1 Problem

Consider a source charge Q located at the origin $P_Q = (0\,\mathrm{m}; 0\,\mathrm{m})$ of a Cartesian coordinate system. A test charge q starts at $P_q = (4\,\mathrm{m}; 4\,\mathrm{m})$. The charge q can move along one of three possible curves (see Figure 1.1):

- C_1: along a straight line from P_q to $P_1 = (2\,\mathrm{m}; 2\,\mathrm{m})$
- C_2: along a straight line from P_q to $P_2 = (6\,\mathrm{m}; 6\,\mathrm{m})$
- C_3: along an the arc from P_q to $P_3 = (0\,\mathrm{m}; \sqrt{8}\,\mathrm{m})$

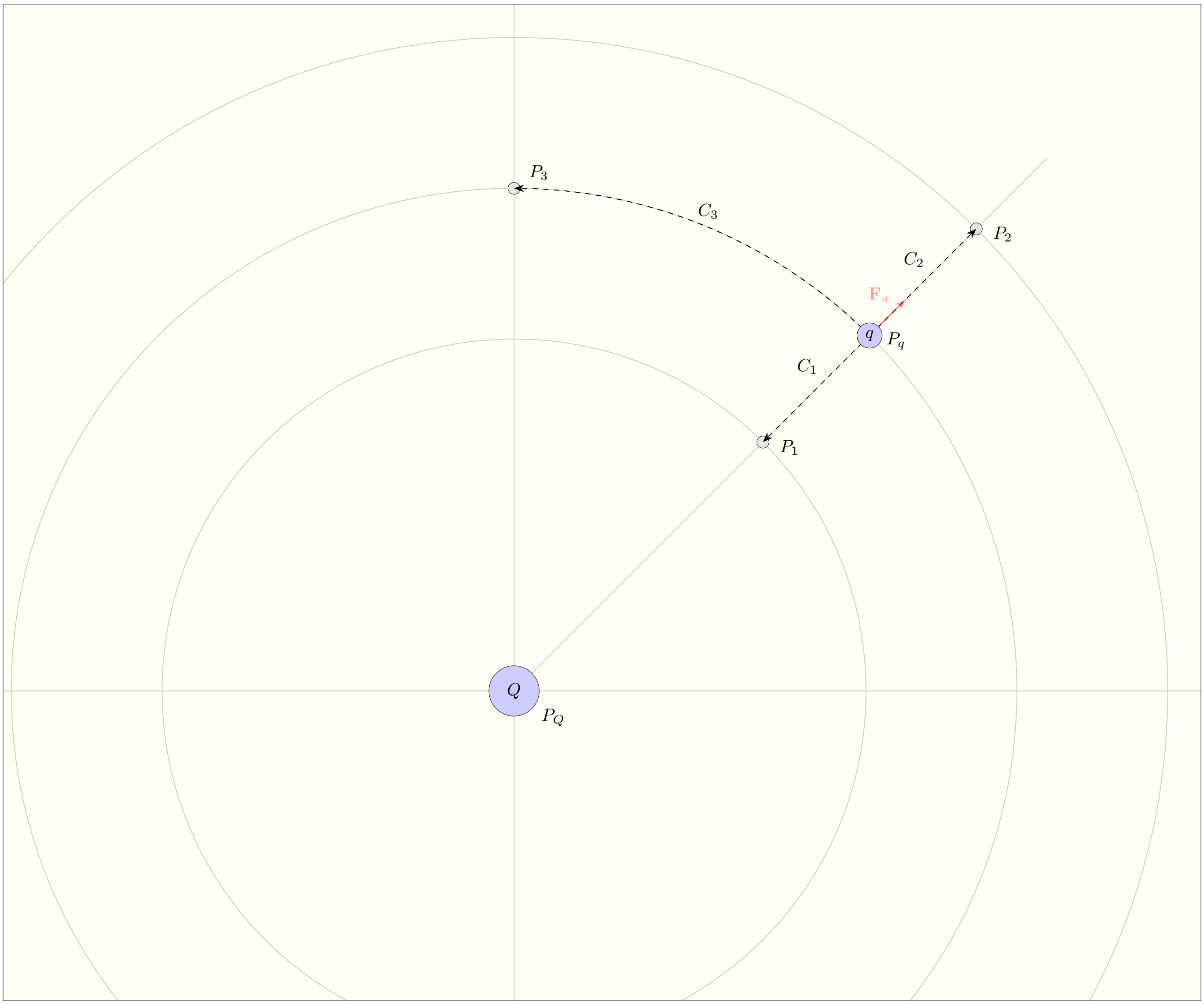

Figure 1.1: The work done in moving a test charge q along three distinct curves.

