Calculating Logarithms By Hand

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Abstract

This details methods by which we can calculate logarithms by hand.

1 Definition and Basic Properties

A logarithm can be defined as follows: if $b^x = y$, then $x = \log_b y$. In other words, the logarithm of y to base b is the exponent we must raise b to in order to get y as the result. Exponentiation will "undo" a logarithm, and vice versa: $b^{\log_b x} = \log_b (b^x) = x$. The logarithm inherits certain useful properties directly from exponents.

1.1 Property 1: Sum of Logs

We have the property

$$\log_b x + \log_b y = \log_b (x \cdot y)$$

Notice here that both logs are to the same base b. That is an absolute requirement of this property. It is a result of the property $b^x \cdot b^y = b^{x+y}$, as shown in the following:

$$\begin{aligned} b^{\log_b x + \log_b y} &= b^{\log_b x} b^{\log_b y} \\ b^{\log_b x + \log_b y} &= x \cdot y \\ b^{\log_b x + \log_b y} &= b^{\log_b (xy)} \\ \log_b x + \log_b y &= \log_b (xy) \end{aligned}$$

In the last step, we noted that the algebra works for arbitrary base b, which is only possible if the exponents are equal.

1.2 Property 2: Logarithms with Exponents within the Argument

The *argument* in any function is the input, which is properly written in brackets after the function, though brackets are often omitted if there is little or no chance of confusion. (For example, $\log_b y$ is properly written as $\log_b(y)$.) If the argument (y in this case) has an exponent, that exponent can be extracted from the logarithm:

$$\log_b y^a = a \log_b y$$

This property can be shown in two ways. For positive whole number values of a, we can apply property 1 in reverse as follows:

$$\log_b y^2 = \log_b (y \cdot y) = \log_b y + \log_b y = 2 \log_b y$$

(This can be easily extended from the specific $a = 2$ case to all positive whole number exponents through proof by induction, if you know how that works.) However, it is entirely possible that the exponent a is neither positive nor a whole number. For this, we use the property of exponents that $(b^x)^a = b^{ax} = b^{xa}$.

$$b^{\log_b (y^a)} = y^a = (y)^a = \left(b^{\log_b y}\right)^a = b^{a \log_b y}$$

Again, if this is true for arbitrary base b, then the exponents on the far left and far right must be equal, and the property is proven.

1.3 Property 3: The Difference of Logarithms

The third property is as follows:

$$\log_b x - \log_b y = log_b \left(\frac{x}{y}\right)$$

If we combine our first two results with the property of exponents that $x^{-a} = \frac{1}{x^a}$, then we are left with the following:

$$\log_b x - \log_b y = \log_b x + (-1)(\log_b y) = \log_b x + \log_b y^{-1} = \log_b \left(x \cdot y^{-1}\right) = log_b \left(\frac{x}{y}\right)$$

which is exactly what we want.

1.4 Property 4: Changing the Base of a Logarithm

The final property of logarithms is the following:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

This one can be proven using only results and properties we have already used.

$$a^{\frac{\log_b x}{\log_b a}} = \left(b^{\log_b a}\right)^{\frac{\log_b x}{\log_b a}} = b^{\log_b x \frac{\log_b a}{\log_b a}} = b^{\log_b x} = x = a^{\log_a x}$$

Again, we compare exponents of the first and last steps, noting that a is arbitrary, to complete the proof.

2 Method 1: Euler's Method

Most techniques for calculating logarithms by hand require reference to the natural logarithms of base e, and some sort of means to evaluate one particular base (often 10) to keep as a reference. To my mind, if one of the steps in a procedure to teach someone how to calculate logarithms by hand is "memorize the fact that $\ln 10 = 2.302585092994...$" then you aren't really learning how to calculate logarithms by hand. Thankfully, Leonhard Euler[1] developed a means to calculate logarithms using square roots and the properties above.

Imagine we wish to calculate the logarithm of 64 to the base 3. We start by using property 4 of logarithms:

$$\log_3 64 = \frac{\log_{10} 64}{\log_{10} 3}$$

This turns the logarithms into logarithms of base 10. Euler's method makes logarithms to base 10 remarkably easy to calculate. We then simply need to take the ratio of the two numbers.

Let's start by calculating $\log_{10} 64$. The general idea is to pin the logarithm down between two logarithms that are easy to calculate, and systematically narrow the range of the endpoints to make things easier for us. We have two approaches that we can try here. One is to leave the 64 as is, and pin it down between the logarithms of 10 and 100. This will lead to some nasty square roots down the road, so we'll use the first property of logarithms to break it down instead:

$$\log_{10} 64 = \log_{10} (10 \cdot 6.4) = \log_{10} 10 + \log_{10} 6.4 = 1 + \log_{10} 6.4$$

Now we only need to compute the logarithm of 6.4, which makes for easier square roots to deal with. We know that $1 < 6.4 < 10$. This means that $\log_{10} 1 < \log_{10} 6.4 < \log_{10} 10$. The next step is the one that illustrates Euler's mathematical brilliance:[2]

$$\log\left(\sqrt{xy}\right) = \log\sqrt{x} + \log\sqrt{y} = \log x^{\frac{1}{2}} + \log y^{\frac{1}{2}} = \frac{\log x + \log y}{2}$$

Now, we get down to the nitty gritty: calculating $\log_{10} 6.4$. We know it's between $\log_{10} 1 = 0$ and $\log_{10} 10 = 1$. Euler's observation tells us that $\log_{10}\sqrt{1 \cdot 10}$ is also in this range. Well, we know[3] that $\sqrt{10} \approx 3.16228$. Additionally,

$$\log_{10}\sqrt{10} = \frac{\log_{10} 1 + \log_{10} 10}{2} = \frac{0+1}{2} = \frac{1}{2} = 0.5$$

[1] "Euler" is pronounced "oiler," which is why virtually every sports team assembled by the University of Alberta's math department has been named "the Edmonton Eulers."

