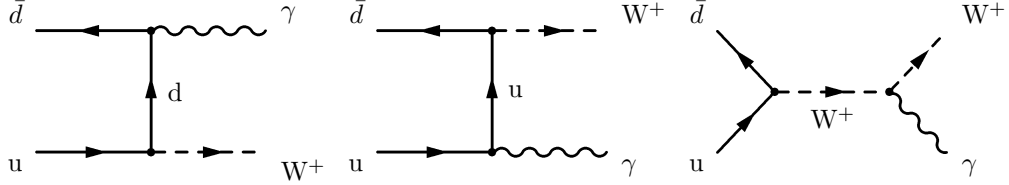


# 1 Introduction

We want to calculate the total cross-section  $\sigma$  at leading order for the process

$$u + \bar{d} \rightarrow W^+ + \gamma.$$

The following three Feynman diagrams contribute to the total matrix element:



The (anti)quark masses are neglected, otherwise we would have additional diagrams involving (anti)quarks of the second and third generation. The total matrix element (at lowest order)  $\mathcal{M}$  is the sum of the three matrix elements  $\mathcal{M}_1$ ,  $\mathcal{M}_2$  and  $\mathcal{M}_3$ . In order to calculate the total cross-section, we first have to calculate  $\langle |\mathcal{M}|^2 \rangle$ :

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \langle |\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3|^2 \rangle \\ &= \langle \mathcal{M}_1 \mathcal{M}_1^* + \mathcal{M}_2 \mathcal{M}_2^* + \mathcal{M}_3 \mathcal{M}_3^* + \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1 \mathcal{M}_3^* + \mathcal{M}_2 \mathcal{M}_3^* \\ &\quad + \mathcal{M}_2 \mathcal{M}_1^* + \mathcal{M}_3 \mathcal{M}_1^* + \mathcal{M}_3 \mathcal{M}_2^* \rangle \\ &= \frac{1}{36} \sum_{S,C} [\mathcal{M}_1 \mathcal{M}_1^* + \mathcal{M}_2 \mathcal{M}_2^* + \mathcal{M}_3 \mathcal{M}_3^* + \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1 \mathcal{M}_3^* + \mathcal{M}_2 \mathcal{M}_3^* \\ &\quad + (\mathcal{M}_1 \mathcal{M}_2^*)^* + (\mathcal{M}_1 \mathcal{M}_3^*)^* + (\mathcal{M}_2 \mathcal{M}_3^*)^*] . \end{aligned}$$

Here  $S = \{s_1, s_2, s_3, s_4\}$  and  $C = \{c_1, c_2\}$ . The factor in front effectuates the averaging over the unknown spin and color properties of the incoming particles. As there are two possible spin polarizations and three options for the color for both quarks, this factor equals  $1/(2 \times 2 \times 3 \times 3) = 1/36$ .

Let us define all the four-momenta without losing any generality. We will work in the center of mass frame and we choose the incoming particles to move along the z-axis and for the direction of the outgoing particles we introduce the spherical angles  $\theta$  and  $\phi$ . For the incoming quarks we have:

$$p_1 = \begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} E_1 \\ 0 \\ 0 \\ -E_1 \end{pmatrix}$$

and for the outgoing particles we define the four-momenta as

$$p_3 = \begin{pmatrix} E_3 \\ E_3 \sin \theta \cos \phi \\ E_3 \sin \theta \sin \phi \\ E_3 \cos \theta \end{pmatrix}, \quad p_4 = \begin{pmatrix} \sqrt{E_3^2 + m_w^2} \\ -E_3 \sin \theta \cos \phi \\ -E_3 \sin \theta \sin \phi \\ -E_3 \cos \theta \end{pmatrix}.$$

By definition of the center of mass frame the total momentum equals zero.

