embody a notion of diffeomorphism invariance, or at the very least,should exhibit a suitable notion of diffeomorphism invariance in the classical limit.

2.3 Einstein's Field Equations

Einstein's equations equate the curvature of spacetime with the energy density of the matter and fields present in the spacetime. Defining the Ricci tensor $R_{\rho\nu} = R^\mu{}_{\rho\mu\nu}$ and the Ricci scalar $R = R^\nu{}_\nu$ (i.e. it is the trace of the Ricci tensor, taken after raising an index using the metric $g_{\mu\nu}$). The relationship between energy density and spacetime curvature is then given by

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi \mathcal{G} T^{\mu\nu}. \tag{2.9}$$

where $\mathcal{G}$ is Newton's constant, and the coefficient Λ is the cosmological constant, which prior to the 1990s was believed to be identically zero. The tensor $T^{\mu\nu}$ is the stress-energy tensor. We will not discuss it in detail, but its components describe the flux of energy and momentum (i.e. 4-momentum) across various timelike and spacelike surfaces[3]. The stress-energy tensor can be defined as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_{\text{matter}}}{\delta g^{\mu\nu}} \tag{2.10}$$

[3]The presence of the stress-energy tensor is related to the fact that it is not merely the mass of matter that creates gravity, but its momentum, as required to maintain consistency when transforming between various Lorentz-boosted frames

where $g = \det(g^{\mu\nu})$, and $L_{\rm matter}$ is a lagrangian encoding the presence of matter. It is sometimes preferable to write equation (2.9) in the form

$$G^{\mu\nu} = 8\pi \mathcal{G} T^{\mu\nu} - \Lambda g^{\mu\nu} \tag{2.11}$$

where the *Einstein tensor* $G^{\mu\nu} = R^{\mu\nu} - R g^{\mu\nu}/2$ is the divergence-free part of the Ricci tensor. The explicit form of equation (2.9) emphasises the relationship between mass-energy, and spacetime curvature. All the quantities related to the structure of the spacetime (i.e. $R^{\mu\nu}$, R, $g^{\mu\nu}$) are on the left-hand side. The quantity related to the presence of matter and energy, $T^{\mu\nu}$, is on the right-hand side. For now it remains a question of interpretation whether this means that mass-energy is equivalent to spacetime curvature, or identical to it. Perhaps more importantly the form of the Einstein Field Equations makes it clear that GR is a theory of dynamical spacetime. As matter and energy move, so the curvature of the spacetime in their vicinity changes.

It is worth noting (without proof, see for instance [18]) that the gravitational field in the simplest case, a static, spherically-symmetric field around a mass M defines a line element of the form derived by Schwarzschild,

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2\mathcal{G}M}{c^2 r}\right)\mathrm{d}t^2 + \left(1 - \frac{2\mathcal{G}M}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{2.12}$$

For weak gravitational fields, and test masses moving at low velocities ($v \ll c$) the majority of the deviation from the line element in empty space is caused by the coefficent of the $\mathrm{d}t^2$ term on the right. This situation also coincides with the limit in which Newtonian gravity becomes a good description of the mechanics. In the Newtonian picture the force of gravity can be written as the gradient of a potential:

$$\vec{F} = \nabla V. \tag{2.13}$$

It can be shown that

$$\partial g_{00} \propto \nabla V, \tag{2.14}$$

implying that gravity in the Newtonian or weak-field limit can be understood, primarily, as the amount of distortion in the local "speed" of time caused by the presence of matter.

2.4 The Einstein-Hilbert Action

From classical mechanics we know that dynamics can be described either in the Hamiltonian or the Lagrangian frameworks. The benefits of a Lagrangian framework are that it provides us with a covariant perspective on the dynamics and connects with the path-integral approach to the quantum field theory of the given system. The Hamiltonian approach, on the other hand, provides us with a phase space picture and access to the Schrodinger method for quantization. Each has its advantages and difficulties and thus it is prudent to be familiar with both frameworks.

The form of the Lagrangian, and hence the action, can be determined by requirements of covariance and simplicity. The volume form $d^n x$ over which the lagrangian is integrated, must be supplemented by a factor of $\sqrt{-g}$ (where g is the determinant of the metric $g^{\mu\nu}$) in order to remain invariant under arbitrary co-ordinate transformations:

$$d^n x\sqrt{-g(x)} \rightarrow d^n x'\sqrt{-g(x')} \tag{2.15}$$

Maybe discuss this a bit earlier, to set the scene, and "warm up" the thinking about volume forms and the metric.Now out of the dynamical elements of geometry - the metric and the connection - we can construct a limited number of quantities which are invariant under co-ordinate transformations (state what is the simplest criterion to make them invariant?). These quantities must be constructed out of the Riemann curvature tensor or its derivatives. These possibilities are of the form: $\{R, R_{\mu\nu}R^{\mu\nu}, R^2, \nabla_\mu R\nabla^\mu R, \ldots\}$. The simplest of these is the Ricci scalar $R = R_{\mu\nu\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$. As it turns out this term is sufficient to fully describe Einstein's general relativity.

This allows us to construct the simplest lagrangian which describes the coupling of geometry to matter:

$$S_{\text{EH+M}} = \frac{1}{\kappa}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}L_{\text{matter}} \tag{2.16}$$

where L_{matter} is the lagrangian for the matter fields that may be present and κ is a constant, to be determined. If the matter lagrangian is omitted, one obtains the usual vacuum field equations of GR. This action (omitting the matter term) is known as the Einstein-Hilbert action.

It is worth digressing to prove (at least in outline form) that the Einstein field equations (EFE) can be found from S_{EH}. The variation of the action (2.16) yields a classical solution which, by the action principle, is chosen to be zero,

$$\delta S = 0 = \int d^4x \left[\frac{1}{\kappa}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \frac{1}{\kappa}\sqrt{-g}\frac{\delta R}{\delta g^{\mu\nu}} + \frac{\delta\sqrt{-g}L_{\text{matter}}}{\delta g^{\mu\nu}}\right] \tag{2.17}$$

which implies that

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \frac{\delta R}{\delta g^{\mu\nu}} = -\kappa\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}L_{\text{matter}}}{\delta g^{\mu\nu}}\,. \tag{2.18}$$

From equation (2.10) we can immediately see that

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}R + \frac{\delta R}{\delta g^{\mu\nu}} = -\frac{\kappa}{2}T^{\mu\nu}\,. \tag{2.19}$$

We now need to work out the variation of the terms on the left-hand-side. Omitting the details, which can be found elsewhere (see e.g. the appendix of [18]), we find that

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta\sqrt{g} = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\delta g_{\mu\nu}) = -\frac{1}{2}\sqrt{-g}(g_{\mu\nu}\delta g^{\mu\nu}) \tag{2.20}$$

thanks to Jacobi's formula for the derivative of a determinant. The variation of the Ricci scalar can be found by differentiating the Riemann tensor, and contracting on two indices to find the variation of the Ricci tensor. Then, since the Ricci scalar is given by $R = g^{\mu\nu} R_{\mu\nu}$ we find that

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \,. \tag{2.21}$$

The second term on the right may be neglected when the variation of the metric vanishes at infinity, and we obtain $\delta R / \delta g^{\mu\nu} = R_{\mu\nu}$. Plugging these results into eq. (2.19) we find that

$$-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} = -\frac{\kappa}{2} T^{\mu\nu} \tag{2.22}$$

which yields the Einstein equations if we set $\kappa = -16\pi\mathcal{G}$.

2.5 The ADM splitting

Since General Relativity is a theory of a dynamical spacetime, we will want to describe the dynamics of spacetime in terms of some variables which make computations as tractable as possible. The Hamiltonian formulation is well suited to a wide range of physical systems, and the ADM (Arnowitt-Deser-Misner) formalism allows us to apply it to General Relativity. We can think of the action (2.16), which is clearly written in the form of an integral of a lagrangian, as a stepping-stone to this hamiltonian approach. This hamiltonian formulation of GR takes us to the close of our discussion of classical gravity, and will be used as the jumping -off point for the quantisation of gravity, to be undertaken in Section 4.

We do not wish to provide a comprehensive discussion of the ADM procedure, which is used to obtain the 3+1 Hamiltonian description of general relativity, but only to describe its salient features and emphasize the aspects relevant to the canonical quantization program. Further details about the ADM splitting and canonical quantization can be found in [18] (in the metric formulation), [19] (in the connection formulation).

The ADM formalism involves foliating spacetime into a set of three-dimensional spacelike hypersurfaces, and picking an ordering for these hypersurfaces which plays the role of time, so that the hypersurfaces are level surfaces of the parameter t. This is a necessary feature of the hamiltonian formulation of a dynamical system, although it seems at odds with the way GR treats space and time as interchangable parts of spacetime. However this time direction is actually a "fiducial time"[4] and will turn out not to affect the dynamics. It is essentially a parameter used as a scaffold, which in the absence of a metric is not directly related to the passage of time as measured by a clock.

To begin, we will suppose that the 4-dimensional spacetime is embedded within a manifold $\mathcal{M}$ (which may be $\mathbb{R}^4$ or any other suitable manifold). Next we choose a local foliation[5] $\{\Sigma_t, t\}$ of $\mathcal{M}$, where Σ_t is a leaf of foliation. The topology of the original four-dimensional spacetime is then $\Sigma \otimes \mathbb{R}$, while $t(s)$ is a parametrization of the set of geodesics orthogonal to Σ_t with s being the affine parameter along each geodesic, c.f. (Fig. 3). In

[4]The term "fiducial" refers to a standard of reference, as used in surveying, or a standard established on a basis of faith or trust.

[5]Generally one assumes that our 4 manifolds can always be foliated by a set of spacelike 3 manifolds. For a general theory of quantum gravity the assumption of trivial topologies must be dropped. In the presence of topological defects in the 4 manifold, in general, there will exist inequivalent foliations in the vicinity of a given defect. This distinction can be disregarded in the following discussion for the time being.

addition at each point of a leaf we have a unit time-like vector n^μ (with $n^\mu n_\mu = -1$) which defines the normal at each point on the leaf.

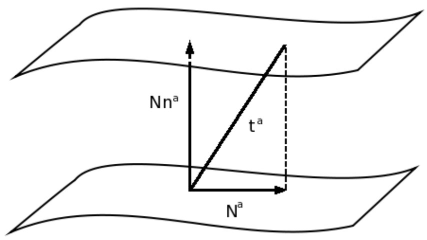


Figure 3: When performing the ADM splitting, the lapse function N and shift vector N_i define how points on successive hypersurfaces are mapped together.

Given the full four-metric $g_{\mu\nu}$ on $\mathcal{M}$ and the vector field n^μ the foliation is completely determined by the requirement that the surfaces Σ_t of constant "time" are normal to n^μ.

The diffeomorphism invariance of general relativity implies that there is no canonical choice of the time-like vector field t^μ which maps a point x^μ on a leaf Σ_t to the point x'^μ on the leaf $\Sigma_{t+\delta t}$, i.e. which generates time evolution of the geometry. This property is in fact the gauge symmetry of general relativity. It implies that we can choose any vector field t^μ as long as it is time-like. Such a vector field can be projected onto the three-manifold to obtain the shift vector $N^a = t_\parallel$ which is the part tangent to the surface, while the component of t^μ normal to the three-manifold is then identified as the lapse function $N = t_\perp$. t^μ can thus be written as:

$$t^\mu = N n^\mu + N^\mu \tag{2.23}$$

where, though we have written the shift as four-vector, it is understood that $N^0 = 0$ in a local basis of co-ordinates adapted to the splitting.

By recognising that $n^\mu n_\nu = -1$, as n is timelike, we identify $g_{\mu\nu}+n_\mu n_\nu$ as the projection operator that takes any 4-vector and projects out its component normal to the leaf Σ_t, leaving only the part tangential to Σ_t. Writing a general four-vector as a sum of terms parallel and perpendicular to the surface $v^\mu = v_\perp n^\mu + v_\parallel \frac{N^\mu}{|N|}$ (where $|N| = N^\mu N_\mu$ is the norm of the shift vector) and acting on it with the projector we have:

$$\begin{aligned}(g_{\mu\nu} + n_\mu n_\nu)\left(v_\perp n^\nu + v_\parallel \frac{N^\nu}{|N|}\right) =& v_\perp n_\mu (1 + n^\nu n_\nu) + \frac{v_\parallel}{|N|}(N_\mu + n^\nu N_\nu) \\ =& v_\parallel \frac{N_\mu}{|N|}\end{aligned} \tag{2.24}$$

Since n^μ is a time-like unit vector ($n^\mu n_\mu = -1$) the first term on the right hand side of the first line vanishes. In the second term $n^\nu N_\nu = 0$ by definition and we are left with only the component of v^μ parallel to Σ_t. We see that this operator projects any vector from the manifold $\mathcal{M}$ down into the subspace defined by a given leaf of foliation.

Now we can determine the components of the four-metric in a basis adapated to the splitting as follows:

$$\begin{aligned} g_{00} = & g_{\mu\nu} t^\mu t^\nu \\ = & g_{\mu\nu} \left(N n^\mu + N^\mu \right) \left(N n^\nu + N^\nu \right) \\ = & N^2 n^\mu n_\mu + N^\mu N_\mu + 2N(N^\mu n_\mu) \\ = & - N^2 + N^\mu N_\mu \end{aligned} \tag{2.25}$$

where we have used $n^\mu n_\mu = -1$ and $N^\mu n_\mu = 0$ in the third line. Working in a co-ordinate basis where $N^\mu = (0, N^a)$, we have $g_{00} = -N^2 + N^a N_a$[6]. Similarly to obtain the other components of the metric we project along the time-space and the space-space directions:

$$g_{\mu\nu} t^\mu N^\mu = N^\mu N_\mu \equiv N^a N_a \tag{2.26}$$

Since, by definition $g_{0\nu} \equiv g_{\mu\nu} t^\mu$, this implies that $g_{0a} = N_a$. The space-space components of $g_{\mu\nu}$ are simply given by the intrinsic three-metric h_{ab} of the leaf Σ_t. Thus the full metric $g_{\mu\nu}$ can be written schematically as:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & \boldsymbol{N} \\ \boldsymbol{N}^T & h_{ab} \end{pmatrix} \tag{2.27}$$

where a, $b \in \{1, 2, 3\}$ and $\boldsymbol{N} \equiv \{N^a\}$. The 4D line-element can then be read off from the above expression:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (-N(t)^2 + N^a N_a) dt^2 + 2N^a dt\, dx_a + h_{ab} dx^a dx^b \tag{2.28}$$

where again a, $b \in \{1, 2, 3\}$ are spatial indices on Σ (hereafter we drop the t superscript as we will deal with only one, representative leaf of the foliation)

The spacelike hypersurfaces Σ will in general have an intrinsic curvature, measured by the curvature tensor constructed from the spatial metric h^{ab}. Here the indices $a, b, c, \ldots$ on a tensor are used to indicate that the only non-zero components are those which live on ${}^3\Sigma$. ${}^3\Sigma$ will also have a curvature associated with their embedding in $\mathcal{M}$, as shown in Fig. 4. This is known as the extrinsic curvature, and measured by taking the gradient of the normal vectors to the hypersurface, symmetrised over the choice of directions.

$$k_{\mu\nu} = \nabla_\mu n_\nu + \nabla_\nu n_\mu \tag{2.29}$$

The reader can verify that, as with the intrinsic metric, $k_{\mu\nu} n^\mu = 0$, making the extrinsic curvature a quantity with only spacelike indices: k_{ab}. Note that due to the properties of the Lie derivative (see Appendix B) and the purely spatial character of the extrinsic curvature we see that $k_{ab} = \mathcal{L}_n h_{ab}$. So the extrinsic curvature is the Lie derivative of the

[6] From this expression we can also see that $g_{00} = -N^2 + N^a N_a$ is a measure of the *local* speed of time evolution and hence is a measure of the *local* gravitational energy density.

