253: Notes on Open interval Max Min Problems

A continuous function on an open interval may or may not have a maximum and a minimum; it may have one but not the other, it may have both, or it may have neither. The one thing which is sure is that any max or min it does have must occur at a critical point. But we need more tests to guarantee that we have a max or a min. The following theorems sum up what we will use.

Theorem: Let f be continuous on (a,b). Then:

(0) On any interval not containing a critical point, f is either strictly increasing or strictly decreasing; in particular if there are no critical points in (a,b) then f is either everywhere increasing or everywhere decreasing on (a,b), hence has neither a global max nor a global min.

(1) If $\lim_{x\to a^+} f(x) = +\infty = \lim_{x\to b^-} f(x)$, then f has a global minimum on (a,b), necessarily occurring at one of the critical points.
If $\lim_{x\to a^+} f(x) = -\infty = \lim_{x\to b^-} f(x)$, then f has a global maximum on (a,b), necessarily occurring at one of the critical points. [We shall call this the "limit test", for extrema on open intervals.]

Assume further $c_1 < \ldots < c_n$ are the only critical points of f on (a,b). Then:

(2) If there is a point s with $a < s < c_1$, such that $f'(s) < 0$, and a point t with $c_n < t < b$, such that $f'(s) > 0$, then f has a global minimum on (a,b), necessarily occurring at one of the critical points.
If there is a point s with $a < s < c_1$, such that $f'(s) > 0$, and a point t with $c_n < t < b$, such that $f'(s) < 0$, then f has a global maximum on (a,b), necessarily occurring at one of the critical points. [We shall call this the "first derivative test", for global extrema on open intervals.]

(3) If there is a point s with $a < s < c_1$, such that $f(s) > f(c_1)$, and a point t with $c_n < t < b$, such that $f(t) > f(c_n)$, then f has a global minimum on (a,b), necessarily occurring at one of the critical points.
If there is a point s with $a < s < c_1$, such that $f(s) < f(c_1)$, and a point t with $c_n < t < b$, such that $f(t) < f(c_n)$, then f has a global maximum on (a,b), necessarily occurring at one of the critical points. [We shall call this the "zeroth derivative test", for global extrema on open intervals.]

Words to the wise: The limit test is often the easiest and fastest test to use, once you get good at taking limits, because you do not have to plug any numbers into any formulas nor evaluate anything. The next easiest is the first derivative test, because you only have two numbers to plug in, one at each end of your interval, and you only need to know if the result is positive or negative. The difficulty here may be insuring that the two numbers you plug in are actually to the left of your first critical point, and to the right of your last critical point respectively. The zeroth derivative test is hardest to use because you must plug four appropriately chosen points (three, if $c_1 = c_n$) into your original function, evaluate it at all of them, and then compare the sizes of some rather difficult looking pairs of numbers in practice.

A combination of the limit test and the first derivative test may be easier than either of those to use, in some cases.

Thm: Let f be continuous on (a,b).
If $\lim_{x\to a^+} f'(x) = -\infty$, and $\lim_{x\to b^-} f'(x) = +\infty$, then f has a global minimum on (a,b), necessarily occurring at one of the critical points.
If $\lim_{x\to a^+} f'(x) = +\infty$, and $\lim_{x\to b^-} f'(x) = -\infty$, then f has a global maximum on (a,b), necessarily occurring at one of the critical points.
[We shall call this the "modified first derivative test", for extrema on open intervals.]

More good advice: It is impossible to understand these criteria just by reading them, and impossible to memorize them without understanding them. Probably the best way to get a grip on them is to draw pictures of what is going on with the graphs in each case, and then learn to picture them in your mind's eye.