From energy conservation we obtain the following relation between  $E_1$  and  $E_3$ :

$$2 E_1 = E_3 + \sqrt{E_3^2 + m_{\text{w}}^2},$$

from which it follows that

$$E_3 = E_1 - \frac{m_{\text{w}}^2}{4 E_1}.$$

Eventually, we want to express the total cross-section in the total center of mass energy  $E$ . The energies  $E_1$  and  $E_3$  are expressed in  $E$  as follows:

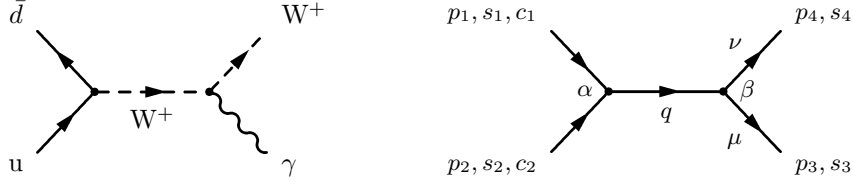
$$E_1 = \frac{E}{2}$$

and

$$E_3 = \frac{E^2 - m_{\text{w}}^2}{2 E}.$$

## 2 The Third Diagram

For the third diagram we make the following definitions:



The Feynman rules read:

$$\begin{aligned}
\text{incoming } \bar{d} &: \sqrt{\hbar} \bar{v}_{(1)} \\
\text{incoming } u &: \sqrt{\hbar} u_{(2)} \\
\text{outgoing } \gamma &: \sqrt{\hbar} \epsilon_{(3)}^{\mu*} \\
\text{outgoing } W^+ &: \sqrt{\hbar} \epsilon_{(4)}^{\nu*} \\
\text{propagator } W^+ &: - \frac{i\hbar (g^{\alpha\beta} - q^\alpha q^\beta / m_w^2)}{q^2 - m_w^2} \\
\text{vertex } u \bar{d} W^+ &: \frac{ig_w}{\hbar} (1 + \gamma^5) \gamma_\alpha \\
\text{vertex } W^+ W^+ \gamma &: \frac{iQ_w}{\hbar} \left( g_{\nu\beta} (q + p_4)_\mu + g_{\mu\nu} (p_3 - p_4)_\beta - g_{\beta\mu} (q + p_3)_\nu \right)
\end{aligned}$$

The corresponding matrix element  $\mathcal{M}_3$  takes the form:

$$\begin{aligned}
\mathcal{M}_3 &= \sqrt{\hbar} \bar{v}_{(1)} \left[ \frac{ig_w}{\hbar} (1 + \gamma^5) \gamma_\alpha \right] \left[ - \frac{i\hbar (g^{\alpha\beta} - q^\alpha q^\beta / m_w^2)}{q^2 - m_w^2} \right] \sqrt{\hbar} \epsilon_{(4)}^{\nu*} \\
&\times \left[ \frac{iQ_w}{\hbar} \left( g_{\nu\beta} (q + p_4)_\mu + g_{\mu\nu} (p_3 - p_4)_\beta - g_{\beta\mu} (q + p_3)_\nu \right) \right] \sqrt{\hbar} \epsilon_{(3)}^{\mu*} \sqrt{\hbar} u_{(2)} \\
&= \frac{i\hbar g_w Q_w}{(p_1 + p_2)^2 - m_w^2} \bar{v}_{(1)} (1 + \gamma^5) \gamma_\alpha \left( g^{\alpha\beta} - \frac{(p_1 + p_2)^\alpha (p_1 + p_2)^\beta}{m_w^2} \right) \epsilon_{(4)}^{\nu*} \\
&\times \left[ g_{\nu\beta} (2p_4 + p_3)_\mu + g_{\mu\nu} (p_3 - p_4)_\beta - g_{\beta\mu} (2p_3 + p_4)_\nu \right] \epsilon_{(3)}^{\mu*} u_{(2)} \\
&= \frac{i\hbar g_w Q_w}{2(p_3 \cdot p_4)} \bar{v}_{(1)} (1 + \gamma^5) \gamma_\alpha \left( g^{\alpha\beta} - \frac{(p_1 + p_2)^\alpha (p_1 + p_2)^\beta}{m_w^2} \right) \\
&\times \left[ (\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) (p_3 - p_4)_\beta + \epsilon_{(4)\beta}^* (2p_4 + p_3) \cdot \epsilon_{(3)}^* - \epsilon_{(3)\beta}^* (2p_3 + p_4) \cdot \epsilon_{(4)}^* \right] u_{(2)} \\
&= \frac{i\hbar g_w Q_w}{2(p_3 \cdot p_4)} \bar{v}_{(1)} (1 + \gamma^5) \left\{ ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) \left[ \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right] \right. \\
&\quad \left. + ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \left[ \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right] + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right\} u_{(2)}.
\end{aligned}$$

Here we have introduced the notation  $\not{\epsilon}^* \equiv \epsilon_\mu^* \gamma^\mu$ . Thus the  $\gamma$ -matrix is *not* complex conjugated.