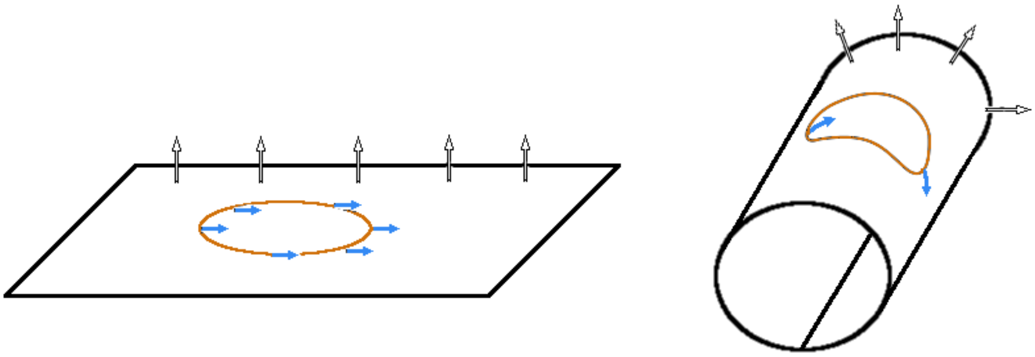


Figure 4: Intrinsiuc curvature measured by parallel transport (*left*), and extrinsic curvature measured by changes in the normal vectors (*right*).

intrinsic metric, i.e., it can interpreted as the *rate of change* of the intrinsic metric along the evolution generated by normal vector field - rather than the actual time-evolution vector t^μ. We might be tempted to identify the extrinsic curvature with the momentum variable to conjugate to "position variable" - the intrinsic metric. This is not far off the mark. As we will see the conjugate momentum will turn to be a function of k_{ab}.

As mentioned above, the lagrangian formulaton of General Relativity is used as a stepping-stone to the hamiltonian formulation. To find the relevant hamiltonian density we proceed in a manner that parallels the approach in classical mechanics or field theory - namely we perform a Legendre transform to obtain the Hamiltonian function from the Lagrangian. In the case of classical mechanics we see that:

$$H[p,\, q] = p\dot{q} - L[q,\, \dot{q}] \quad \text{where} \quad p = \frac{\partial L}{\partial \dot{q}}. \tag{2.30}$$

Similarly, in the case of scalar field theory, we find that

$$H[\pi,\, \phi] = \int d^4x\, \pi\dot{\phi} - L[\phi,\, \dot{\phi}] \tag{2.31}$$

and in the case of General Relativity, taking the intrinsic metric on Σ as our configuration or "position" variable:

$$H[\pi^{\mu\nu},\, h_{\mu\nu}] = \int d^3x\, \pi^{ab}\dot{h}_{ab} - L[h_{ab},\, \dot{h}_{ab}] \tag{2.32}$$

The Einstein-Hilbert action can be re-written in terms of quantities defined on the spatial hypersurfaces, by making two substitutions. Firstly we recognise that the four-dimensional volume form $\sqrt{-g}$ is equal to $N\sqrt{h}$ (that is, the three-dimensional volume form multiplied by the distance between hypersurfaces). Analogously to g, we write h for the determinant of h^{ab}. Secondly, using the Gauss-Codazzi equation[7], the four-dimensional

[7]a derivation of which can be found on pg. 13 of [7]

Ricci curvature scalar R can be re-written in terms of the three-dimensional Ricci scalar of Σ, ${}^{(3)}R$, and the extrinsic curvature of Σ as:

$$R = {}^{(3)}R + k^{ab}k_{ab} - k^2 \tag{2.33}$$

where k is the trace of the extrinsic curvature taken with respect to the 3-metric $k := k^{ab}h_{ab}$. The Gauss-Codazzi relation is a very general result which is true in an arbitrary number of dimensions. The reader with too much time on their hands may wish to derive it for themselves by using the definition of the Ricci scalar in terms of the Christoffel connection and using the 3-metric h^μ_ν to project quantities in 3+1 dimensions down to three dimensions of Σ.

Using these substitutions, the Einstein-Hilbert action can be rewritten in a form that is convenient for identifying the parts that depend only on Σ:

$$S_{\mathrm{EH}} = \int dt\, d^3x\, N\sqrt{h}\left({}^{(3)}R + k^{ab}k_{ab} - k^2\right) = \int dt\, L_{\mathrm{EH}} \tag{2.34}$$

We next need to find $\dot{h}_{ab}$, which is obtained by taking the Lie derivative (Appendix B) with respect to the vector field t^μ which generates time-translations:

$$\dot{h}_{ab} = \pounds_{\vec{t}}h_{ab} = 2Nk_{ab} + \pounds_{\vec{N}}h_{ab} \tag{2.35}$$

The conjugate momentum is then found to be:

$$\pi^{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \sqrt{h}(k^{ab}k_{ab} - k^2) \tag{2.36}$$

Substituting these results into eqn. (2.32) we obtain

$$H[\pi^{ab},\, h_{ab}] = \int d^3x\, \pi^{ab}\dot{h}_{ab} - L[h_{ab},\, \dot{h}_{ab}] \tag{2.37a}$$

$$= \int d^3x\, N\left(-\sqrt{h}^{(3)}R + \frac{1}{\sqrt{h}}(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2)\right) - 2N_a D_b \pi^{ab} \tag{2.37b}$$

$$= \int d^3x\, N\mathcal{H} - N_a\mathcal{C}^a \tag{2.37c}$$

where for brevity we have adopted the notation

$$\mathcal{H} = \left(-\sqrt{h}^{(3)}R + \tfrac{1}{\sqrt{h}}(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2)\right) \quad \text{(Hamiltonian constraint)} \tag{2.38a}$$

$$\mathcal{C}^a = 2D_b\pi^{ab} \quad \text{(Diffeomorphism constraint)} \tag{2.38b}$$

where π is the trace of π^{ab}.

We can reverse the Legendre transform to rewrite the action for GR as:

$$S_{EH} = \int dt L_{EH} = \int dt d^3x \left(\pi^{ab}\dot{h}_{ab} - H[\pi^{ab}, h_{ab}]\right) \tag{2.39a}$$

$$= \int dt d^3x \left(\pi^{ab}\dot{h}_{ab} - N\mathcal{H} + N_a\mathcal{C}^a\right) \tag{2.39b}$$

It is now apparent that the action written in this form is a function of the lapse and shift but *not* their time derivatives. Consequently the Euler-Lagrange equations of motion obtained by varying S_{EH} w.r.t the lapse and shift are:

$$\frac{\delta S_{EH}}{\delta N} = -\mathcal{H} = 0 \tag{2.40a}$$

$$\frac{\delta S_{EH}}{\delta N_a} = \mathcal{C}^a = 0 \tag{2.40b}$$

implying that $\mathcal{H}$ and $\mathcal{C}^a$ are identically zero and are thus to be interpreted as constraints on the phase space! This is nothing more than the usual prescription of Lagrange multipliers - when an action depends only on a configuration variable q but not on the corresponding momentum p, the terms multiplying the configuration variable are constraints on the phase space.

$\mathcal{C}^a$ and $\mathcal{H}$ are referred to as the vector (or diffeomorphism) constraint and the scalar (or "Hamiltonian") constraint, respectively. The diffeomorphism constraint generates diffeomorphisms within the spatial hypersurfaces Σ_t. The Hamiltonian constraint generates the time evolution which takes the geometry of Σ_t to Σ_{t+1}. A little later, when we cast GR in the first order formulation we will encounter a third constraint, referred to as the Gauss constraint.

We see that the Hamiltonian density H_{EH} in eqn. (2.37c), obtained after performing the $3+1$ split of the Einstein-Hilbert action via the ADM procedure [19], is a sum of constraints, i.e. $H_{EH} = N\mathcal{H} - N_a\mathcal{C}^a = 0$. This is a generic feature of diffeomorphism invariant theories.

2.6 Connection Formulation

In the previous section we worked with GR in *second-order* form, i.e. with the metric $g_{\mu\nu}$ as the only configuration variable, with the Christoffel connection $\Gamma^{\alpha}{}_{\mu\nu}$ being determined by the metric compatibility condition:

$$\nabla g_{\mu\nu} = 0 \tag{2.41}$$

The passage to the quantum theory is facilitated by switching to a *first-order* formulation of G, in which *both* the metric and the Christoffel connection are treated as independent configuration variables. An example is the Palatini framework. In this approach the metric compatibility condition 2.41 arises as the equation of motion obtained by varying the action w.r.t the connection:

$$\frac{\delta S[g,\Gamma]}{\delta \Gamma} = 0 \Rightarrow \nabla g = 0 \tag{2.42}$$

These variables are however inconvenient for the program for quantization. Therefore we transition to a first-order formulation of gravity in terms of a tetrad or "frame-field" and a gauge connection both of which take values in the Lie algebra of the Lorentz group.

The connection formulation exposes a hidden symmetry of geometry as illustrated by the following analogy. The introduction of spinors in quantum mechanics (and the corresponding Dirac equation) allows us to express a scalar field $\phi(x)$ as the "square" of a spinor $\phi = \Psi^i \Psi_i$. In a similar manner the use of the vierbien allows us to write the metric as a square $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$. The transition from the metric to connection variables in GR is analogous to the transition from the Klein-Gordon to the Dirac equation in field theory.

The connection is a Lie-algebra valued one-form $A_\mu{}^{IJ}\tau_{IJ}$ where τ_{IJ} are the generators of the Lorentz group. Our configuration space is then spanned by a tetrad and a connection pair: $\{e^I_\mu, A^\mu_{IJ}\}$. As we shall see, tetrads are naturally identified with basis vectors of a lie algebra ($\mathfrak{sl}(2,\mathbb{C})$) in which case the $\{I, J, \ldots\}$ become lie algebra indices.

2.6.1 Tetrads

We begin by considering the four dimensional manifold $\mathcal{M}$, introduced in section 2.5, above. As we know, any sufficiently small region of a curved manifold will look flat[8] and so we may define a tangent space to any point P in $\mathcal{M}$. Such a tangent space will be a flat Minkowski spacetime, and the point P may be regarded as part of the worldline of an observer, without loss of generality. This tangent space will be spanned by four vectors, e_μ. Each basis vector will have four components, referred to the locally-defined reference frame (the "laboratory frame" of the observer who's worldline passes through P, with lengths and angles measured using the Minkowski metric), e^I_μ, where $I \in \{0,\ 1,\ 2,\ 3\}$. As noted back in section 1, such a set of four basis vectors is referred to as a tetrad or vierbein (German for "four legs")[9] Since the tetrads live in Minkowski space, their dot product is taken using the Minkowski metric. But the dot product of basis vectors is just the metric itself, so the metric of $\mathcal{M}$ at any point is just given by

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ} \tag{2.43}$$

where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Taking the determinant of both sides we find that:

$$\det(g) = -\det(e)^2 \tag{2.44}$$

where the minus sign on the RHS comes from the determinant of the Minkowski metric. Alternatively we can write $\sqrt{-g} = e$, where $g \equiv \det(g)$ and $e \equiv \det(e)$. Due to this fact the tetrad can be thought of as the "square-root" of the metric.

Tetrads can also be interpreted as the transformation matrices that map between two sets of coordinates, as can be seen by comparing Eq (2.43) with the standard form for a coordinate transformation, $x^*_i = x^j A^i_j = x^j \partial \hat{u}_j / \partial \hat{u}^*_i$ (where the $\hat{u}^*$ and $\hat{u}$ are the basis vectors of the two coordinate systems). In fact the tetrads perform two roles. They facilitate the transformation of vector and tensor quantities, just as the Christoffel symbols do, by encoding information about the tangent space, and they facilitate the transformation of spinor quantities to vectorial ones and vice-versa, by mapping the tangent space in which vectors live to the space of spinors[10], at a given point. It is this fact which makes the tetrads a useful tool in modern formulations of GR.

[8] So long as the manifold is continuous, not discrete. This is an important point to keep in mind for later.

[9] The similar word *vielbein* ("any legs") is used for the generalisation of this concept to an arbitrary number of dimensions (e.g. triads, pentads).

[10] which be more elegantly stated in the language of fiber bundles, see e.g., [20]

Any vector v^μ can be written as an $\mathfrak{sl}(2,\mathbb{C})$ spinor v_{ab} as:

$$v_{ab} := v_\mu e^\mu{}_I \sigma^I{}_{ab} \tag{2.45}$$

where $\sigma^I = \{\mathbf{1}, \sigma_x, \sigma_y, \sigma_z\}$ is a basis of the lie-algebra $\mathfrak{sl}(2,\mathbb{C})$ and a, b are the spinorial matrix indices shown explicitly for clarity.

2.6.2 Spin Connection

Our ultimate goal is to cast general relativity in the mold of gauge field theories such as Maxwell or Yang-Mills. Though the Christoffel connection $\Gamma^\alpha_{\beta\delta}$ is an affine connection it does not transform as a tensor under arbitrary co-ordinate transformations (c.f. [18, chapter 4]) and thus cannot play the role of a gauge connection which should be a covariant quantity. Γ allows us to parallel transport vectors v^μ and, in general, arbitrary tensors $T_{\mu\nu\ldots}{}^{\alpha\beta\cdots}$, i.e. it allows us to map the tangent space T_p at point p to the tangent space $T_{p'}$ at the point p'. Of course the map depends on the path connecting p and p'. It is this aspect that allows us to use curvature to measure *local* geometric properties of a manifold. However, in order to allow the parallel transport of elementary particles the Christoffel connection is not sufficient.

The Christoffel connection does not "know" about lie-algebra value vector fields of the form $\psi_\mu{}^I$ (where I is a lie-algebra index). A theory of quantum gravity which does not know about fermions would not be very useful. Thus we need an alternative to the Christoffel connection which has both these properties: covariance w.r.t. co-ordinate transformations and coupling with spinors.

The simplest candidate for such a quantity is an $\mathfrak{sl}(2,\mathbb{C})$ valued connection $A_\mu{}^{IJ}$. $SL(2,\mathbf{C})$, or the Lorentz group, is the local gauge group of general relativity. While dynamics on a flat spacetime can be described by the Poincare group, in a general curved spacetime translational symmetry is broken and only local Lorentz invariance remains as an unbroken symmetry in general relativity. Thus the choice of an $\mathfrak{sl}(2,\mathbb{C})$ valued *spin connection*[11] would seem to be a logical candidate for casting GR as a gauge theory.

Outline for this section:

- Palatini action: [21] Section 2.3 - Equality of internal and spacetime Riemann curvature.
- Self-dual action: [22] Chap 7, Sec 7.3.4 3+1 decomposition of Self-dual action
- Barbero-Immirzi parameter: [23, 24, 25, 26] - generalizing from Ashtekar's self-dual connection to arbitrary connections.

[11]The term "spin connection" may cause some confusion, however it is analogous to the Christoffel connection familiar from classical GR, with the added functionality that it allows us to parallel transport spinors around paths in spacetime. This terminology can occasionally trick newcomers into thinking they have to learn a new concept, when it fact this is nothing more than the notion of parallel transport of a particle along a Wilson line.