The task is to determine the minimum work W_{you} (Definition 1.1) you must exert to move q along each of these curves.

$$W_{\text{you}} = -\int_{u=a}^{u=b} \mathbf{F}_{\text{el}}(\mathbf{r}(u)) \cdot \mathrm{d}\mathbf{r}(u) = -q \cdot \int_{u=a}^{u=b} \mathbf{E}(\mathbf{r}(u)) \cdot \mathrm{d}\mathbf{r}(u) \tag{1.1}$$

$$\text{with} \quad \mathbf{r}(u) = \Big\langle x(u), y(u) \Big\rangle \quad \text{and} \quad \mathrm{d}\mathbf{r}(u) = \left\langle \frac{\mathrm{d}x(u)}{\mathrm{d}u} \cdot \mathrm{d}u, \frac{\mathrm{d}y(u)}{\mathrm{d}u} \cdot \mathrm{d}u \right\rangle$$

Note that, to minimize writing effort in the subsequent text x, y, $\mathbf{r}$ and $\mathrm{d}\mathbf{r}$ are used as shorthand for $x(u)$, $y(u)$, $\mathbf{r}(u)$ and $\mathrm{d}\mathbf{r}(u)$, respectively: $x \equiv x(u)$, $y \equiv y(u)$, $\mathbf{r} \equiv \mathbf{r}(u)$ $\mathrm{d}\mathbf{r} \equiv \mathrm{d}\mathbf{r}(u)$.

1.1 Solution - The Required Work To Move A Charge q Along the Curve C_1

To calculate the required work using Definition 1.1 to move a charge q along the curve C_1 from P_q to P_1, as illustrated in Figure 1.1, it is necessary to determine the infinitesimal displacement vector d$\mathbf{r}$ and the electric force $\mathbf{F}_{\text{el}}(\mathbf{r})$. First, the curve C_1 is described by an oriented parametric position vector $\mathbf{r}$, as shown in Equation 1.2. In this equation $u \in \mathbb{R}$ is the parameter that starts at $u = 2$ and decreases to $u = 1$.

$$\mathbf{r} = \Big\langle x, y \Big\rangle = \Big\langle u \cdot 2\,\mathrm{m}, u \cdot 2\,\mathrm{m} \Big\rangle \quad \text{with} \quad 1 \le u \le 2 \tag{1.2}$$

The magnitude of $\mathbf{r}$ is:

$$r = \sqrt{x^2 + y^2} = \sqrt{u^2 \cdot 2^2\,\mathrm{m}^2 + u^2 \cdot 2^2\,\mathrm{m}^2} = \sqrt{2 \cdot u^2 \cdot 2^2\,\mathrm{m}^2} = \sqrt{2} \cdot u \cdot 2\,\mathrm{m} = u \cdot \sqrt{8}\,\mathrm{m} \tag{1.3}$$

The infinitesimal displacement vector d$\mathbf{r}$ determined as follows

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \Big\langle \frac{\mathrm{d}x}{\mathrm{d}u}, \frac{\mathrm{d}y}{\mathrm{d}u} \Big\rangle = \Big\langle 2\,\mathrm{m}, 2\,\mathrm{m} \Big\rangle \tag{1.4}$$

Thus,

$$\mathrm{d}\mathbf{r} = \Big\langle \frac{\mathrm{d}x}{\mathrm{d}u} \cdot \mathrm{d}u, \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \mathrm{d}u \Big\rangle = \Big\langle 2\,\mathrm{m} \cdot \mathrm{d}u, 2\,\mathrm{m} \cdot \mathrm{d}u \Big\rangle = 2\,\mathrm{m} \cdot \mathrm{d}u \cdot \Big\langle 1, 1 \Big\rangle \tag{1.5}$$