### 3 Useful Formulas

#### 3.1 Completeness relations

In our calculations we need several completeness relations for polarizations vectors. For simplicity the momentum of every particle is denoted by  $p$  and the spin by  $s$ .

Quarks (massless):

$$\sum_{s,c} u_{(s,c)} \overline{u_{(s,c)}} = \not{p}$$

Antiquarks (massless):

$$\sum_{s,c} v_{(s,c)} \overline{v_{(s,c)}} = \not{p}$$

Photons:

$$\sum_s \epsilon_{(s)}^\mu \epsilon_{(s)}^{\nu*} = -g^{\mu\nu} + \frac{p^\mu r^\nu + r^\mu p^\nu}{p \cdot r},$$

where  $r^\mu$  (the gauge vector) is an arbitrary massless vector that is not parallel to  $p^\mu$ . By making the following choice

$$r = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix},$$

a lot of terms will drop out in the calculation. All inner products containing  $r$  appear in one the following forms:

$$\frac{p_1 \cdot r}{p_3 \cdot r} = 0, \quad \frac{p_2 \cdot r}{p_3 \cdot r} = 0, \quad \frac{p_3 \cdot r}{p_3 \cdot r} = 1, \quad \frac{p_4 \cdot r}{p_3 \cdot r} = -1.$$

Hence by choosing for this specific gauge we can save ourselves a lot of time!

$$\begin{aligned} \sum_s \not{\epsilon}_{(s)} \not{\epsilon}_{(s)}^* &= \left( \sum_s \epsilon_{(s)}^\mu \epsilon_{(s)}^{\nu*} \right) \gamma_\mu \not{p} \gamma_\nu \\ &= \left( -g^{\mu\nu} + \frac{p^\mu r^\nu + r^\mu p^\nu}{p \cdot r} \right) \gamma_\mu \not{p} \gamma_\nu \\ &= -\gamma^\nu \not{p} \gamma_\nu + \frac{\not{p} \not{r} + \not{r} \not{p}}{p \cdot r} \\ &= 2 \not{p} + \frac{2(a \cdot p) \not{r} - \not{p} \not{r} + 2(a \cdot r) \not{p} - \not{r} \not{p}}{p \cdot r} \\ &= 2 \not{p} + \frac{2[(a \cdot p) \not{r} + (a \cdot r) \not{p}]}{p \cdot r} - \frac{\not{p} [\not{p} \not{r} + 2(p \cdot r) - \not{p} \not{r}]}{p \cdot r} \\ &= 2 \left( \frac{a \cdot p}{p \cdot r} \right) \not{r} + 2 \left( \frac{a \cdot r}{p \cdot r} \right) \not{p} \end{aligned}$$

$$\begin{aligned}
\sum_s (a \cdot \epsilon_{(s)}) \not{\epsilon}_{(s)}^* &= a_\mu \left( \sum_s \epsilon_{(s)}^\mu \epsilon_{(s)}^{\nu *} \right) \gamma_\nu \\
&= a_\mu \left( -g^{\mu\nu} + \frac{p^\mu r^\nu + r^\mu p^\nu}{p \cdot r} \right) \gamma_\nu \\
&= -\not{a} + \left( \frac{a \cdot p}{p \cdot r} \right) \not{r} + \left( \frac{a \cdot r}{p \cdot r} \right) \not{p}
\end{aligned}$$

W bosons:

$$\sum_s \epsilon_{(s)}^\mu \epsilon_{(s)}^{\nu *} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_w^2}$$

$$\begin{aligned}
\sum_s \not{\epsilon}_{(s)} \not{a} \not{\epsilon}_{(s)}^* &= \left( \sum_s \epsilon_{(s)}^\nu \epsilon_{(s)}^{\mu *} \right) \gamma_\mu \not{a} \gamma_\nu \\
&= \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_w^2} \right) \gamma_\mu \not{a} \gamma_\nu \\
&= -\gamma^\nu \not{a} \gamma_\nu + \frac{\not{p} \not{a} \not{p}}{m_w^2} \\
&= 2 \not{a} + \frac{2(a \cdot p) \not{p}}{m_w^2} - \frac{\not{a} \not{p} \not{p}}{m_w^2} \\
&= \not{a} + 2 \left( \frac{a \cdot p}{m_w^2} \right) \not{p}
\end{aligned}$$