2.6.3 Preliminaries

In order to be able to parallel transport objects with spinorial indices we need a suitable extension of the notion of covariant derivative from objects with spacetime indices to spinorial objects (we follow [21, Appendix B]). Given the tetrad e_I^μ, the Christoffel connection $\Gamma^\gamma_{\alpha\beta}$ and the spin-connection ω_α^{IJ}, the *generalised* derivative operator on $\mathcal{M}$ is defined such that it annihilates the tetrad:

$$\mathcal{D}_\alpha e^I_\beta = \partial_\alpha e^I_\beta - \Gamma^\gamma_{\alpha\beta} e^I_\gamma + \omega^I_{\alpha J} e^J_\beta = 0 \tag{2.46}$$

Now one would expect that this operator should also annihilate the (internal) Minkowski metric $\eta_{IJ} = e_{\alpha I} e^\alpha_I$ and the spacetime metric $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$. One can check that requiring this to the be case yields that the spin-connection is anti-symmetric $\omega_\alpha^{(IJ)} = 0$ and the Christoffel connection is symmetric $\Gamma^\alpha_{[\beta\gamma]} = 0$.

We can solve for $\Gamma^\alpha_{\beta\gamma}$ in the usual manner (see for e.g. [18]) to obtain:

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right) \tag{2.47}$$

Inserting the above into 2.46 we can solve for ω to obtain:

$$\omega_\alpha^{IJ} = \frac{1}{2} e^{\delta[I} \left(\partial_{[\alpha} e^{J]}_{\delta]} + e^{|\beta|J]} e^K_\alpha \partial_\beta e_{\delta K} \right) \tag{2.48}$$

Note that in the above expression the Christoffel connection does not occur.

In the definition of $\mathcal{D}$ we have include the Christoffel connection. Ideally, in a gauge theory of gravity, we would not want any dependence on the spacetime connection. That this is the case can be seen by noting that all derivatives that appear in the Lagrangian or in expressions for physical observables are *exterior* derivatives, i.e. of the form $\mathcal{D}_{[\alpha} e^I_{\beta]}$. The anti-symmetrization in the spacetime indices and the symmetry of the Christoffel connection $\Gamma^\gamma_{[\alpha\beta]} = 0$ implies that the exterior derivative of the tetrad can be written without any reference to Γ:

$$\mathcal{D}_{[\alpha} e^I_{\beta]} = \partial_{[\alpha} e^I_{\beta]} + \omega_{[\alpha}{}^{IL} e_{\beta]L} = 0 \tag{2.49}$$

We can solve for ω by a trick similar to one used in solving for the Christoffel connection. Following [21, Appendix B], first contract the above expression with $e^\alpha_J e^\beta_K$ to obtain:

$$e^\alpha_J e^\beta_K \left(\partial_{[\alpha} e^I_{\beta]} + \omega_{[\alpha}{}^{IL} e_{\beta]L} \right) = 0 \tag{2.50}$$

Now let us define $\Omega_{IJK} = e^\alpha_I e^\beta_J \partial_{[\alpha} e_{\beta]K}$. Performing a cyclic permutation of the indices I, J, K in the above expression, adding the first two terms thus obtained and subtracting the third term we are left with:

$$\Omega_{JKI} + \Omega_{IJK} - \Omega_{KIJ} + 2 e^\alpha_J \omega_{\alpha IK} = 0 \tag{2.51}$$

This can be solved for ω to yield:

$$\omega_{\alpha IJ} = \frac{1}{2} e^K_\alpha [\Omega_{KIJ} + \Omega_{JKI} - \Omega_{IJK}] \tag{2.52}$$

which is equivalent to the previous expression 2.48 for ω.

Next we consider the curvature tensors for the Christoffel and the spin connections and show the fundamental identity that allows us to write the Einstein-Hilbert action solely in terms of the tetrad and the spin-connection. The Riemann tensor for the spacetime and the spin connections, respectively is defined as:

$$\mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}v_\gamma = R_{\alpha\beta\gamma}{}^{\delta}v_\delta; \qquad \mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}v_I = R_{\alpha\beta I}{}^{J}v_J \tag{2.53}$$

Writing $v_\gamma = e^I_\gamma v_I$ and inserting into the first expression we obtain:

$$R_{\alpha\beta\gamma}{}^{\delta}v_\delta = \mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}v_\gamma = \mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}e^I_\gamma v_I = e^I_\gamma R_{\alpha\beta I}{}^{J}v_J = e^I_\gamma R_{\alpha\beta I}{}^{J}e^\delta_J v_\delta \tag{2.54}$$

where we have used the fact that $\mathcal{D}_\mu e^I_\nu = 0$. Since the above is true for all v_δ, we obtain:

$$R_{\alpha\beta\gamma}{}^{\delta} = R_{\alpha\beta I}{}^{J}e^I_\gamma e^\delta_J \tag{2.55}$$

The Ricci scalar is given by $R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R_{\mu\delta\nu}{}^{\delta}$. Using the previous expression we find:

$$R_{\mu\delta\nu}{}^{\delta} = R_{\mu\delta I}{}^{J}e^I_\nu e^\delta_J \tag{2.56}$$

Contracting over the remaining two spacetime indices then allows us to write the Ricci scalar in terms of the curvature of the spin-connection and the tetrads:

$$R = R_{\mu\nu}{}^{IJ}e^\mu_I e^\nu_J \tag{2.57}$$

2.6.4 Palatini Action

Using 2.57 and the fact that $\sqrt{-g} = e$ we can write down the EH action for GR in terms of the connection and tetrad:

$$\begin{aligned} S_{EH}[e,\omega] &= \frac{1}{2\kappa}\int d^4x \star (e^I \wedge e^J) \wedge F^{KL}\,\epsilon_{IJKL} \\ &= \frac{1}{4\kappa}\int d^4x\, \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}\, e_\mu{}^I e_\nu{}^J F_{\alpha\beta}{}^{KL} \end{aligned} \tag{2.58}$$

where $F^{KL}{}_{\gamma\delta}$ is the curvature of the spin-connection:

$$F^{KL}{}_{\gamma\delta} = \partial_{[\gamma}\omega_{\delta]}{}^{KL} + \frac{1}{2}\left[\omega_\gamma{}^{KM}, \omega_{\delta\,M}{}^{L}\right] \tag{2.59}$$

The integrand in 2.58 is a four-form, which can therefore be integrated over a four-dimensional manifold. Thus this action is valid only for four-dimensional manifolds.

At this point $F_{\mu\nu}{}^{IJ}$ is the curvature of ω, but does not yet satisfy the identity 2.57. The equations of motion obtained by varying the Palatini action are:

$$\frac{\delta S_P}{\delta\omega_\nu{}^{IJ}} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}\, D_\nu\left(e_\alpha{}^I e_\beta{}^J\right) = 0 \tag{2.60a}$$

$$\frac{\delta S}{\delta e_I{}^\mu} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{IJKL}\, e_\nu{}^J F_{\alpha\beta}{}^{KL} = 0 \tag{2.60b}$$

For the first equation we have utilized the fact that $F[\omega+\delta\omega] = F[\omega] + \mathcal{D}_{[\omega]}(\delta\omega)$, where $\mathcal{D}_{[\omega]}$ is the covariant derivative defined with respect to the unperturbed connection ω as in 2.49. The resulting equation of motion 2.60a is then the *torsion-free* or *metric-compatibility* condition which tells us that the tetrad is parallel transported by the connection ω. This then implies that 2.57 holds, i.e. $F_{\mu\nu}{}^{IJ} \equiv R_{\mu\nu}{}^{IJ}$. The second equation of motion can be obtained by inspection, since F does not depend on the tetrad. Already we see dramatic technical simplification compared to when we had to vary the Einstein-Hilbert action with respect to the metric as in 2.17.

In order to show that 2.60b is equivalent to Einstein's vacuum equations, we first note that the volume form can be written as

$$\epsilon_{\mu\nu\alpha\beta} = \frac{1}{4!}\epsilon_{PQRS}\, e_{[\mu}{}^{P} e_{\nu}{}^{Q} e_{\alpha}{}^{R} e_{\beta]}{}^{S} \tag{2.61}$$

Contracting both sides with $e^{\nu}{}_{J}$ we find:

$$\begin{aligned}\epsilon_{\mu\nu\alpha\beta}\, e^{\nu}{}_{J} &= \frac{1}{4!}\epsilon_{PQRS}\, e_{[\mu}{}^{P} e_{\nu}{}^{Q} e_{\alpha}{}^{R} e_{\beta]}{}^{S} e^{\nu}{}_{J} \\ &= -\frac{1}{3!}\epsilon_{JPQR}\, e_{[\mu}{}^{P} e_{\alpha}{}^{Q} e_{\beta]}{}^{R}\end{aligned} \tag{2.62}$$

where in the second line we have switched some dummy indices and relabeled others. Inserting the right hand side of the above in 2.60b and using the fact that 2.57 implies $F_{\mu\nu}{}^{IJ} \equiv R_{\mu\nu}{}^{IJ}$, we find:

$$\begin{aligned}\frac{\delta S}{\delta e_I{}^{\mu}} &= \epsilon^{\mu\nu\alpha\beta}\, e_{\nu}{}^{J} \epsilon_{IJKL}\, R_{\alpha\beta}{}^{KL} \\ &= -\frac{1}{3!}\epsilon^{JPQR}\, \epsilon_{IJKL}\, e^{[\mu}_{P}\, e^{\alpha}_{Q}\, e^{\beta]}_{R}\, R_{\alpha\beta}{}^{KL} \\ &= \delta^{P}_{[I}\, \delta^{Q}_{K}\, \delta^{R}_{L]}\, e^{\mu}_{P}\, e^{\alpha}_{Q}\, e^{\beta}_{R}\, R_{\alpha\beta}{}^{KL} \\ &= e^{\mu}_{[I}\, e^{\alpha}_{K}\, e^{\beta}_{L]}\, R_{\alpha\beta}{}^{KL} \\ &= \left(e^{\mu}_{I} e^{\alpha}_{K} e^{\beta}_{L} + e^{\mu}_{K} e^{\alpha}_{L} e^{\beta}_{I} + e^{\mu}_{L} e^{\alpha}_{I} e^{\beta}_{K}\right) R_{\alpha\beta}{}^{KL} \\ &= e^{\mu}_{I} R + e^{\beta}_{I} R_{\alpha\beta}{}^{\mu\alpha} + e^{\alpha}_{I} R_{\alpha\beta}{}^{\beta\mu} \\ &= e^{\mu}_{I} R - 2e^{\beta}_{I} R_{\beta}{}^{\mu} = 0\end{aligned} \tag{2.63}$$

In the first step we have used the result in 2.62. In the second step we have used the fact that the contraction of two epsilon tensors can be written in terms of anti-symmetrized products of Kronecker deltas. In the third and fourth steps we have simply contracted some indices using the Kronecker deltas and expanded the anti-symmetrized product explicitly. In the fifth and sixth steps we have made use of 2.55 and the definition of the Ricci tensor as the trace of the Riemann tensor: $R_{\beta}{}^{\mu} = \sum_{\alpha} R_{\alpha\beta}{}^{\alpha\mu}$. Contracting the last line of the above with $e^{\nu I}$ and using the fact that $g_{\mu\nu} = e^{I}_{\mu} e^{J}_{\nu} \eta_{IJ}$ we find:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{2.64}$$

Thus the tetradic action in the first-order formulation - where the connection and tetrad are independent variables - is completely equivalent to classical general relativity.

2.6.5 Palatini Hamiltonian & Constraints

We can perform a $3+1$ split of the Palatini action 2.58 and obtain a hamiltonian which, once again, is a sum of constraints. However, while the resulting formulation appears simpler than that in terms of the metric variables, there are some second class constraints which when solved [21, Section 2.4] yield the same set of constraints as obtained in the ADM framework. Thus, the Palatini approach does not appear to yield any substantial improvements over the ADM version as far as canonical quantization is concerned. For this we must transition to the description in terms of complex, self-dual connections.

2.7 Ashtekar Formulation - "New Variables" for General Relativity

At the heart of the formulation of general relativity as a gauge theory lies a canonical transformation from the phase space variables of the Palatini picture (e^a_i, Γ^i_a) (which are the intrinsic metric of the spacelike manifold Σ and its extrinsic curvature respectively) to the "new" or Ashtekar variables $(\frac{1}{\gamma}e^a_i, A^i_a)$ where γ is the so-called Immirzi parameter A is the Ashtekar-Barbero connection:

$$\Gamma^i_a \to \Gamma^i_a + \gamma K^i_a \qquad e^a_i \to \frac{1}{\gamma} e^a_i \tag{2.65}$$

We begin with tetradic GR whose action is written in terms of a tetrad and connection in the Palatini form as:

$$S_{EH}\left[e,\omega\right] = \frac{1}{2\kappa}\int d^4x\, \tilde{e}_I{}^{\mu}\tilde{e}_J{}^{\nu} F[\omega]_{\mu\nu}{}^{IJ} \tag{2.66}$$

This action is equivalent to the usual Einstein-Hilbert action *on-shell*, i.e. for configurations which satisfy Einstein's field equations. as shown in the previous subsection. It is easier to work with connections and tetrads rather than metrics as shown above. The constraints arising from the $3+1$ decomposition are also simpler than the original ADM versions. However, the Hamiltonian constraint is still a complicated non-polynomial function and canonical quantization does not appear to be any easier in this formalism.

Ashtekar made the remarkable observation that if instead of the *real* connection $\omega_\mu{}^{IJ}$ one works with a *complex*, self/anti-self dual connection ${}^{\pm}A = \omega \pm i \star \omega$, the form of the constraints simplifies dramatically[12]:

$$\mathcal{H} = \epsilon^{ij}{}_k e^a_i e^b_j {}^{\pm}F^k{}_{ab} \qquad \text{(Hamiltonian constraint)} \tag{2.67a}$$

$$\mathcal{C}_a = e^b_i {}^{\pm}F^i{}_{ab} \qquad \text{(Diffeomorphism constraint)} \tag{2.67b}$$

$$\mathcal{G}_i = D_a e^a_i \qquad \text{(Gauss constraint)} \tag{2.67c}$$

where ${}^{\pm}F^k{}_{ab}$ is the curvature of the self(anti-self)-dual connection ${}^{\pm}A$. The second class constraints which were present in the Palatini framework must now vanish due to the Bianchi identity and the diffeomorphism constraint becomes a polynomial quadratic function of the momentum variables - in this case the triad. The phase space configuration

[12] for the detailed derivation of these constraints starting with the self-dual Lagrangian see for e.g. [19, Section 6.2]

and momentum variables are the three dimensional triad ${e_a}^i$ and the spatial connection ${A^i}_a$. Here R^i_{ab} is the curvature of ${A^i}_a$. It is instructive to compare the above form of the constraints to their metric counterparts in 2.38a which are reproduced below for the reader's convenience:

$$\begin{aligned}
\mathcal{H} &= \left(-\sqrt{h}^{(3)}R + \frac{1}{\sqrt{h}}(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2)\right) \\
\mathcal{C}^a &= 2D_b\pi^{ab}
\end{aligned}$$

The price to be paid for this simplification is that the theory we are left with is no longer the theory we started with - general relativity with a manifestly real metric geometry. The connection ${}^{\pm}A$ is now a complex connection. However the new concoction is also not too far from the original theory and can be derived from an action. That this is the case was shown independently by Jacobson and Smolin [27] and by Samuel [28]. They completed the analysis by writing down the Lagrangian from which Ashtekar's form of the constraints would result:

$$S_{\pm}\left[e, A\right] = \frac{1}{4\kappa}\int d^4x\, {}^{\pm}{\Sigma^{\mu\nu}}_{IJ}\, {}^{\pm}{F_{\mu\nu}}^{IJ} \tag{2.68}$$

Here ${}^{\pm}F$ is the curvature of a *self-dual* (anti-self-dual) four-dimensional connection ${}^{\pm}A$ one-form. The field ${}^{\pm}\Sigma$ is the self-dual (anti-self-dual) portion of the two-form $\tilde{e}^I \wedge \tilde{e}^J$. The Palatini action then simply given by the real part of the the self-dual (or anti-self-dual) action.

$$S_P = \mathbf{Re}[S_{\pm}] \tag{2.69}$$

We obtain a form for the constraints which is polynomial in the coordinates and momenta and thus amenable to methods of quantization used for quantizing gauge theories such as Yang-Mills. The resulting expression for the Einstein-Hilbert-Ashtekar hamiltonian of GR is:

$$\mathcal{H}_{eha} = N^a_i \mathcal{C}^i_a + N\mathcal{H} + T^i\mathcal{G}_{\rangle} = 0 \tag{2.70}$$

where $\mathcal{C}^i_a$, $\mathcal{H}$ and $\mathcal{G}_i$ are the vector, scalar and Gauss constraints respectively.