Then the electric force $\mathbf{F}_{\text{el}}(\mathbf{r})$ is

$$\mathbf{F}_{\text{el}}(\mathbf{r}) = \frac{k}{r^2} \cdot \frac{\mathbf{r}}{r} = \frac{k}{r^3} \cdot \mathbf{r} = \frac{k}{(u \cdot \sqrt{8}\,\mathrm{m})^3} \cdot \Big\langle u \cdot 2\,\mathrm{m}, u \cdot 2\,\mathrm{m} \Big\rangle \quad \text{with} \quad k \equiv \frac{Q \cdot q}{4 \cdot \pi \cdot \epsilon_0} \tag{1.6}$$

$$= \frac{k \cdot u \cdot 2\,\mathrm{m}}{u^3 \cdot 8^{3/2}\,\mathrm{m}^3} \cdot \Big\langle 1, 1 \Big\rangle = \frac{k \cdot 2}{u^2 \cdot 8 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot \Big\langle 1, 1 \Big\rangle = \frac{k}{u^2 \cdot 4 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot \Big\langle 1, 1 \Big\rangle \tag{1.7}$$

and the work:

$$W_{\text{you}} = -\int_{u=2}^{u=1} \mathbf{F}_{\text{el}}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = -\int_{u=2}^{u=1} \frac{k}{u^2 \cdot 4 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot \Big\langle 1, 1 \Big\rangle \cdot 2\,\mathrm{m} \cdot \mathrm{d}u \cdot \Big\langle 1, 1 \Big\rangle \tag{1.8}$$

$$= -\int_{u=2}^{u=1} \frac{k \cdot 2\,\mathrm{m}}{u^2 \cdot 4 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot \Big\langle 1, 1 \Big\rangle \cdot \Big\langle 1, 1 \Big\rangle \cdot \mathrm{d}u = -\int_{u=2}^{u=1} \frac{k \cdot 2\,\mathrm{m}}{u^2 \cdot 4 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot 2 \cdot \mathrm{d}u \tag{1.9}$$

$$= -\int_{u=2}^{u=1} \frac{k \cdot 4\,\mathrm{m}}{u^2 \cdot 4 \cdot \sqrt{8}\,\mathrm{m}^2} \cdot \mathrm{d}u = -\int_{u=2}^{u=1} \frac{k}{u^2 \cdot \sqrt{8}\,\mathrm{m}} \cdot \mathrm{d}u = -\frac{k}{\sqrt{8}\,\mathrm{m}} \cdot \int_{u=2}^{u=1} \frac{1}{u^2} \cdot \mathrm{d}u \tag{1.10}$$

$$= -\frac{k}{\sqrt{8}\,\mathrm{m}} \cdot \left[-\frac{1}{u}\right]_{u=2}^{u=1} = -\frac{k}{\sqrt{8}\,\mathrm{m}} \cdot \left[-\frac{1}{1} - \left[-\frac{1}{2}\right]\right] = -\frac{k}{\sqrt{8}\,\mathrm{m}} \cdot \left[-\frac{1}{2}\right] = \frac{k}{2 \cdot \sqrt{8}\,\mathrm{m}} = \boxed{\frac{k}{\sqrt{32}\,\mathrm{m}}} \tag{1.11}$$

1.2 Solution - The Required Work to move a Charge q Along the Curve C_2

Similarly, the work required to move q along the curve C_2 from P_q to P_2, as shown in Figure 1.1, also requires calculating the infinitesimal displacement vector d$\mathbf{r}$ and the electric force $\mathbf{F}_{\text{el}}(\mathbf{r})$. Here, the curve C_2 is expressed by the parametric position vector $\mathbf{r}$, given in Equation 1.12, where $u \in \mathbb{R}$ starts at $u = 1$ and increases to $u = 1.5$.

$$\mathbf{r} = \left\langle x, y \right\rangle = \left\langle u \cdot 4\,\mathrm{m}, u \cdot 4\,\mathrm{m} \right\rangle \quad \text{with} \quad 1 \leq u \leq 1.5 \tag{1.12}$$

The magnitude of $\mathbf{r}$ is:

$$r = \sqrt{x^2 + y^2} = \sqrt{u^2 \cdot 4^2\,\mathrm{m}^2 + u^2 \cdot 4^2\,\mathrm{m}^2} = \sqrt{2 \cdot u^2 \cdot 4^2\,\mathrm{m}^2} = u \cdot \sqrt{32}\,\mathrm{m} \tag{1.13}$$