$$\begin{aligned}
\sum_s (a \cdot \epsilon_{(s)}) \not{\epsilon}_{(s)}^* &= a_\mu \left( \sum_s \epsilon_{(s)}^\mu \epsilon_{(s)}^{\nu *} \right) \gamma_\nu \\
&= a_\mu \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_w^2} \right) \gamma_\nu \\
&= -\not{a} + \left( \frac{a \cdot p}{m_w^2} \right) \not{p}
\end{aligned}$$

### 3.2 Trace identities

$$\begin{aligned}
\text{Tr}(\gamma^\mu \gamma^\nu) &= \text{Tr}(\gamma^\nu \gamma^\mu) \\
&= \frac{1}{2} \text{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \\
&= \text{Tr}(g^{\mu\nu} 1) \\
&= 4 g^{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\text{Tr}(\not{a} \not{b}) &= \text{Tr}(a^\mu b_\nu \gamma_\mu \gamma^\nu) \\
&= a^\mu b_\nu \text{Tr}(\gamma_\mu \gamma^\nu) \\
&= a^\mu b_\mu \text{Tr}(1) \\
&= 4(a \cdot b)
\end{aligned}$$

$$\begin{aligned}
\text{Tr}(\not{a}\not{b}\gamma^5) &= \text{Tr}(a_\mu b_\nu \gamma^\mu \gamma^\nu \gamma^5) \\
&= a_\mu b_\nu \text{Tr}(\gamma^\mu \gamma^\nu \gamma^5) \\
&= 0
\end{aligned}$$