2.8 (anti)self-dual connections

Let us now show the relation between the (anti)self-dual four-dimensional connection and its restriction to the spatial hypersurface Σ. We begin by writing the full connection in terms of the generators $\{\gamma^I\}$ of the Lorentz lie-algebra: ${}^{\pm}\boldsymbol{A} := A^{IJ}_\mu \gamma_I\gamma_J$ and expanding the sum (see [29, Section 2] and A.1):

$$\begin{aligned}
A^{IJ}_\mu \gamma_I\gamma_J &= A^{i0}_\mu \gamma_i\gamma_0 + A^{0i}_\mu \gamma_0\gamma_i + A^{ij}_\mu \gamma_i\gamma_j \\
&= 2A^{0i}_\mu \gamma_0\gamma_i + A^{ij}_\mu \gamma_i\gamma_j \\
&= 2A^{0i}_\mu \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} + iA^{jk}_\mu \epsilon^{ijk} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}
\end{aligned} \tag{2.71}$$

In the second line we have used the fact that A^{IJ}_μ is antisymmetric in the internal indices and that the gamma matrices anticommute. In the third we have used the expressions for the gamma matrices given in Appendix A.1 to expand out the matrix products. Now, the (anti)self-duality of the connection implies that $A_\mu{}^{IJ} = \pm\frac{i}{2}\epsilon^{IJ}{}_{KL}A_\mu{}^{KL}$. For $I = 0, J \in \{1,2,3\}$ this gives:

$$A_\mu{}^{0i} = \frac{i}{2}\epsilon^{0i}{}_{jk}A_\mu{}^{jk}$$

This allows us to write the last line in the above expression in the form:

$$\boldsymbol{A} = A^{IJ}_\mu \gamma_I \gamma_J = 2i \begin{pmatrix} A^{i+}_\mu \sigma_i & 0 \\ 0 & A^{i-}_\mu \sigma_i \end{pmatrix} \tag{2.72}$$

where:

$$A^{i+}_\mu = \frac{1}{2}\epsilon^{ijk}A^{jk}_\mu + iA^{0i}_\mu \tag{2.73a}$$

$$A^{i-}_\mu = \frac{1}{2}\epsilon^{ijk}A^{jk}_\mu - iA^{0i}_\mu \tag{2.73b}$$

2.9 Phase space structure

2.10 Immirzi Parameter

Now there are several problems with the above prescription which must be addressed before the formulation in terms of constraints and self/anti-self-dual connections becomes a faithful representation of classical Einstein gravity (CEg). The first obvious problem is that in CEg, both the metric $g_{\mu\nu}$ and the Christoffel connection $\Gamma^\alpha{}_{\beta\gamma}$ are both manifestly real objects. Ashtekar's original approach, involved complexifying the phase-space of general relativity in order to simplify the form of the constraints. Consequently, the bivector ${}^\pm\Sigma_{IJ}{}^{\mu\nu}$ (which encodes the metric degrees of freedom) and the gauge connection ${}^\pm A_\mu{}^{IJ}$ ()which determines how the tetrads change from point to point on the manifold and thus encodes the curvature of the manifold) are both complex. In order to recover the usual *real* general relativity we are thus forced to an extra constraint in addition to those listed above 2.67a. These are the so-called "reality conditions":

(2.74)

question is that of the so-called *reality* conditions

There are several subtleties involved with the crucial step 2.69 where we take the

2.11 Symmetries of GR

It is a truth universally acknowledged, that a student encountering the connection variables of Loop Quantum Gravity will be in search of an explanation for why these should be labeled by irreps of SU(2). We will attempt to motivate this choice of gauge group, by noting that it is an appropriately-chosen subgroup of SL(2, C). The significance of SL(2, C) will now be described.

If we wish to construct a theory that encompasses GR under the framework of gauge field theories we should anticipate that the local symmetries of spacetime will define the gauge group of our quantum gravity theory. The causal structure of spacetime defines a future light-cone and past light-cone at each event. The past light-cone of an observer at any given value of time is the celestial sphere at a fixed distance from the observer. The celestial sphere can be parametrised by the angles θ, ϕ, and any point on a sphere can be stereographically projected onto a plane. For our purposes, this shall be taken to be the complex plane, so that any point on the celestial sphere corresponds with a complex number $\zeta = X + iY$. We can write this as the ratio of two complex numbers $\zeta = \alpha/\beta$, which can (if we so desire) be written as functions of θ, ϕ. A change of the complex coordinates (which is equivalent to a coordinate transformation of the real angles θ, ϕ) can be effected by acting on the 2-vector with components α, β with a linear transformation, written in the form of a 2×2 matrix with complex components. If we take the determinant of this matrix to be $+1$ (which we can do, without loss of generality) this is an SL(2, $\mathbb{C}$) transformation.

3 Quantum Field theory

Quantum Field theory should be familiar to most (if not all) modern physicists, however we feel it is worth mentioning the basic details here, in order to emphasize the similarities between QFT and GR, and hence illustrate how GR can be written as a gauge theory. In short, we will see that a local change of phase of the wavefunction is equivalent to the position-dependent change of basis we considered in the case of GR. Just as the partial derivative of a vector gave (via the chain rule) a derivative term corresponding to the change in basis, we will see that a derivative term arises corresponding to the change in phase of the quantum field. This introduces a connection and a covariant derivative defined in terms of the connection.

3.1 Covariant Derivative and Curvature

We require that the action for a gauge theory (such as QED) be invariant under local gauge transformations. This condition is exactly analogous to the freedom to assign a local basis at each point of space, as we did in GR, and similarly we will find that when we differentiate a wavefunction along a path, a connection term arises.

Consider for instance the action for a Dirac field ψ of mass m:

$$S = \int d^4x\, \bar{\psi}(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi \tag{3.1}$$

A *global* gauge transformation corresponds to rotating ψ by a *constant* phase $\psi \to e^{i\theta}\psi$. Under this change we can see that the value of the action

$$S \to \int d^4x\, \bar{\psi}e^{-i\theta}(i\hbar c\gamma^\mu \partial_\mu - mc^2)e^{i\theta}\psi \tag{3.2}$$

does not change because the factor of $e^{i\theta}$ acting on ψ and the corresponding factor of $e^{-i\theta}$ acting on $\bar{\psi}$ pass through the partial derivative unaffected, and cancel out. However if θ

is allowed to vary and becomes a function $\theta(x)$ of position, then we speak of a *local* gauge transformation, due to which the partial derivative becomes

$$\partial_\mu \left(e^{i\theta}\psi\right) = e^{i\theta}\left(\partial_\mu + i(\partial_\mu\theta)\right)\psi \tag{3.3}$$

leading to a modification of the action $S \to S - \int d^4x\hbar c\gamma^\mu(\partial_\mu\theta)\bar{\psi}\psi$. Eq. (3.3), analogously to eq. (2.3), is simply a consequence of the chain rule for differentiation. Now, consider compensating for the effect of the partial derivative by the addition of a 1-form A_μ, with the transformation property $A_\mu(x) \to A_\mu(x) - g^{-1}\partial_\mu\theta$ as $\psi \to e^{i\theta}\psi$. Then the covariant derivative operator $D_\mu = \partial_\mu + igA_\mu$ satisfies all the properties required of a derivative operator (linearity, Leibniz's rule, etc.), and

$$\begin{aligned} D_\mu\psi \to D_\mu\left(e^{i\theta}\psi\right) &= \partial_\mu e^{i\theta}\psi + ig\left(A_\mu(x) - \frac{1}{g}\partial_\mu\theta\right)e^{i\theta}\psi \\ &= e^{i\theta}\left(\partial_\mu + i(\partial_\mu\theta)\right)\psi + ige^{i\theta}A_\mu(x)\psi - ie^{i\theta}\partial_\mu\theta\psi \\ &= e^{i\theta}\left(\partial_\mu + igA_\mu(x)\right)\psi \\ &= e^{i\theta}D_\mu\psi \end{aligned} \tag{3.4a}$$

And so the phase factor passes through the covariant derivative as desired. It is now trivial to show that the Dirac action defined in terms of the covariant derivative,

$$S_{\text{Dirac}} = \int d^4x\, \bar{\psi}(i\hbar c\gamma^\mu D_\mu - mc^2)\psi \tag{3.5}$$

is invariant under local phase transformations of the form $\psi \to e^{i\theta(x)}\psi$, $\bar{\psi} \to \bar{\psi}e^{-i\theta(x)}$, so long as $A_\mu(x)$ transforms as above. The requirement that the action be invariant under *local* gauge transformations has introduced a connection A_μ which tells us how the phase of the wavefunction at each point corresponds to the phase at a different point, in analogy to the connection in GR which told us how coordinate bases varied from point to point.

The discussion above has been restricted to the case of a simple rotation of the phase (that is, $e^{i\theta} \in U(1)$, the rotation group of the plane). In GR, by contrast, the local bases at different points may be rotated in three dimensions relative to each other (that is, the basis vectors are acted upon by elements of $SO(3)$). We can accordingly generalise the discussion above to include phase rotations arising from more elaborate groups, for instance if we replace the wavefunction ψ by a Dirac doublet

$$\psi \to \psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{3.6}$$

and act upon this with transformations of the form

$$U(x) = \exp(i\theta^I(x)t^I). \tag{3.7}$$

Here $t^I = \sigma^I/2$, (with σ^I the I^{th} Pauli matrix) and in general the t^I will be the appropriate generators of the symmetry group, and $I = 1, 2, \ldots N$. In this case the covariant derivative becomes

$$D_\mu = \partial_\mu - igA^I_\mu t^I \tag{3.8}$$

(summation on the repeated index is implied). In analogy to the case discussed above, for GR, we can form the commutator of covariant derivatives. In this case, we obtain the field strength tensor $F_{\mu\nu}$, the analogue of the Riemann curvature tensor,

$$[D_\mu, D_\nu] = -igF^I_{\mu\nu}t^I \tag{3.9}$$

where we can see (by applying the standard commutation relations for the Pauli matrices) that

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g\epsilon^{IJK} A^J_\mu A^K_\nu. \tag{3.10}$$

When our gauge group is abelian (as in QED) all the generators of the corresponding Lie algebra commute with each other and thus the structure constants of the group (ϵ^{IJK} in the above example of $\mathfrak{su}(2)$) vanish. In this event the field strength simplifies to:

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \tag{3.11}$$

For $U(1)$ there is only one group generator and so we can drop the index I in the above expression.

Now, what we have so far is an action 3.5 which describes the dynamics of spinorial fields, interactions between which are mediated by the gauge field. The gauge field *itself* is not yet a dynamic quantity. In any gauge theory, consistency demands that the final action should also include terms which describe the dynamics of the gauge field alone. We know this to be true from our experience with QED where the gauge field becomes a particle called the photon. From classical electrodynamics Maxwell's equations possess propagating solutions of the gauge field - or more simply *electromagnetic waves*. This basic postulate of gravitational theory - the equivalence principle - according to which:

> "Matter tells geometry how to curve and geometry tells matter how to move"

has a parallel statement in the language of gauge theory. In a gauge theory, matter is represented by the fields ψ whereas the "geometry" (not of the background spacetime, but of the interactions between the particles) is determined by the configurations of the gauge field. The equivalence principle of GR can then be cast into field theoretic terms:

> "Matter tells gauge fields how to curve and gauge fields tells matter how to move"

The field strength $F^I_{\mu\nu}$ itself is gauge *covariant* but not gauge *invariant*. Under an infinitesimal gauge transformation $A_0 \to A_0 + \delta A$ the field strength also changes by $F[A_0] \to F[A_0 + \delta A] = F_0 + \delta F$ where the variation in field strength is given by $\delta F = D_\mu[A_0]$ as the user can easily verify by substituting and expanding in 3.11 or 3.10 Here $D_\mu[A_0]$ denotes that the covariant derivative is taken with respect to the original connection A_0.

The term giving the dynamics of the gauge field can be uniquely determined from the requirement of gauge invariance. We need to construct out of the field strength an expression with no indices. This can be achieved by contracting $F^I_{\mu\nu}$ with itself and then taking the trace over the Lie algebra indices. Doing this we get the term:

$$S_{gauge} = -\frac{1}{4}\int d^4x Tr\left[F^{\mu\nu}F_{\mu\nu}\right] \tag{3.12}$$

which in combination with 3.5 gives us the complete action for a gauge field interacting with matter:

$$S = S_{gauge} + S_{Dirac} = \int d^4x \left\{ -\frac{1}{4} Tr\left[F^{\mu\nu} F_{\mu\nu}\right] + \bar{\psi}(i\hbar c\gamma^\mu D_\mu - mc^2)\psi \right\} \tag{3.13}$$

3.2 Wilson Loops and Holonomies

In section 2 we defined a holonomy, as a measure of how much the initial and final values of a spinor or vector transported around a closed loop differ. The name holonomy is also used within the LQG community to refer to a closed loop itself. For this reason we will, as a compromise, adopt the name "piecewise holonomy" to refer to a path or loop along which a spinor is transported. The piecewise holonomies are therefore another way of referring to the generalised coordinates (spin connection) we mentioned above. The conjugate momenta are the metrics, given by tetrads defined along the piecewise holonomies.

Given these constructions of the covariant derivative and the field strength, we must now proceed to determining the form of the gauge-invariant quantities that will serve as physical observables and will allow us to calculate measurable quantities such as scattering amplitudes.

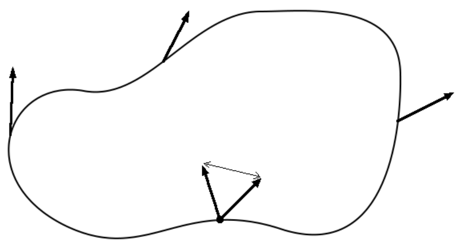


Figure 5: Parallel transporting an object (vector, spinor, etc.) around a closed loop measures the curvature of a surface bounded by the loop. After a complete loop if the object returns to its original state the curvature is zero. If not, then the "angular" change in its state is a measure of the curvature.

