1. The problem statement, all variables and given/known data

Derive the Representation Formula for Harmonic Functions (i.e. $\nabla^2 u = 0$) in 2-D, which is:

$$u(\vec{x_0}) = \frac{1}{2\pi}\int_{\partial D}[u(\vec{x})\frac{\partial}{\partial n}(ln|\vec{x}-\vec{x_0}| - ln|\vec{x}-\vec{x_0}|\frac{\partial u}{\partial n}]dS$$

2. Relevant equations

We have been directed to make use of Green's Theorem/Identity for the 2-D case:

$$\int_D\int(u\nabla^2 v - v\nabla^2 u)dD = \int_{\partial D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}dS$$

3. The attempt at a solution

Firstly, it should be noted that I have been having difficulties following this professor through examples in concepts in class. He has walked us through the 3-D example, and I am attempting to apply roughly the same method to the 2-D case. We are making use of Green's theorem as the second part of the question is asking us to derive the green's functions, but for now I will be satisfied if I can understand this problem. Here is what I have attempted, as well as where I am getting stuck:

Suppose that u satisfies $\nabla^2 u = 0$. Then, we show that u has the form detailed above. By Green's Theorem it suffices for us to show that the function

$$v = \frac{ln|\vec{x}-\vec{x_0}|}{2\pi}$$

.

To show this, we firstly separate our domain into two pieces, as we anticipate a singularity as $\vec{x}$ approaches $\vec{x_0}$. Let $D = D_1 + D_\epsilon$. Then, from Green's Theorem, we have that our surface integral is:

$$\int_{\partial D}[u\frac{\partial}{\partial n}ln(r) - ln(r)\frac{\partial u}{\partial n}]dS + \int_{\partial D_\epsilon}[u\frac{\partial}{\partial r}ln(r) - ln(r)\frac{\partial u}{\partial r}]dS = 0$$

Given that $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ and $r = |\vec{x}-\vec{x_0}|$. Then, on the disk D_ϵ we have that

$$\frac{\partial}{\partial r}ln(r) = \frac{1}{r} = \frac{1}{\epsilon}$$

Now, define the following: $\overline{u} = \frac{1}{2\pi\epsilon}\int_{D_\epsilon} udS$ and $\overline{\frac{\partial u}{\partial r}} = \frac{1}{2\pi\epsilon}\int_{D_\epsilon}\frac{\partial u}{\partial r}dS$

Then, plugging these into the equation above gives:

$$\int_{\partial D_\epsilon} u\frac{\partial}{\partial r}ln(r)dS - \int_{\partial D_\epsilon} ln(r)\frac{\partial u}{\partial r}dS = 0$$

And if we then pull out the values of $ln(r)$ and $\frac{\partial}{\partial r}ln(r)$ we have:

$$= \frac{1}{\epsilon}\int_{\partial D_\epsilon} udS - ln(\epsilon)\int_{\partial D_\epsilon}\frac{\partial u}{\partial r}dS = 0$$

Lastly, we can equate this to our values of u bar and partial u bar as follows:$= 2\pi\overline{u} - 2\pi ******$

This is where I am getting stuck. In the 3-D example we walked through in class, we were able to write the second integral as being some number times 1/r times the integral, and we could then set that equal to the partial u bar value. This allowed us to discern what the value of v was for the 3-D case. Here, we end up getting that

$$ln(\epsilon)$$

is equal to something multiplying the partial u bar term. However, as epsilon is going to zero,

$$ln(\epsilon)$$

is going to infinity, which is not what I was expecting. So either this is correct, and we utilize that fact, or I've done something entirely wrong here. I can't quite figure out which one it is.

Any help would be immensely appreciated, as I haven't quite been able to get much help from my notes or the text book.