#### 4 $\mathcal{M}_3 \mathcal{M}_3^*$

$$\begin{aligned}
\langle |\mathcal{M}_3|^2 \rangle &= \frac{1}{36} \sum_{S,C} \mathcal{M}_3 \mathcal{M}_3^* \\
&= \frac{1}{36} \sum_{S,C} \left\{ \frac{i\hbar g_w Q_w}{2(p_3 \cdot p_4)} \overline{v_{(1)}} (1 + \gamma^5) \left[ \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \times ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) + \left. \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \right. \\
&\quad \left. \left. + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] u_{(2)} \right\} \times \left\{ \frac{i\hbar g_w Q_w}{2(p_3 \cdot p_4)} \overline{v_{(1)}} (1 + \gamma^5) \left[ ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) \right. \right. \\
&\quad \times \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) + ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \\
&\quad \left. \left. \times \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] u_{(2)} \right\}^* \\
&= \frac{\hbar^2 g_w^2 Q_w^2}{72(p_3 \cdot p_4)^2} \sum_{S,C} \left\{ \overline{v_{(1)}} (1 + \gamma^5) \left[ \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \times ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) + \left. \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \right. \\
&\quad \left. \left. + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] u_{(2)} \right\} \times \left\{ \overline{u_{(2)}} \left[ (p_4 \cdot \epsilon_{(3)}) \left( \not{\epsilon}_{(4)} - \frac{p_3 \cdot \epsilon_{(4)}}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \left. \left. + (p_3 \cdot \epsilon_{(4)}) \left( \frac{p_4 \cdot \epsilon_{(3)}}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)} \right) + (\epsilon_{(3)} \cdot \epsilon_{(4)}) \not{p}_3 \right] (1 - \gamma^5) v_{(1)} \right\} \\
&= \frac{\hbar^2 g_w^2 Q_w^2}{72(p_3 \cdot p_4)^2} \sum_{s_1, s_3, s_4, c_1} \left\{ \overline{v_{(1)}} (1 + \gamma^5) \left[ \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \times ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) + \left. \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \right. \\
&\quad \left. \left. + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] \not{p}_2 \left[ (p_4 \cdot \epsilon_{(3)}) \not{\epsilon}_{(4)} - (p_3 \cdot \epsilon_{(4)}) \not{\epsilon}_{(3)} + (\epsilon_{(3)} \cdot \epsilon_{(4)}) \not{p}_3 \right] (1 - \gamma^5) v_{(1)} \right\} \\
&= \frac{\hbar^2 g_w^2 Q_w^2}{72(p_3 \cdot p_4)^2} \sum_{s_1, s_3, s_4, c_1} \text{Tr} \left\{ v_{(1)} \overline{v_{(1)}} (1 + \gamma^5) \left[ \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \times ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) + \left. \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \right. \\
&\quad \left. \left. + 2(\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] \not{p}_2 \left[ (p_4 \cdot \epsilon_{(3)}) \not{\epsilon}_{(4)} - (p_3 \cdot \epsilon_{(4)}) \not{\epsilon}_{(3)} + (\epsilon_{(3)} \cdot \epsilon_{(4)}) \not{p}_3 \right] (1 - \gamma^5) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\hbar^2 g_w^2 Q_w^2}{36 (p_3 \cdot p_4)^2} \sum_{s_3, s_4} \text{Tr} \left\{ \not{p}_1 \left[ ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) \left( \not{\epsilon}_{(4)}^* - \frac{(p_1 + p_2) \cdot \epsilon_{(4)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) \right) \right. \right. \\
&\quad \left. \left. + ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \left( \frac{(p_1 + p_2) \cdot \epsilon_{(3)}^*}{m_w^2} (\not{p}_1 + \not{p}_2) - \not{\epsilon}_{(3)}^* \right) + 2 (\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] \not{p}_2 \right. \\
&\quad \left. \times \left[ (p_4 \cdot \epsilon_{(3)}) \not{\epsilon}_{(4)} - (p_3 \cdot \epsilon_{(4)}) \not{\epsilon}_{(3)} + (\epsilon_{(3)} \cdot \epsilon_{(4)}) \not{p}_3 \right] (1 - \gamma^5) \right\} \\
&= \frac{\hbar^2 g_w^2 Q_w^2}{36 (p_3 \cdot p_4)^2} \sum_{s_3, s_4} \text{Tr} \left\{ (1 - \gamma^5) \not{p}_1 \left[ ((2p_4 + p_3) \cdot \epsilon_{(3)}^*) \not{\epsilon}_{(4)}^* - ((2p_3 + p_4) \cdot \epsilon_{(4)}^*) \not{\epsilon}_{(3)}^* \right. \right. \\
&\quad \left. \left. + 2 (\epsilon_{(3)}^* \cdot \epsilon_{(4)}^*) \not{p}_3 \right] \not{p}_2 \left[ (p_4 \cdot \epsilon_{(3)}) \not{\epsilon}_{(4)} - (p_3 \cdot \epsilon_{(4)}) \not{\epsilon}_{(3)} + (\epsilon_{(3)} \cdot \epsilon_{(4)}) \not{p}_3 \right] \right\} \\
&\vdots \\
&= \frac{2 \hbar^2 g_w^2 Q_w^2}{9} \left[ \frac{E^4 (E^2 - 5 m_w^2)}{m_w^2 (E^2 - m_w^2)^2} + \sin^2 \theta \right]
\end{aligned}$$