The principle tool used in such efforts has already been introduced in Section 2 and goes by the name of *holonomy*. The condition for parallel transport of a vector is that its covariant derivative with respect to the Christoffel connection should vanish, i.e.:

$$\nabla_k v^i = \partial_k v^i + v^j \Gamma^i_{jk} = 0$$

Similarly the condition for parallel transport of a spinor requires that its covariant derivative with respect to the gauge connection should vanish:

$$D_\mu \psi = \partial_\mu \psi + \imath g A_\mu \psi = 0$$

Here $A_\mu \equiv A^I_\mu t^I$ is the gauge connection. Now we can formally write down a solution to the parallel transport equation (see e.g. pages 66-68 of [30]). A path γ_τ, is given by a map from the unit interval on the real line to the manifold M, parameterized by $\tau \in [0,1]$ such that γ_0 and γ_1 are respectively the beginning and end-points of the path. Then the result of parallel transporting the spinor [13] at $\psi_{|(\tau=0)}$ along γ is given by:

$$\psi_{|(\tau=1)} = \mathcal{P}\exp\left\{-\int_\gamma d\tau'(x) A_\mu{}^I(x) T_I n^\mu(x)\right\}\psi_{|(\tau=0)} \tag{3.14}$$

where the $\mathcal{P}$ tells us that the integral must be *path ordered* [14] and n^μ is the *unit* tangent vector to γ at the point x. The path-ordered exponential of the connection gives us an element of the relevant gauge group. Thus ψ at $\tau = 1$ is related to the spinor at $\tau = 0$ by a gauge rotation $\psi_{|(\tau=1)} = U_\gamma \psi_{|(\tau=0)}$, where U_γ is the *holonomy* of the connection along the path.

Now consider the situation when the path γ is a closed loop, i.e. its beginning and its end-points coincide. If the gauge connection vanishes along this path then the gauge rotation is simply the identity matrix and ψ returns to its original value after being parallel transported along the loop. In general, however, the connection will *not* vanish and therefore the associated holonomy U_γ need not be trivial. Analogously to the situation for a curved manifold, where the parallel transport of a vector along a closed path gives us a measure of the curvature of the spacetime bounded by that path, the parallel transport of a spinor around a closed path yields a measure of the *gauge* curvature living on a surface bounded by this path.

3.3 Observables

Configuration variable: $\mathfrak{su}(2)$ valued connection 1-form A_a^i Gauge invariant Wilson loops (Holonomies):

$$g_\gamma[A] = \mathcal{P}\exp\left\{-\int_{\gamma_0}^{\gamma_1} ds\, n^a(s)\, A_a{}^i \tau_i\right\} \tag{3.15}$$

where γ is the curve along which the holonomy is evaluated, s is an affine parameter along that curve and n^a is the tangent to the curve at s. and $g_\gamma[A] \in SU(2)$ (for GR). A calculation that will be useful later on is the functional derivative of the holonomy w.r.t. the connection:

$$\frac{\delta}{\delta A_a{}^i}\, g_\gamma[A] = n^a(s)\tau_i g_\gamma[A] \tag{3.16}$$

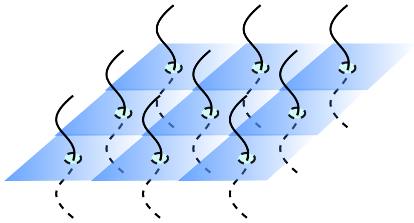
Momentum variable - “electric field” $\Sigma_{ab}^i = \eta_{abc}e^{ci}$ Regularized by smearing on 2-surfaces to obtain “flux” variables:

$$\Phi_{(S,f)} = \int_S d^2x\, f(x) E_a^i E^{ai}\eta_{ij}$$

---

[13] In an exactly analogous way 2.5 has a solution given by $v^\mu_{|(\tau=1)} = \mathcal{P}\left\{e^{-\int_\gamma d\tau' \Gamma^\mu_{\alpha\nu} n^\alpha}\right\} v^\nu_{|(\tau=0)} = U^\mu{}_\nu v^\nu_{|(\tau=0)}$ where the holonomy $U^\mu{}_\nu$ is now an element of $GL(4,\mathbb{R})$.

[14] see Appendix D for the definition of a “path ordered” exponential



4 First steps to a theory of Quantum Gravity

A significant obstacle to the development of a theory of quantum gravity is the fact that GR is not renormalizable. The gravitational coupling constant G (or equivalently $1/M^2_{\text{Planck}}$ in dimensionless units where $G = c = \hbar = 1$) is not dimensionless, unlike the fine-structure constant α in QED. This means that successive terms in any perturbative series have increasing powers of momenta in the numerator.

4.1 Lagrangian (or Path Integral) Approach

In the path-integral approach to quantum field theory the basic element is the propagator (or the partition function when $\mathcal{M}$ is a Euclidean manifold) which allows us to calculate the probability amplitudes between pairs of initial and final states of our Hilbert space. The prototypical example is that of the non-relativistic point particle in flat space moving under the influence of an external potential $V(x)$for which the action is given by:

$$S_{pp}[\gamma] = \int_\gamma d^3x dt \left(\frac{1}{2} m\dot{x}^2 - V(x)\right) \tag{4.1}$$

Note that the potential term must be replaced by a gauge field A_μ the relativistic case, in which case the action takes the form:

$$S_{Rel}[\gamma] = \int_\gamma d^3x dt \, \frac{(p^\mu + A^\mu)(p_\mu + A_\mu)}{m_0} \tag{4.2}$$

where p^μ is the energy-momentum 4-vector of the particle and m_0 is its rest mass. This is the familiar action for a charged point particle moving under the influence of an external potential encoded in the abelian gauge potential A_μ. It is important to keep in mind that the action integral *depends* on the choice of the path γ taken by the system as it evolves from the initial to final states in question. The action can be evaluated for *any* such path and not just the ones which extremize the variation of the action. This allows us to assign a complex amplitude (or real probability in the Euclidean case) to any path γ by:

$$\exp\{iS[\gamma]\} \tag{4.3}$$

Using this complex amplitude as a weighting function we can calculate matrix elements for transitions between an arbitrary pair of initial $\Psi_i(t)$ and final $\Psi_f(t')$ states by summing all paths or *histories* which interpolate between the two states:

$$\left\langle \Psi_i(t) \mid \Psi_f(t') \right\rangle = \int \mathcal{D}[\psi] \exp\{iS[\gamma]\} \tag{4.4}$$

Here $\mathcal{D}[\psi]$ is an appropriate measure on the space of allowed field configurations.

For the point-particle $|q,t\rangle$ represents a state where the particle is localized at position q at time t. The matrix-element between states at two different times then takes the form:

$$\langle q,t \mid q,t'\rangle = \int \mathcal{D}[\psi] \exp\{iS_{pp}[\gamma]\} \tag{4.5}$$

As show in Section 2.4 the field equations (2.9) for gravitation can be derived from a Lagrangian formulation. This is necessary in order to make contact with the path-integral or sum-over-histories approach. For the reader's convenience let us recall the form of the Einstein-Hilbert action for GR on a manifold $\mathcal{M}$ without matter:

$$S_{EH} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\,R \tag{4.6}$$

where $\kappa = -16\pi G$ as before, g is the determinant of the metric tensor and R is the Ricci scalar. For gravity, it is this action which is used in order to calculate the matrix-elements (as in 4.4) for transitions between two states of geometry.

In gravity we are interested in calculating the transition amplitudes between states of geometry defined on two spacelike 3-manifolds Σ_t and $\Sigma_{t'}$ (see Fig. 3). Let us represent the quantum states corresponding to the two hypersurfaces as $|h_{ab},t\rangle$ and $|h'_{ab},t'\rangle$. Then the probability that evolving the geometry according to the quantum Einstein equations will lead to a transition between these two states is given by:

$$\langle h_{ab},t|h'_{ab},t'\rangle = \int \mathcal{D}[g_{\mu\nu}] \exp\{iS_{EH}(g_{\mu\nu})\} \tag{4.7}$$

where the action is evaluated over all 4-metrics $g_{\mu\nu}$ interpolating between the two hypersurfaces Σ_t and $\Sigma_{t'}$. $\mathcal{D}[g_{\mu\nu}]$ is the appropriate measure on the space of 4-metrics.

4.2 Canonical Quantization

The alternative to the path-integral approach is the Hamiltonian method. In the context of general relativity, where the Hamiltonian is a sum of constraints, the Dirac procedure for quantization of constrained systems comes into play. This approach is generally referred to as "canonical" quantization.

In the Hamiltonian formulation one works with a phase space spanned by a set of generalized coordinates $\mathbf{q}_i$, and a set of (generalized) momenta $\mathbf{p}_i$. For the case of general relativity, the generalised coordinate is the intrinsic metric h_{ab} of the spatial 3-manifold ${}^3\Sigma$ and its extrinsic curvature k_{ab} induced by its embedding in ${}^4\mathcal{M}$ is the corresponding generalized momentum. For comparison the phase spaces of various classical systems are listed in Table 4.2

System	**Co-ordinate**	**Momentum**
Simple Harmonic Oscillator	x	p
Ideal Rotor	θ	L_θ
Scalar Field	$\phi(x,t)$	$\pi(x,t)$
Geometrodynamics	h_{ab}	$k_{ab} = \mathcal{L}_t h_{ab}$
Connection- dynamics	$A_a{}^i$	$E^a{}_i$

Now, given our phase space co-ordinatized by $\{h_{ab}, \pi^{ab}\}$ and the explicit form of the Hamiltonian of GR in terms of the Hamiltonian 2.38a and diffeomorphism 2.38b constraints, we can proceed directly to quantization by promoting the Poisson brackets on the classical phase space to commutation relations between the operator acting on a Hilbert space H_{GR}:

$$h_{ab} \to \hat{h}_{ab} \qquad \pi^{ab} \to i\hbar\frac{\delta}{\delta h_{ab}} \tag{4.8a}$$

$$\left\{h_{ab}(x), \pi^{a'b'}(x')\right\} = \delta(x - x')\delta^{a'}{}_a\delta^{b'}{}_b \quad \to \quad \left[\hat{h}_{ab}, i\hbar\frac{\delta}{\delta h_{ab}}\right] = i\hbar\delta^{a'}{}_a\delta^{b'}{}_b \tag{4.8b}$$

$$f[h_{ab}] \quad \to \quad |\Psi_{h_{ab}}\rangle \tag{4.8c}$$

It then remains to write the constraints $\mathcal{H}$ and $\mathcal{C}^\mu$ in operator form using the above substitutions and find states - functionals of the three-metric $|h_{ab}\rangle$ - which are annihilated by the resulting operators:

$$\mathcal{H}, \mathcal{C}^a \quad \to \quad \hat{\mathcal{H}}, \hat{\mathcal{C}}^a \tag{4.9a}$$

$$\hat{\mathcal{H}}|\Psi_{h_{ab}}\rangle = 0; \qquad \hat{\mathcal{C}}^a|\Psi_{h_{ab}}\rangle = 0 \tag{4.9b}$$

States $|\Psi_q\rangle$ which satisfy the above constraints would then be identified with the *physical* states of quantum gravity. The physical Hilbert space is a subset of the kinematic Hilbert space which consists of all functionals of the 3-metrics: $|\Psi_{q'}\rangle \in \mathrm{H}_{phys} \subset \mathrm{H}_{kin}$.

The above prescription is only formal in nature and we run into severe difficulties when we try to implement this recipe. The primary obstacle is the fact that the Hamiltonian constraint 2.38a has a *non-polynomial* dependence on the 3-metric via the Ricci curvature ${}^3\mathcal{R}$. We can see this schematically by noting that ${}^3\mathcal{R}$ is a function of the Christoffel connection Γ which in turn is a complicated function of h_{ab}:

$${}^3\mathcal{R} \sim (\partial\Gamma)^2 + (\Gamma)^2; \qquad \Gamma \sim q\partial q \Rightarrow \partial\Gamma \sim \partial q\partial q + q\partial^2 q \tag{4.10}$$

This complicated form of the constraints raises questions about operator ordering and is also very non-trivial to quantize. Therefore, in this form, the constraints of general relativity are not amenable to quantization.

This is in contrast to the situation with the Maxwell and Yang-Mills fields, which being gauge fields can be quantized in terms of holonomies[15], which form a complete set gauge invariant variables. An optimist might believe that were we able to cast general relativity as a theory of a gauge field, we could make considerably more progress towards quantization than in the metric formulation. This does turn out to be the case as we see in the following sections.

4.3 Loop Quantization

The following exposition only gives us a bird's eye view of the process of canonical quantization. The reader interested in the mathematical details of and the history behind the canonical quantization program is referred to [31].

[15]Holonomies were mentioned briefly in Section 1 and will be covered in greater detail in Section 3.2

The program of Loop Quantum Gravity is as follows. The notion of *background independence*[16], which is central to General Relativity, is considered sacrosanct.

1. Write GR in connection and tetrad variables (in first order form).

2. Perform $3+1$ decomposition to obtain the Einstein-Hilbert-Ashtekar Hamiltonian $\mathcal{H}_{eha}$ which turns out to be a sum of constraints. Therefore, the action of the quantized version of this Hamiltonian on elements of the physical space of states yields $\mathcal{H}_{eha}|\Psi\rangle = 0$. We find that these states are represented by graphs whose edges are labeled by representations of the gauge group (for GR this is $SU(2)$).

5 Kinematical Hilbert Space

A state is given by a graph Γ with edges p_i labeled by elements of $SU(2)$

$$\Psi_\Gamma = \psi(g_1, g_2, \ldots, g_n) \tag{5.1}$$

where g_i is holonomy of A along the i^{th} edge. The inner-product of two different states on the **same** graph can be defined using the Haar measure on group:

$$\langle \Theta_\Gamma | \Psi_\Gamma \rangle = \int_{\mathcal{G}^n} d\mu_1 \ldots d\mu_n \Theta(g_1, \ldots, g_n) \bar{\Psi}(g_1, \ldots, g_n) \tag{5.2}$$

For e.g. $L^2(\mathcal{G})$ - the space of square integrable functions on the manifold of the group $\mathcal{G}$ - constitutes the kinematical space of states for a single edge.

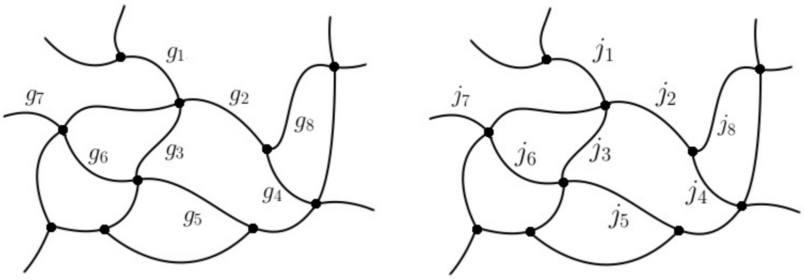
5.1 Spin Networks

So what are spin-networks? Briefly, they are graphs with representations ("spins") of some gauge group (generally SU(2) or SL(2,C) in LQG) living on each edge. At each non-trivial vertex, one has three or more edges meeting up. What is the simplest purpose of the intertwiner? It is to ensure that angular momentum is conserved at each vertex. For the case of four-valent edge we have four spins: (j_1, j_2, j_3, j_4). There is a simple visual picture of the intertwiner in this case.

Picture a tetrahedron enclosing the given vertex, such that each edge pierces precisely one face of the tetrahedron. Now, the natural prescription for what happens when a surface is punctured by a spin is to associate the Casimir of that spin $\mathbf{J}^2$ with the puncture. The Casimir for spin j has eigenvalues $j(j+1)$. You can also see these as energy eigenvalues for the quantum rotor model. These eigenvalues are identified with the area associated with a puncture.

In order for the said edges and vertices to correspond to a consistent geometry it is important that certain constraints be satisfied. For instance, for a triangle we require that

[16]It is important to mention one aspect of background independence that is *not* implemented, *a priori*, in the LQG framework. This is the question of the topological degrees of freedom of geometry. On general grounds, one would expect any four dimensional theory of quantum gravity to contain non-trivial topological excitations at the quantum level. Classically, these excitations would correspond to defects which would lead to deviations from smoothness of any coarse-grained geometry.