The infinitesimal displacement vector d$\mathbf{r}(u)$ determined as follows

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \left\langle \frac{\mathrm{d}x}{\mathrm{d}u}, \frac{\mathrm{d}y}{\mathrm{d}u} \right\rangle = \left\langle 4\,\mathrm{m}, 4\,\mathrm{m} \right\rangle \tag{1.14}$$

Thus,

$$\mathrm{d}\mathbf{r} = \left\langle \frac{\mathrm{d}x}{\mathrm{d}u} \cdot \mathrm{d}u, \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \mathrm{d}u \right\rangle = \left\langle 4\,\mathrm{m} \cdot \mathrm{d}u, 4\,\mathrm{m} \cdot \mathrm{d}u \right\rangle = 4\,\mathrm{m} \cdot \mathrm{d}u \cdot \left\langle 1, 1 \right\rangle \tag{1.15}$$

Then the electric force $\mathbf{F}_{\text{el}}(\mathbf{r})$ is

$$\mathbf{F}_{\text{el}}(\mathbf{r}) = \frac{k}{r^2} \cdot \frac{\mathbf{r}}{r} = \frac{k}{r^3} \cdot \mathbf{r} = \frac{k}{(u \cdot \sqrt{32}\,\mathrm{m})^3} \cdot \left\langle u \cdot 4\,\mathrm{m}, u \cdot 4\,\mathrm{m} \right\rangle \quad \text{with } k \text{ already defined in Equation 1.6} \tag{1.16}$$

$$= \frac{k \cdot u \cdot 4\,\mathrm{m}}{u^3 \cdot 32^{3/2}\,\mathrm{m}^3} \cdot \left\langle 1, 1 \right\rangle = \frac{k \cdot 4}{u^2 \cdot 32 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot \left\langle 1, 1 \right\rangle = \frac{k}{u^2 \cdot 16 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot \left\langle 1, 1 \right\rangle \tag{1.17}$$

and the work:

$$W_{\text{you}} = -\int_{u=1}^{u=1.5} \mathbf{F}_{\text{el}}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = -\int_{u=1}^{u=1.5} \frac{k}{u^2 \cdot 16 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot \left\langle 1, 1 \right\rangle \cdot 4\,\mathrm{m} \cdot \mathrm{d}u \cdot \left\langle 1, 1 \right\rangle \tag{1.18}$$

$$= -\int_{u=1}^{u=1.5} \frac{k \cdot 4\,\mathrm{m}}{u^2 \cdot 16 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot \left\langle 1, 1 \right\rangle \cdot \left\langle 1, 1 \right\rangle \cdot \mathrm{d}u = -\int_{u=1}^{u=1.5} \frac{k \cdot 1\,\mathrm{m}}{u^2 \cdot 4 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot 2 \cdot \mathrm{d}u \tag{1.19}$$

$$= -\int_{u=1}^{u=1.5} \frac{k \cdot 1\,\mathrm{m}}{u^2 \cdot 2 \cdot \sqrt{32}\,\mathrm{m}^2} \cdot \mathrm{d}u = -\int_{u=1}^{u=1.5} \frac{k}{u^2 \cdot 2 \cdot \sqrt{32}\,\mathrm{m}} \cdot \mathrm{d}u = -\frac{k}{\sqrt{128}\,\mathrm{m}} \cdot \int_{u=1}^{u=1.5} \frac{1}{u^2} \cdot \mathrm{d}u \tag{1.20}$$

$$= -\frac{k}{\sqrt{128}\,\mathrm{m}} \cdot \left[-\frac{1}{u} \right]_{u=1}^{u=1.5} = -\frac{k}{\sqrt{128}\,\mathrm{m}} \cdot \left[-\frac{2}{3} - \left[-\frac{1}{1} \right] \right] = -\frac{k}{\sqrt{128}\,\mathrm{m}} \cdot \frac{1}{3} = \boxed{-\frac{k}{\sqrt{1125}\,\mathrm{m}}} \tag{1.21}$$

1.3 Solution - The Required Work to move a Charge q Along the Curve C_3

in progress