## 5 The Total Matrix Element Squared

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{36} \sum_{S,C} [\mathcal{M}_1 \mathcal{M}_1^* + \mathcal{M}_2 \mathcal{M}_2^* + \mathcal{M}_3 \mathcal{M}_3^* + \mathcal{M}_1 \mathcal{M}_2^* + \mathcal{M}_1 \mathcal{M}_3^* + \mathcal{M}_2 \mathcal{M}_3^* \\
&\quad + (\mathcal{M}_1 \mathcal{M}_2^*)^* + (\mathcal{M}_1 \mathcal{M}_3^*)^* + (\mathcal{M}_2 \mathcal{M}_3^*)^*] \\
&= -\frac{4\hbar^2 g_w^2 Q_d^2}{9} \left[ \left( \frac{m_w^2}{E^2 - m_w^2} \right) \csc^2 \left( \frac{\theta}{2} \right) + 1 \right] \\
&\quad - \frac{4\hbar^2 g_w^2 Q_u^2}{9} \left[ \left( \frac{m_w^2}{E^2 - m_w^2} \right) \sec^2 \left( \frac{\theta}{2} \right) + 1 \right] \\
&\quad + \frac{2\hbar^2 g_w^2 Q_w^2}{9} \left[ \frac{E^4 (E^2 - 5m_w^2)}{m_w^2 (E^2 - m_w^2)^2} + \sin^2 \theta \right] \\
&\quad + \frac{8\hbar^2 g_w^2 Q_d Q_u}{9} \left[ \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \theta + i f_{12} \right] \\
&\quad + \frac{2\hbar^2 g_w^2 Q_d Q_w}{9} \left[ \frac{E^4 - m_w^4}{(E^2 - m_w^2)^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \left( \frac{\theta}{2} \right) - \cos \theta + i f_{13} \right] \\
&\quad - \frac{2\hbar^2 g_w^2 Q_u Q_w}{9} \left[ \frac{2m_w^2}{E^2 - m_w^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \sec^2 \left( \frac{\theta}{2} \right) + \cos \theta + i f_{23} \right] \\
&\quad + \frac{8\hbar^2 g_w^2 Q_d Q_u}{9} \left[ \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \theta + i f_{12} \right]^* \\
&\quad + \frac{2\hbar^2 g_w^2 Q_d Q_w}{9} \left[ \frac{E^4 - m_w^4}{(E^2 - m_w^2)^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \left( \frac{\theta}{2} \right) - \cos \theta + i f_{13} \right]^* \\
&\quad - \frac{2\hbar^2 g_w^2 Q_u Q_w}{9} \left[ \frac{2m_w^2}{E^2 - m_w^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \sec^2 \left( \frac{\theta}{2} \right) + \cos \theta + i f_{23} \right]^* \\
&= \frac{2\hbar^2 g_w^2}{9} \left\{ -2Q_d^2 \left[ \left( \frac{m_w^2}{E^2 - m_w^2} \right) \csc^2 \left( \frac{\theta}{2} \right) + 1 \right] - 2Q_u^2 \left[ \left( \frac{m_w^2}{E^2 - m_w^2} \right) \sec^2 \left( \frac{\theta}{2} \right) + 1 \right] \right. \\
&\quad + Q_w^2 \left[ \frac{E^4 (E^2 - 5m_w^2)}{m_w^2 (E^2 - m_w^2)^2} + \sin^2 \theta \right] + 8Q_d Q_u \left[ \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \theta \right] \\
&\quad + 2Q_d Q_w \left[ \frac{E^4 - m_w^4}{(E^2 - m_w^2)^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \csc^2 \left( \frac{\theta}{2} \right) - \cos \theta \right] \\
&\quad \left. - 2Q_u Q_w \left[ \frac{2m_w^2}{E^2 - m_w^2} + \left( \frac{E^2 (E^2 + m_w^2)}{(E^2 - m_w^2)^2} \right) \sec^2 \left( \frac{\theta}{2} \right) + \cos \theta \right] \right\} \\
&= \frac{\pi^2 \alpha^2 m_z^2}{162 m_w^2 (E^2 - m_w^2)^2 (m_w^2 - m_z^2) \sin^2 \theta} \left[ -36 E^6 + 9 E^4 m_w^2 - 122 E^2 m_w^4 - 27 m_w^6 \right. \\
&\quad + 12 m_w^2 (7 E^4 + 2 E^2 m_w^2 + 7 m_w^4) \cos \theta + 4 (9 E^6 - 40 E^4 m_w^2 + 26 E^2 m_w^4 - 31 m_w^6) \cos 2\theta \\
&\quad \left. + 12 m_w^2 (E^4 - 2 E^2 m_w^2 + m_w^4) \cos 3\theta - 9 m_w^2 (E^4 - 2 E^2 m_w^2 + m_w^4) \cos 4\theta \right].
\end{aligned}$$

In the last step of the previous page we applied the following relations:

$$Q_{\text{w}} = -\frac{e}{\hbar}, \quad Q_{\text{d}} = -\frac{1}{3} \frac{e}{\hbar}, \quad Q_{\text{u}} = \frac{2}{3} \frac{e}{\hbar}.$$

As well as

$$g_{\text{w}} = \frac{e}{\sqrt{8} s_{\text{w}}}, \quad s_{\text{w}} = \sqrt{1 - c_{\text{w}}^2}, \quad c_{\text{w}} = \frac{m_{\text{w}}}{m_{\text{z}}}.$$

Setting  $\epsilon_0 = \hbar = c = 1$  yields the following expression for the elementary charge  $e$ :

$$e = \sqrt{4\pi\alpha},$$

where  $\alpha$  is the (dimensionless) fine structure constant.