(a) Labelling of edges by group elements(b) Labelling of edges by group representations

Figure 6: States of quantum geometry are given by arbitrary graphs whose edges are labeled by group elements representing the holonomy along each edge. The Peter-Weyl theorem allows us to decompose these states in terms of *spin-network* states, where edges are now labelled by group representations (angular momenta).

the edge lengths satisfy the triangle inequality $a + b < c$ and the angles should add up to $\angle a + \angle b + \angle c = \kappa\pi$, with $\kappa = 1$ if the triangle is embedded in a flat space and $\kappa \neq 1$ denoting the deviation of the space from zero curvature (positively or negatively curved).

In a similar manner, for a classical tetrahedron, now it is the sums of the areas of the faces which should satisfy "closure" constraints. For a quantum tetrahedron these constraints translate into relations between the operators j_i which endow the faces with area.

Now for a triangle giving its three edge lengths (a, b, c) completely fixes the angles and there is no more freedom. However, specifying all four areas of a tetrahedron *does not* fix all the freedom. The tetrahedron can still be bent and distorted in ways that preserve the closure constraints (not so for a triangle!). These are the physical degrees of freedom that an intertwiner possesses - the various shapes that are consistent with a tetrahedron with a given set of face areas. More generally a polyhedron with n faces represents an intertwiner between the edges piercing each one of the faces.

5.2 Operators for Quantum Geometry

References: [32, 33, 34, 35, 36, 37]

5.2.1 Area Operator

The area operator in quantum geometry is defined in analogy with the classical definition of the area of a two-dimensional surface S embedded in some higher dimensional manifold M. In the simplest case S is a piece of $\mathbf{R}^2$ embedded in $\mathbf{R}^3$. To each point $p \in S$ we can associate a triad - "frame field" - i.e. a set of vectors which form a basis for tangent space T_p at that point: $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3\}$.

In abstract index notation this basis can also be written more succinctly as $\{e_a{}^i\}_p$ where $a, b, c \ldots$ index the vectors and $i, j, k \ldots$ label the components of each individual vector in the active or "chosen" co-ordinate system. Here it is understood that the basis need not be the same for all points on S, i.e. the parallel transport of the frame field, as given by the gauge connection A_a, is non-zero.

The area of a two-dimensional surface S embedded in ${}^3\Sigma$ is given by:

$$A_S = \int d^2x \sqrt{h} \tag{5.3}$$

where h_{ab} is the metric on S, induced by the three-dimensional metric g_{ab} on ${}^3\Sigma$, and h is its determinant. Given an orthonormal triad field $\{e_a{}^i\}$ on Σ, we can always apply a local gauge rotation to obtain a new triad basis $\{e'_a{}^i\}$, such that two of its legs - or "dyad" - $\{e'_x{}^i, e'_y{}^j\}$ are tangent to the surface S and $e'_z{}^k$ is normal to S. Then the components of the two-dimensional metric h_{AB} ($A, B \in \{x, y\}$ are purely spatial indices) can be written in terms of the dyad basis $\{e_A{}^I\}$ [17] as:

$$h_{AB} = e_A{}^I e_B{}^J \delta_{IJ} \tag{5.4}$$

The above expression with all indices shown explicitly becomes:

$$h_{AB} := \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} = \begin{pmatrix} e_x{}^I e_x{}^J & e_x{}^I e_y{}^J \\ e_y{}^I e_x{}^J & e_y{}^I e_y{}^J \end{pmatrix} \delta_{IJ} \tag{5.5}$$

Now, the determinant of a $n \times n$ matrix A_{ij} can be written as:

$$\det(A) = \sum_{i_1 \ldots i_n \in \mathcal{P}} A_{1\, i_i} A_{2\, i_2} \ldots A_{n\, i_n} \epsilon^{i_1 i_2 \ldots i_n} \tag{5.6}$$

where the sum is over all elements of the permutation group $\mathcal{P}$ of the set of indices $\{i_m\}$ and $\epsilon^{i_1 i_2 \ldots i_n}$ is the completely anti-symmetric tensor. For a 2×2 matrix h_{AB} this expression reduces to:

$$\det(h) = \sum_{i_1, i_2} h_{1\, i_1} h_{2\, i_2} \epsilon^{i_1\, i_2} = h_{11} h_{22} - h_{12} h_{21} \tag{5.7}$$

as the reader can easily check.

In terms of the dyad basis $\{e_A{}^I\}$, adapted to the surface S, the above expression becomes:

$$\begin{aligned} \det(h) &= \left(e_x{}^i e_x{}^j e_y{}^k e_y{}^l - e_x{}^i e_y{}^j e_y{}^k e_x{}^l \right) \delta_{ij} \delta_{kl} \\ &= \left(\epsilon^{ik}{}_m \epsilon^{jl}{}_n - \epsilon^{ij}{}_m \epsilon^{kl}{}_n \right) e_z{}^m e_z{}^n\, \delta_{ij} \delta_{kl} \\ &= \epsilon^{ik}{}_m\, \epsilon_{ikn}\, e_z{}^m e_z{}^n \\ &= \delta_{mn}\, e_z{}^m e_z{}^n \end{aligned} \tag{5.8}$$

where in the first step we have used the fact that for an *orthornormal* triad $\epsilon^{ij}{}_k e_z{}^k = e_x{}^i e_y{}^j$. In the second we have used $\epsilon^{ij}{}_m\, \delta_{ij} = 0$ and in the third step we have used the fact that the

[17] $I, J \in \{0, 1\}$ label generators of the group of rotations $SO(2)$ in two dimensions. They are what is left of the "internal" $\mathfrak{su}(2)$ degrees of freedom of the triad when it is projected down to $\mathcal{S}$.

contraction of two completely anti-symmetric tensors can be written in terms of products of Kronecker deltas.

Thus the classical expression for the area becomes[18]

$$A_S = \int_S d^2x \sqrt{\vec{e_z} \cdot \vec{e_z}} \tag{5.11}$$

where $\vec{e_z} \cdot \vec{e_z} \equiv {e_z}^i {e_z}^j \delta_{ij}$. With the classical version in hand it is straightforward to write down the quantum expression for the area operator. In the connection representation, the classical vierbein plays the role of the momenta. So just as in usual quantum mechanics where the quantum operator for the momenta corresponds to derivation w.r.t. the position co-ordinate $p \to \hat{p} = -i\hbar \frac{\partial}{\partial q}$, the quantum operator for the vierbien is given by the derivative w.r.t. the connection ${e_a}^i \to -i\hbar \frac{\partial}{\partial {A_a}^i}$, giving us:

$$\hat{A}_S = \int_S d^2x \sqrt{\delta_{ij} \frac{\delta}{\delta {A_z}^i} \frac{\delta}{\delta {A_z}^j}} \tag{5.12}$$

In order to determine the action of this operator on spin-network state, let us recall the form of the state 5.1:

$$\Psi_\Gamma = \psi(g_1, g_2, \ldots, g_n)$$

where g_n is the holonomy along the i^{th} edge of the graph. Let the edges of the graph Γ intersect the surface S at exactly k locations. For the time being let us ignore the cases when an edge is tangent to S or when a vertex of the graph happens to lie on S. Then, evidently, the action of 5.12 on the state Ψ_Γ will give us a non-zero result only in the vicinity of the punctures[19]. Thus:

$$\hat{A}_S \Psi_\Gamma \equiv \sum_k \sqrt{\delta_{ij} \frac{\delta}{\delta {A_z}^i} \frac{\delta}{\delta {A_z}^j}} \Psi_\Gamma \tag{5.13}$$

At the k^{th} puncture, the operator will act only the holonomy g_k. From the definition of the holonomy 3.15 and using 3.16 we can see that:

$$\frac{\delta}{\delta {A_a}^i} \psi(g_1, \ldots, g_k, \ldots, g_n) = n^a \tau_i \psi(g_1, \ldots, g_k, \ldots, g_n) \tag{5.14}$$

where n^a is the unit vector tangent to the edge at the location of the puncture. Thus we have:

$$\frac{\delta}{\delta {A_a}^i} \frac{\delta}{\delta {A_b}^j} \psi = n^a n^b \tau_i \tau_j \psi \tag{5.15}$$

[18]This is only valid for the case when Σ is a three-dimensional manifold. In a general n-dimensional manifold, the area is a tensor:

$$A_{\mu\nu}{}^{ij} = e_{[\mu}{}^i e_{\nu]}{}^j \tag{5.9}$$

In order to extract a single number - the "area" - from this tensor we project onto a two-dimensional plane spanned by $\{{n_1}^i, {n_2}^j\}$ and contract the Lie algebra indices:

$$A[S] \tag{5.10}$$

[19]since the connection is *defined* only along those edges and nowhere else!

Performing the contractions over the spatial and internal indices, noting that $n^a n_a = 1$, we finally obtain:

$$\hat{A}_S \Psi_\Gamma \equiv \sum_k \sqrt{\delta^{ij} \hat{J}_i \hat{J}_j} \Psi_\Gamma = \sum_k \sqrt{\boldsymbol{J}^2} \Psi_\Gamma \tag{5.16}$$

where $\hat{J}_i$ is the i^{th} component of the angular momentum operator acting on the spin assigned to a given edge. $\boldsymbol{J}^2$ is the usual Casimir of the rotation group whose action upon a given spin state gives us:

$$\boldsymbol{J}^2 |j\rangle = j(j+1)|j\rangle \tag{5.17}$$

This gives us the final expression for the area of S in terms of the angular momentum label j_k assigned to each edge of Γ which happens to intersect S:

$$\hat{A}_S \Psi_\Gamma = l_p^2 \sum_k \sqrt{j_k(j_k+1)} \Psi_\Gamma \tag{5.18}$$

where l_p^2 (a unit of area given as the square of the Planck length) is inserted in order for both sides to have the correct dimensions.

5.2.2 Volume Operator

First, let us fix some notation: Consider an ensemble $\{\Gamma_i\}$ of spin-networks which corresponds to a semiclassical geometry $\{\mathcal{M}, g_{ab}\}$ in the thermodynamic limit [20].

The volume of a given region S is given by the action of the volume operator $\hat{V}_S$ on the vertices of Γ which lie in S, i.e. for $v \in S \cap \Gamma$. γ is the Immirzi parameter and l_p the Planck length. C_{reg} is a regularization constant and $X^a_{e_I}$ is the operator given by:

$$X^a_{e_I} \Psi_\Gamma = \frac{d}{dt} f \left(U_{e_1}[A^{(\gamma)}], \ldots, e^{t\tau_a} U_{e_I}[A^{(\gamma)}], \ldots, U_{e_E}[A^{(\gamma)}] \right)$$

where the derivative is taken at $t = 0$. $X^a_{e_I} \tau_a$ is thus the left-invariant vector field in the lie-algebra $su(2)$ evaluated along the given edge e_I.

The Rovelli-Smolin version [37] of the volume operator is:

$$\hat{V}_S^{RS} \Psi_\Gamma = \gamma^{3/2} l_p^3 \sum_{v \in S \cap \Gamma} \sum_{I,J,K} \left| \frac{i C_{reg}}{8} \epsilon_{abc} X^a_{v,e_I} X^b_{v,e_J} X^c_{v,e_K} \right|^{1/2} \Psi_\Gamma \tag{5.19}$$

where ϵ_{abc} is the alternating tensor.

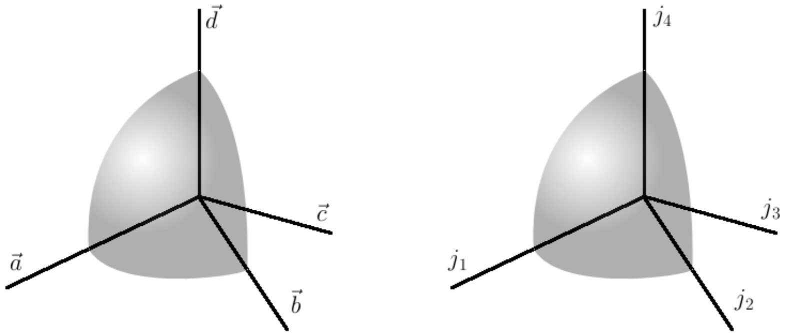
The Ashtekar-Lewandowski [34] version is:

$$\hat{V}_S^{AL} \Psi_\Gamma = \gamma^{3/2} l_p^3 \sum_{v \in S \cap \Gamma} \left| \frac{i C_{reg}}{8} \sum_{I,J,K} \epsilon_v(e_I, e_J, e_K) \epsilon_{abc} X^a_{v,e_I} X^b_{v,e_J} X^c_{v,e_K} \right|^{1/2} \Psi_\Gamma \tag{5.20}$$

Here $\epsilon_v(e_I, e_J, e_K) \in -1, 1, 0$ is the orientation of the three tangent vectors at v to the curves/edges e_I, e_J, e_K. The key difference between the two version lies in this term. The RS operator does not take into account the orientation of the edges which come into the vertex. This fact is taken into account in the AL version. and it allows us to speak of a

Flesh out following section and mention *work in progress*

[20] when the number of degrees of freedom $N \to \infty$, the volume $V \to \infty$ and the number density $N/V \to n$ where n is bounded above



(a) Volume around node in classical geometry. Edges are labeled by vectors of the form $a\hat{x} + b\hat{y} + c\hat{z} \in \mathbb{R}^3$

(b) Volume operator in quantum geometry. Edges are labeled by elements of the form $\alpha\sigma_x + \beta\sigma_y + \gamma\sigma_z \in \mathfrak{sl}(2, \mathbb{C})$

Figure 7: In order to calculate the volume around the vertex we must sum over the volume contained in the solid angles between each unique triple of edges. Classically this volume can be determined by the usual prescription $\vec{a} \cdot (\vec{b} \times \vec{c})$, where $\vec{a}, \vec{b}, \vec{c}$ are the vectors along each edge in the triple. In quantum geometry these vectors are replaced by irreps of $SU(2)$ but the basic idea remains the same.

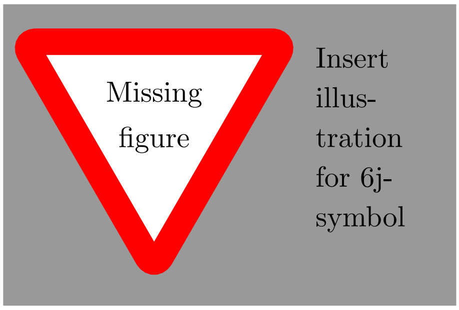
phase transition from a state of geometry at high-temperature ($T > T_c$)where the volume operator averages to zero for all graphs (which are "large" in some suitable sense) and a low-temperature ($T < T_c$) state where a geometric condensate forms and the volume operator gains a non-zero expectation value for states on all graphs. The key point here is that the AL version takes into account the "sign" of the volume contribution from any triplet of edges meeting at a vertex. Given any such triplet of edges e_I, e_J, e_K, by flipping the orientation of any one edge we flip the sign of the corresponding contribution to $\hat{V}_S^{AL}$. If we take the orientation of an edge as our random variable for the purposes of constructing a thermal ensemble, then it is clear, that in the limit of high-temperature these orientations will flip randomly and the sum over the triplets of edges in $\hat{V}_S^{AL}$ will give zero for most (if not all) graph states. As we lower the temperature the system begins to anneal and for some temperature $T = T_c$ the system should reach a critical point where the volume operator spontaneously develops a non-zero expectation on most (if not all) graph states.