[2] I'm not exaggerating. I've taken no less than 13 University level math courses, and Euler's name showed up in at least 10 of them. The man's contributions to mathematics are probably more significant than Einstein's contributions to physics. Find him on Wikipedia when you have a minute or thirty.

[3] Or can calculate using the method detailed in `http://www.bureau42.com/view/7396`.

Well, $\sqrt{10} < 6.4 < 10$, so $0.5 < \log_{10} 6.4 < 1$. We've effectively cut the range of values that we need to search by half: 6.4 does not fall between 1 and $\sqrt{10}$, so that interval can now be ignored. We can do so again, by working out the logarithm and value of $\sqrt{3.166228 \cdot 10}$.

$$\begin{aligned} \sqrt{31.66228} &\approx 5.623413 \\ \log_{10} \sqrt{3.166228 \cdot 10} &= \frac{0.5+1}{2} = 0.75 \end{aligned}$$

Well, $5.623413 < 6.4 < 10$, so we can narrow it down further:

$$\begin{aligned} \sqrt{56.23413} &\approx 7.49894 \\ \log_{10} \sqrt{5.623413 \cdot 10} &= \frac{0.75+1}{2} = 0.875 \end{aligned}$$

We now work with $5.623413 < 6.4 < 7.49894$, and so find that:

$$\begin{aligned} \sqrt{5.623413 \times 7.49894} &\approx 6.4938 \\ \log_{10} \sqrt{5.623413 \cdot 7.49894} &= \frac{0.75+0.875}{2} = 0.65625 \end{aligned}$$

The process would continue with $5.623413 < 6.4 < 6.4938$ and so forth, narrowing the region with each step. We ultimately find that $\log_{10} 6.4 \approx 0.80618$ and $\log_{10} 3 \approx 0.47712$, so that

$$\log_3 64 = \frac{\log_{10} 64}{\log_{10} 3} = \frac{1.80618}{0.47712} \approx 3.78558$$

It takes a *lot* of iterations to really start to pin the value of our logarithm down, but we can certainly get there. Of course, we can also do this to any base: we could have, instead, used the base 3 to begin with, and calculated the logarithm of 64 by saying it's between $3^3 = 27$ and $3^4 = 81$. Base 3 logs are not always the most convenient logs to work with, but if you are more familiar with them, it could be more comfortable to work with. While certainly advantageous for something like $\log_2 3$, given that $2^1 < 3 < 2^2$, it could be hard to work in the function's original logarithm if you need to calculate something like $\log_{17} 2489$.

3 Method 2: Taylor Expansions

There are some methods which can arrive at an answer in fewer iterations, but they require some special circumstances. For example, if we work with the natural logarithm $\ln x = \log_e x$,[4] then we can exploit the Taylor expansion[5][6]

$$\ln{(x)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

This has one major caveat, however: this Taylor expansion only converges[7] if $0 < x \leq 2$. What if we want to calculate $\log_3 64$ this way?

This is where logarithm property 2 comes into play: if $x > 2$, then $0 < \frac{1}{x} = x^{-1} \leq 2$. We can combine properties 2 and 4 of logarithms to arrive at the following means of calculating the logarithm in question:

$$\log_3 64 = \frac{\ln 64}{\ln 3} = \frac{-\ln \frac{1}{64}}{-\ln \frac{1}{3}} = \frac{\ln \frac{1}{64}}{\ln \frac{1}{3}}$$

Note that, not only have we transformed our arguments into numbers between 0 and 2, but we've canceled the negative signs because both numbers in question were reciprocated. We can now expand the Taylor series above to as many digits as needed to achieve the required accuracy. Unfortunately, the closer x gets to 0 or 2, the longer it takes the series to converge to something useful. We find that $\ln \frac{1}{3}$ rounds accurately to 4 decimal places (-1.0986) after 20 iterations, while $\ln \frac{1}{64}$ rounds accurately to 4 decimal places (-4.1589) after a whopping 377 iterations! If, however, we look for $\ln 1$, the series is exactly accurate in a single term. $\ln 1.1$ is accurate to 4 decimal places after a mere 3 iterations.

So, the methods for calculating logarithms by hand are out there, and they work. The actual algebra involved is not terribly complicated. However, there's going to be a lot of it. If you need the logarithm of a number close to 1, use method 2. For others, method 1 is more likely to reach a satisfactory conclusion more quickly.

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[4] If you aren't familiar with e, know that it appears naturally in calculus. If you aren't familiar with calculus, don't worry. You'll have to take parts of the explanation on faith, but you'll still be able to use the method.

[5] If you aren't familiar with Taylor expansions either, you only need to know this: a Taylor expansion is an infinitely long polynomial that approximates another function. Essentially, if we take the first few terms of this polynomial, we'll be accurate to the first few decimal places. If you need more accuracy, take more terms.

[6] If you are not familiar with the series notation used here, you can find it explained in `http://www.bureau42.com/view/7363`.

[7] A series that converges is a series that adds up to a finite, and therefore useful, number.