Notes:

a. Since the result of the volume operator acting on a vertex depends on the signs $\epsilon(e_I, e_J, e_K)$ of each triplet of edges. A simple dynamical system would then consist of a fixed graph with fixed spin assignments (j_e) to edges but with orientations that can flip, i.e. $j_e \leftrightarrow -j_e$ (much like a spin).

b. The Hamiltonian must be a hermitian operator. This fixes the various term one can include in it. We must also include all terms consistent with all the allowed symmetries in our model.

c. The simplest tri-valent spin-network has one vertex with three edges, e.g. a vertex of the hexagonal lattice. One can generalize the action of the vol. op. on graphs which have vertices with valence v (number of connecting edges) greater than 3. [The vol. op. gives zero on vertices with $v \leq 2$ so these are excluded] To do so we use the fundamental identity which allows to decompose the state describing a vertex with $v \geq 4$ into a sum over states with $v = 3$. One example of the decomposition of a four valent vertex into two three-valent vertices is in the following figure:



d. This model can help us understand how a macroscopic geometry can emerge from the "spin" or manybody system described byy a Hamiltonian, which contains terms with the volume and area operators, on a spin-network.

5.3 Spin-Foams

Spin-foams correspond to histories which connect two spin-networks states. On a given spin-network one can perform certain operations on edges and vertices which leave the state in the kinematical Hilbert space. These involve moves which split or join edges and vertices and those which change the connectivity (as in the "star-triangle transformation"). One can "formally" view a spin-foam as a succession of states $\{|\Psi(t_i)\rangle\}$ obtained by the repeated action of the scalar constraint:

$$\begin{aligned} |\Psi(t_1)\rangle &\sim \exp^{-i\mathcal{H}_{eha}\delta t}|\Psi(t_0)\rangle \\ |\Psi(t_2)\rangle &\sim \exp^{-i\mathcal{H}_{eha}\delta t}|\Psi(t_1)\rangle \ldots \end{aligned} \tag{5.21}$$

and so on [38, 39].

6 Applications

Ultimately, the value of any theory is judged by its relevance for the *real* world. While the question of black hole entropy is, as yet, an abstract problem, it is concrete enough to serve as a test-bed for testing theories of quantum gravity. The ideas of *quantum geometry*, which have their origins in the canonical quantisation approach to quantum gravity, allow us to speak of the microstates of geometry in terms of which we can give a statistical mechanical description of a black hole horizon.

In addition to the Bekenstein area law, mentioned in 1, by investigating the behaviour of a scalar field in the curved background geometry near a black hole horizon it was determined [15] that all black holes behave as almost perfect black bodies radiating at a temperature inversely proportional to the mass of the black hole: $T \propto 1/M_{BH}$. This thermal flux is named Hawking radiation after its discoverer. ~~Since the mass of a black hole determines the area of the black hole's event horizon, the discrete nature of electromagnetic radiation implies that the area of a black hole must also be discrete.~~These properties of a black hole turn out to be completely independent of the nature and constitution of the matter which underwent gravitational collapse to form the black hole in the first place. These developments led to the understanding that a macroscopic black hole, at equilibrium, can be described as a thermal system characterised solely by its mass, charge and angular momentum.

Bekenstein's result has a deep implications for any theory of quantum gravity. The "Bekenstein bound" refers to the fact that 1.1 is the *maximum* number of degrees of freedom - of both, geometry and matter - that can lie within *any region of spacetime* of a given volume V. The argument is straightforward [17]. Consider a region of volume V whose entropy is greater than that of a black hole which would fit inside the given volume. If we add additional matter to the volume, we will eventually trigger gravitational collapse leading to the formation of a black hole, whose entropy will be less than the entropy of the region was initially. However, such a process would violate the second law of thermodynamics and therefore the entropy of a given volume must be at a maximum when that volume is occupied by a black hole. And since the entropy of a black hole is contained entirely on its horizon, one must conclude that the maximum number of degrees of freedom $\mathcal{N}_{max}$ that would be required to describe the physics in a given region of spacetime $\mathcal{M}$, in any theory of quantum gravity, scales not as the volume of the region $V(\mathcal{M})$, but as the area of its boundary [16, 17] $\mathcal{N}_{max} \propto A(\partial\mathcal{M})$.

In view of the independence of the Bekenstein entropy on the matter content of the black hole, the origin of 1.1 must be sought in the properties of the horizon geometry. Assuming that at the Planck scale, geometrical observables such as area are quantized such that there is a minimum possible area element a_0 that the black hole horizon, or any surface for that matter, can be "cut up into", 1.1 can be seen as arising from the number of ways that one can put (or "sew") together $\mathcal{N}$ quanta of area to form a horizon of area $A = k\mathcal{N}a_0$, where k is a constant. In this manner, understanding the thermal properties of a black hole leads us to profound conclusions:

1. In a theory of quantum gravity the physics within a given volume of spacetime $\mathcal{M}$ is completely determined by the values of fields on the boundary of that region $\partial\mathcal{M}$. This is the statement of the *holographic principle*.

2. At the Planck scale (or at whichever scale quantum gravitational effects become relevant) spacetime ceases to be a smooth and continuous entity, i.e. *geometric observables are quantized*.

In LQG, the second feature arises naturally - though not all theorists are convinced that geometry should be "quantized" or that LQG is the right way to do so. One can also argue on general ground, that the first feature - holography - also is present in LQG, though this has not been demonstrated in a conclusive manner. Perhaps this paper might motivate some of its readers to close this gap!

Let us now review the black hole entropy calculation in the framework of LQG.

6.1 Black Hole Entropy

While the question of black hole entropy is, as yet, an abstract problem, it is concrete enough to serve as a test-bed for testing theories of quantum gravity. The ideas of *quantum geometry*, which have their origins in the canonical quantisation approach to quantum gravity, allow us to speak of the microstates of geometry in terms of which we can give a statistical mechanical description of a black hole horizon. For each macroscopic interval of

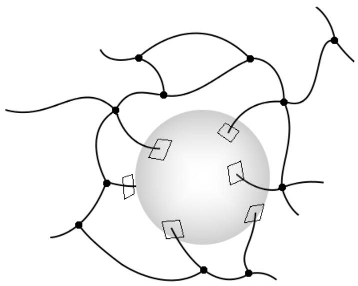


Figure 8: A spin-network corresponding to some state of geometry in the bulk punctures a black-hole horizon at the indicated locations. Each puncture yields a quantum of area $\propto \sqrt{j(j+1)}$ where j is the spin-label on the corresponding edge. The entropy of the black-hole - or, more precisely, of the horizon - can be calculated by counting the number of possible configurations of punctures which add up to give a macroscopic value of the area lying within some finite interval $(A, A+\delta A)$

area $[A+\delta A, A-\delta A]$, entropy S is proportional to log of the number of ways in which we can puncture the sphere to yield an area within that interval. A given set of punctures with labels $\{j_1, j_2, \ldots, j_n\}$ is permissible if:

$$l_p^2 \sum_i \sqrt{j_i(j_1+1)} \in [A+\delta A, A-\delta A]$$

Counting all such configurations compatible with an area $A \gg l_p^2$ yields:

$$S \sim \log(N) \propto A$$

6.2 Loop Quantum Cosmology

One of the first avenues to follow when approaching old problems with new tools is to select the simplest possible scenarios for study, in the hope that the understanding gained in this arena would ultimately lead to a better understanding of more complex systems and processes. In classical GR this corresponds to studying the symmetry reduced solutions of Einstein's equations, such as the FLRW cosmologies and their anisotropic counterparts, and various other exact solutions such as deSitter, anti-deSitter, Schwarzschild, Kerr-Newman

Make table for commonly used metrics

etc. which correspond respectively to a "universe" (in this very restricted sense): with positive cosmological constant ($\Lambda > 0$), a universe with $\Lambda < 0$, a non-rotating black hole and a rotating black hole, both in asymptotically flat spacetimes[21]. In each of these cases the metric has a very small number of local degrees of freedom and hence provides only a "toy model". Of course, in the *real world*, the cosmos is a many-body system and reducing its study to a model such as the FLRW universe is a gross simplification. However, via such models, one can obtain a qualitative grasp of the behavior of the cosmos on the largest scales. Consequently, the simplest quantum cosmological model is that which corresponds to the Friedmann metric whose line-element is given by:

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right) \tag{6.1}$$

where the only dynamical variable is the scale factor $a(t)$ which depends only on the time parameter.

6.2.1 Homogenous Isotropic Models

It is from the application of these ideas to LQG that the first, simplest fully quantum-mechanical cosmological models were born [42] resulting in the framework now known as Loop Quantum Cosmology (LQC).

6.3 Semiclassical Limit

The graviton propogator has a robust quantum version in these models. Its long-distance limit yields the $1/r^2$ behavior [43] expected for gravity and an effective coarse-grained action given by the usual one consisting of the Ricci scalar plus terms containing quantum corrections.

7 Recent Developments

8 Discussion

8.1 Criticisms of LQG

Any fair and balanced review paper on LQG should also mention at least a few of the many objections its critics have presented. A list a few of the more important points of weakness in the framework and brief responses to them follows:

1. *LQG admits a volume extensive entropy and therefore does not respect the Holographic principle*: This misconception arises due to a lack of understanding of the difference between the kinematical and the dynamical phase space of LQG. This critique hinges upon the description of states of quantum gravity as spin-networks which are essentially spin-systems on arbitrary graphs. However, spin-networks only constitute the *kinematical* Hilbert space of LQG. They are solutions of the spatial diffeomorphism

[21] We refer the reader to the extremely comprehensive and well-researched catalog of solutions to Einstein's field equations, in both metric and connection variables, presented in [40]. A somewhat older, but still valuable, catalog of exact solutions is given in [41]

and the gauss constraints but *not* of the Hamiltonian constraint which generates time-evolution.

In order to solve the Hamiltonian constraint we are forced to enlarge the set of states to include *spin-foams* which are histories of spin-networks. The amplitudes associated with a given spin-foam are determined completely by the specification of its boundary state. Physical observables do not depend on the possible internal configurations of a spin-foam but only on its boundary state. In this sense LQG satisfies a far stronger and cleaner version of holography than string theory, where this picture emerges from much more complicated considerations involving graviton scattering from certain extremal black hole solutions.

There are some weak points in my arguments which I have to clarify but the general picture i correct.

2. *LQG violates local Lorentz invariance/picks out a preferred frame of reference*: Lorentz invariance is obeyed in LQG but obviously not in the exact manner as for a continuum geometry. A spin-network/spin-foam state transforms in a well-defined way under boosts and rotations.

fill in details

3. *LQG does not have stable semiclassical geometries as solutions - geometry "crumbles"* - CDT simulations (Renate Loll etc.) show how a stable geometry emerges. The question is exactly how similar is CDT to LQG

fill in details

4. *LQG does not contain fermionic and bosonic excitations that could be identified with members of the Standard Model*: LQG or a suitably modified version which allows braiding between various edges will exhibit invariant topological structures which can be identified with SM particles. In addition in any spin-system - such as LQG - there are effective (emergent) low-energy degrees of freedom which satisfy the equations of motion for Dirac and gauge fields. Xiao-Gang Wen and Michael Levin [44, 45] have investigated so-called "string-nets" and find that the appropriate physical framework is the so-called "tensor category" or "tensor network" theory [46, 47, 48]. In fact string-nets are very similar to spin-networks so Wen and Levin's work - showing that gauge bosons and fermions are quasiparticles of string-net condensates - should carry over into LQG without much modification.

5. *LQG does not exhibit dualities in the manner String Theory does*: Any spin-system exhibits dualities. A graph based model like LQG even more so. One example of a duality is to consider the dual of a spin-network which is a so-called 2-skeleton or simplicial cell-complex. Another is the star-triangle transformation, which can be applied to spin-networks which have certain symmetries, and which leads to a duality between the low and high temperature versions of a theory on a hexagonal and triangular lattice respectively [49].

6. *LQG doesn't admit supersymmetry, wants to avoid extra dimensions, strings, extended objects, etc*: Extra dimensions are supersymmetry are precisely that - "extra" (as in baggage). Occam's razor dictates that a successful physical theory should be founded on the *minimum* number of ingredients. The fact that LQG does not need any such structures does not imply that these structures do not have a natural habitat

in the quantum geometrical picture of LQG. It also bears to note that results from the LHC appear to have ruled out most supersymmetric extensions of the standard model.

7. *LQG has a proliferation of models and a lacks robustness*: Again a lack of extra baggage implies the opposite. LQG is a tightly constrained framework. There are various uniqueness theorems which underlie its foundations and were rigorously proven in the 1990s by Ashtekar, Lewandowski and others. There are questions about the role of the Immirzi parameter and the ambiguity it introduces however these are part and parcel of the broader question of the emergence of semi-classicality from LQG (see Simone Mercuri's papers in this regard). It is in fact, string theory, which suffers from this weakness. There we find not one but at least *five* different effective theories which are all supposed to be emerge as the low-energy of a, so far incompletely understood, "M-theory".

8. *LQG does not contain any well-defined observables and does not allow us to calculate graviton scattering amplitudes*: Several calculations of two-point correlation functions in spin-foams exist in the literature [43] These demonstrate the emergence of an inverse-square law.

8.2 Many body physics and gravitational phenomena

"Quantum Gravity" will ultimately be a theory which describes the interactions of large numbers of quanta of geometry or "atoms" of spacetime and therefore it should be amenable to the application of the methods of many body physics pioneered in condensed matter. However, as practitioners of condensed matter physics are well aware, there is a vast gulf between *knowing* the exact microscopic interaction hamiltonian of a system and in *exploiting* that knowledge to understand the properties of real-world systems. In a related development, our increasing understanding of the renormalization group has led us to understand that in order to describe the dynamical behaviour of macroscopic systems the precise form of the microscopic interaction Hamiltonian is not important. It is instead the symmetries of the Hamiltonian interaction (the spacetime transformations which remains which leave it invariant) that determine the structure of the RG flow and the classification of critical points, lines and surfaces in the phase diagram of the corresponding macroscopic many-body system.

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A Conventions

Uppercase letters $I, J, K, \ldots \in \{0,1,2,3\}$ are "internal" indices which take values in the $\mathfrak{sl}(2,\mathbf{C})$ Lorentz lie-algebra. Greek letters $\mu,\nu,\alpha,\beta \in \{0,1,2,3\}$ are four-dimensional spacetime indices. Lowercase letters $i, j, k, \ldots \in 1,2,3$ are $\mathfrak{su}(2)$ lie-algebra indices and $a, b, c, \ldots \in \{1,2,3\}$ are three-dimensional spatial indices.

A.1 Lorentz Lie-Algebra

For the internal space we use the metric with signature $(-+++)$. For this signature, the generators of the Lorentz lie-algebra are the gamma matrices, which, in the Weyl representation are of the form:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}; \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1}$$

where $i, j, k \in \{1,2,3\}$. These satisfy the commutation relations:

$$\left\{\gamma^I, \gamma^J\right\} = 2\eta_{IJ}; \qquad \left[\gamma^I, \gamma^5\right] = 0 \tag{A.2}$$

where $\eta_{IJ} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric.

B Lie Derivative

Any vector field v^a on a manifold $\mathcal{M}$

Basic results:

$$\mathcal{L}_X T_{\mu_1\ldots\mu_n} = X^\alpha \nabla_\alpha T_{\mu_1\ldots\mu_n} - \sum_{i=1}^{n} T_{\mu_1\ldots\mu_{i-1}\,\alpha\,\mu_{i+1}\mu_n} \nabla_{\mu_i} X^\alpha \tag{B.1}$$

$$\text{in particular...} \quad \mathcal{L}_X T_\mu = X^\alpha \nabla_\alpha T_\mu - T^\alpha \nabla_\alpha X_\mu \equiv [X, T] \tag{B.2}$$

$$\mathcal{L}_X T_{\mu\nu} = X^\alpha \nabla_\alpha T_{\mu\nu} + T^\alpha{}_\nu \nabla_\alpha X_\mu + T_\mu{}^\alpha \nabla_\alpha X_\nu \tag{B.3}$$

$$\Rightarrow \mathcal{L}_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu \tag{B.4}$$

$$\tag{B.5}$$

C Duality

The notion of self/anti-self duality of the gauge field F_{ij} is central to understanding both the topological sector of Yang-Mills theory and the solutions of Einstein's equations in the connection formulation. Let us review this concept.

C.1 Differential Forms

Duality is a notion that emerges naturally from the construction of the space $\oplus_{k=0}^{n}\,{}^n\Lambda_k$ of differential forms on a n-dimensional manifold M. ${}^n\Lambda_k$ denotes the subspace consisting only forms of order k *e.g.* in three dimensions the space of two-forms ${}^3\Lambda_2$ is spanned by the basis $\left\{dx^1 \wedge dx^2, dx^2 \wedge dx^3, dx^3 \wedge dx^1\right\}$ where $\{x^1, x^2, x^3\}$ is some local co-ordinate patch - *i.e.* a mapping from a portion of the given manifold to a region around the origin in $\mathbb{R}^3$.

Now one can show [20, 18] that ${}^n\Lambda_k = {}^n\Lambda_{n-k}$, *i.e.* the space of k-forms is the same as the space of (n-k)-forms. Thus any k-form $F_{a_1 a_2 \ldots a_k}$, defined on an n dimensional manifold, can be mapped to an (n-k)-form $(\star F_{a_1 a_2 \ldots a_{n-k}})$. This is accomplished with the completely antisymmetric tensor $\epsilon_{x_1 \ldots x_n}$ on M:

$$(\star F)^{a_1 \ldots a_{n-k}} = \frac{1}{(n-k)!} \epsilon^{a_1 \ldots a_{n-k}}{}_{a_{n-k+1} \ldots a_n} F^{a_{n-k+1} \ldots a_n} \tag{C.1}$$

A simple illustration of this is the equivalence between one-forms and two-forms on a three-dimensional manifold. Any two, non-degenerate, vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ span a two-dimensional subspace of $\mathbb{R}^3$. Using these two vectors we can construct a third vector $\boldsymbol{c}$ formed by the so-called cross product: $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$, where the components of $\boldsymbol{c}$ are given by $c^i = \epsilon^i{}_{jk} a^j b^k$. This construction is taught to us in elementary algebra courses, but never quite seemed to make complete sense because it seemed to be peculiar to three-dimensions.

We need the language of differential forms to fully comprehend what is happening. In this language the cross-product $\times$ is replaced by the wedge-product denoted by the $\wedge$ symbol. The *wedge product* of two one-forms is a two-form which is then written as: $c = a \wedge b$. In component notation $c_{ij} = a_{[i} b_{j]}$. Now the antisymmetric tensor ϵ^{ijk} on $\mathbb{R}^3$ allows us to find the dual of this two-form. On a three-dimensional manifold this must necessarily be a one-form $(\star c)^i = \epsilon^{ijk} c_{ij}$. It is this one-form that we then identify with the vector $\boldsymbol{c}$.

C.2 Spacetime Duality

In four-dimensions the dual of any two-form is another two-form

$$\star F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu} \tag{C.2}$$

This goes through in for even-dimensional manifold. It is due to this property of even-dimensional manifolds that we can define *self-dual* and *anti-self-dual* n-forms, where a form is self (anti-self) dual if:

$$\star F = \pm F \tag{C.3}$$

Given an arbitrary 2-form $G_{\mu\nu}$ its self-dual part is given by $G + \star G$ and the anti-self-dual part by $G - \star G$. We can check that these satisfy : check and fix this

$$\star \{G \pm \star G\} = \star G \mp G = \pm \{G \pm \star G\}$$

because $\star\star = \mathbf{1}$ in a Euclidean background.

Thus any 2-form can always written as a linear-sum of a self-dual and an anti-self-dual piece:

$$G = \alpha G^+ + \beta G^- \qquad \star G = \alpha G^+ - \beta G^-$$

because $\star G^+ = G^+$ and $\star G^- = -G^-$. Conversely $G^\pm$ can be determined by inverting this relationship

$$G^+ = \frac{G + \star G}{2\alpha} \qquad G^- = \frac{G - \star G}{2\beta}$$

The above results hold for a Euclidean spacetime. For a Lorentzian background we would instead have $\star\star = -\mathbf{1}$ and the dual of a two-form is given by:

$$\star F_{\alpha\beta} = \frac{i}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta} \tag{C.4}$$

and the statement of self (anti-self) duality becomes:

$$\star F = \pm iF \tag{C.5}$$

with the self-dual and anti-self-dual pieces of a two-form G given by $G^{\pm} = G \pm \star iG$

In general, the dual of a k-form $F^{a_1 \ldots a_k}$ is given by a $n-k$-form which can be written as:

C.3 Lie-algebra duality

The previous section discussed self-duality in the context of tensors with spacetime indices $T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}$. In gauge theories based on non-trivial lie-algebras we also have tensors with lie-algebra indices, such as the curvature $F_{\mu\nu}{}^{IJ}$ of the gauge connection $A_\mu{}^{IJ}$ where I, J label generators of the relevant Lie algebra.. The dual of the connection can then be defined using the completely antisymmetric tensor acting on the Lie algebra indices, as in:

$$\star A_\mu{}^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}A_\mu{}^{KL} \tag{C.6}$$

C.4 Yang-Mills

Let us illustrate the utility of the notion of self-duality by examining the classical Yang-Mills action:

$$S_{YM} = \int_{R^4} Tr\left[F \wedge \star F\right]$$

Varying this action w.r.t the connection gives us the equations of motion:

$$dF = 0\,; \quad d\star F = 0$$

which are satisfied if $F = \pm\star F$, i.e. if the gauge curvature is self-dual or anti-self-dual. Thus for self/anti-self dual solutions the Yang-Mills action reduces to:

$$S^{\pm}_{YM} = \pm\int_{R^4} Tr\left[F \wedge F\right]$$

which is a topological invariant of the given manifold and is known as the *Pontryagin index*. Here the sign $\pm$ superscript on the r.h.s. denotes whether the field is self-dual or anti-self-dual. Again this is a matter of convention and we could equally well choose to reverse this identification.

C.5 Geometrical interpretation

Given any (Lie-algebra valued) two-form F^I_{ab} (where $I, J, K \ldots$ are Lie-algebra indices) we can obtain an element of the Lie-algebra by contracting it with a member of the basis of the space of two-forms: $\{dx^i \wedge dx^j\}$ where x^i denotes the i^{th} vector and **not** the components of a vector. The components are suppressed in the differential form notation as explained in the preceding sections. The resulting lie-algebra element is:

$$\Phi^I{}_{ab} = F^I{}_{ab} dx^a \wedge dx^b$$

and $\Phi^I{}_{ab}$ is the *flux* of the field strength through the two-dimensional surface spanned by $\{dx^a, dx^b\}$.

We can also define:

$$\star\Phi^I_{ab} = \star F^I{}_{ab} dx^a \wedge dx^b = \frac{1}{2}\epsilon_{ab}{}^{cd} F^I{}_{cd} dx^a \wedge dx^b$$

which implies that $\star\Phi^I{}_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}\Phi^I{}_{cd}$, *i.e.* the flux of the field strength through the ab plane is equal to the flux of the *dual* field through the cd plane.

D Path Ordered Exponential

The holonomy of a connection along a path (open or closed) γ in a manifold M is defined as (**??**):

$$\psi_{|(\tau=1)} = \mathcal{P}\left\{e^{-\int_\gamma d\tau' A_\mu n^\mu}\right\}\psi_{|(\tau=0)} = U_\gamma\, \psi_{|(\tau=0)} \tag{D.1}$$

The exponential can be formally expressed in terms of a Taylor series expansion:

$$e^{-\int_\gamma d\tau' A_\mu n^\mu} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!}\left\{\int_{\sigma_0=0}^{\sigma_1}\int_0^{\sigma_2}\ldots\int_0^{\sigma_n=1} d\tau_1 d\tau_2 \ldots d\tau_n\ A(\sigma_n)A(\sigma_{n-1})\ldots A(\sigma_1)\right\} \tag{D.2}$$

where for the n^{th} term in the sum, the path γ is broken up into n intervals parametrized by the variables $\{\tau_1, \tau_2, \ldots, \tau_n\}$ over which the integrals are performed. The interested reader is referred to pgs. 66 - 68 of [30].

E Peter-Weyl Theorem

The crucial step involved in going from graph states with edges labeled by holonomies to graph states with edge labeled by group representations (angular momenta) is the Peter-Weyl theorem . This theorem allows the generalization of the notion of Fourier transforms to functions defined on a group manifold for compact, semi-simple Lie groups.

Given a group $\mathcal{G}$, let $D^j(g)_{mn}$ be the matrix representation of any group element $g \in \mathcal{G}$. Then we have (see Chapter 8 of [50]):

Theorem E.1. *The irreducible represenation matrices $D^j(g)$ for the group $SU(2)$ satisfy the following orthonormality condition:*

$$\int d\mu(g) D^\dagger_j(g)^m{}_n D^{j'}(g)^{n'}{}_{m'} = \frac{n_G}{n_j}\delta^{j'}{}_j \delta^{n'}{}_n \delta^{m'}{}_m \tag{E.1}$$

where n_j is the dimensionality of the j^{th} representation of G and n_G is the *order* of the group. For a finite group this is simply the number of elements of the group. For *e.g.*, for $\mathbb{Z}_2$, $n_G = 2$. However a continuous or Lie group such as $SU(2)$ has an uncountable infinity of group elements. n_G in such cases corresponds to the "volume" of the group manifold.

This property allows us to decompose any square-integrable function $f(g) : \mathcal{G} \to \mathbb{C}$ in terms of its components with respect to the matrix coefficients of the group representations:

Theorem E.2. *The irreducible representation functions $D^j(g)^m{}_n$ form a complete basis of (Lebesgue) square-integrable functions defined on the group manifold.*

Any such function $f(g)$ can then be expanded as:

$$f(g) = \sum_{j;mn} f_j{}^{mn} D^j(g)_{mn} \tag{E.2}$$

where $f_j{}^{mn}$ are constants which can be determined by inserting the above expression for $f(g)$ in E.1 and integrating over the group manifold. Thus we obtain:

$$\int d\mu(g) f(g) D^\dagger_j(g)^{mn} = \sum_{j';m'n'} \int d\mu(g) f_{j'}{}^{m'n'} D^{j'}(g)_{m'n'} D^\dagger_j(g)^{mn} = \sum_{j';m'n'} f_{j'}{}^{m'n'} \frac{n_G}{n_j} \delta^{j'}{}_j \delta^{n'}{}_n \delta^{m'}{}_m \tag{E.3}$$

which gives us:

$$f_j{}^{mn} = \sqrt{\frac{n_j}{n_G}} \int d\mu(g) f(g) D^\dagger_j(g)^{mn} \tag{E.4}$$

F Kodama State

The Kodama state is an exact solution of the Hamiltonian constraint for LQG with positive cosmological constant $\Lambda > 0$ and hence is of great importance for the theory. It is given by:

$$\Psi_K(A) = \mathcal{N} e^{\int S_{CS}} \tag{F.1}$$

where $\mathcal{N}$ is a normalization constant; $S_{CS}[A]$ is the Chern-Simons action for the connection A^I_μ on the spatial 3-manifold M, given by:

$$S_{CS} = \frac{2}{3\Lambda} \int Y_{CS}$$

where:

$$Y_{CS} = \frac{1}{2} Tr \left[A \wedge \mathbf{d}A + \frac{2}{3} A \wedge A \wedge A \right]$$

where $\mathbf{d}A \simeq \partial_{[\mu} A^I_{\nu]}$ is the exterior derivative. The wedge product $\wedge$ between two 1-forms P_a and Q_a is:

$$P \wedge Q \simeq P_{[a} Q_{b]}$$

For identical one-forms the wedge product gives zero. That is why for the Chern-Simons to have a non-zero cubic term the connection must be non-abelian. Let us write the various terms in the Chern-Simons density explicitly:

$$A \wedge \mathbf{d}A \equiv A^i_{[p} \partial_q A^j_{r]} T_i T_j \qquad A \wedge A \wedge A \equiv A^i_{[p} A^j_q A^k_{r]} T_i T_j T_k$$

where $p, q, r \ldots$ are worldvolume ("spacetime") indices and $i, j, k \ldots$ are worldsheet ("internal") indices; and T_i are the basis vectors of the lie-algebra/internal space.

Taking the trace over these terms gives us:

$$Y_{CS} = \frac{1}{2} Tr \left[A^i_{[p} \partial_q A^j_{r]} T_i T_j + \frac{2}{3} A^i_{[p} A^j_q A^k_{r]} T_i T_j T_k \right]$$

The trace over the lie-algebra elements gives us:

$$Tr \left[T_i T_j \right] = \delta_{ij} \qquad Tr \left[T_i T_j T_k \right] = f_{ijk}$$

where f_{ijk} are the structure constants of the gauge group.

G 3j-symbols

The Wigner 3j-symbol is related to the Clebsch-Gordan coefficients through:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} \langle j_1, m_1; j_2, m_2 | j_3, m_3 \rangle$$

where the (j_i, m_i) are the orbital and magnetic quantum numbers of the i^{th} particle. $| \, j_1, m_1; j_2, m_2 \rangle$ is the state representing two particles (or systems) each with their separate angular momentum numbers. $| \, j_3, m_3 \rangle$ represents the *total* angular momentum of the system. Classically we have two systems with angular momentum $\vec{L}_1$ and $\vec{L}_2$, then the angular momentum of the combined system is: $\vec{L}_3 = \vec{L}_1 + \vec{L}_2$.

In quantum mechanics, however, the angular momentum of the composite system can be any one of a set of possible allowed choices. Whether or not the angular momentum of the composite system can be specified by quantum numbers j_3, m_3 is determined by whether or not the Clebsch-Gordan coefficient is non-zero.

H Regge Calculus

Regge showed in 1961 that one could obtain the continuum action of general relativity "in 2+1 dimensions" from a discrete version thereof given by [51, 52]:

$$S_i = \sum_{a=1}^{6} l_{i,a} \theta_{i,a}$$

is the Regge action for the i^{th} tetrahedron. Here the sum over a is the sum over the edges of the tetrahedron. $l_{i,a}$ and $\theta_{i,a}$ are the length of the edge and the dihedral deficit angle, respectively, *around* the a^{th} edge of the i^{th} tetrahedron.

The Regge action for a manifold built up gluing such simplices is simply the sum of the above expression over all N simplices:

$$S_{Regge} = \sum_{i=1}^{N} S_i$$

It was later shown by Ponzano and Regge [53] that the in the limit that $j_i \gg 1$, the 6-j symbol corresponds to the cosine of the Regge action [54]:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{12\pi V} \cos\left(\sum_i j_i \theta_i + \frac{\pi}{4} \right)$$

I Glossary

A list of terms which are commonly used in the quantum gravity community but which are likely to be unfamiliar to those without a background in gravitational physics is given below with a brief descriptions of each term.

Symmetry Reduction	the study of solutions of the EFEs possessing strong global symmetries which reduces the effective local degrees of freedom to a small number.
Asymptotic Flatness	a metric with a radial dependence is considered asymptotically flat if it approaches (in a well-defined sense) a flat Minkowski metric as $r \to \infty$

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