Definition
UMP of the free monoid $M(1)$, where $1 = \{\star\}$, (Awodey p.19)
There is a set-function $i: 1 = \{\star\} \to |M(1)|$,
such that for every monoid L and every function $f: 1 \to |L|$, there is a unique $\overline{f}: M(1) \to L$ such that $|\overline{f}| \circ i = f$, where $|\overline{f}|: |M(1)| \to |L|$ (see diagram 1 below and the diagram on p.19)

Problem
Let M and N be monoids and $h: M \to N$ a monoid-homomorphism.
To prove:

"$h: M \to N$ is monic $\Leftrightarrow$ $|h|: |M| \to |N|$ is monic"

$\Rightarrow$) Suppose the monoid-homomorphism $h: M \to N$ is monic and let $x, y \in M$ such that $x \neq y$
We have to prove that $|h|(x) \neq |h|(y)$, then we are ready

We have $x, y \in M$ such that $x \neq y$
This is equivalent with taking two different set-functions $x, y: 1 = \{\star\} \to |M|$
I guess $x(\star) = x$ and $y(\star) = y$ (sorry for the double notation)

Now use the definition of the UMP of the free monoid $M(1)$ [[[not the same M]]] applied to $x: 1 \to |M|$:
[[[in the definition, replace L with M, and $f: 1 \to |L|$ with $x: 1 \to |M|$]]]
There is a unique $\overline{x}: M(1) \to M$ such that $|\overline{x}| \circ i = x$, where $|\overline{x}|: |M(1)| \to |M|$ (See diagram 2)

Similar applied to $y: 1 \to |M|$:
[[[in the definition, replace L with M, and $f: 1 \to |L|$ with $y: 1 \to |M|$]]]
There is a unique $\overline{y}: M(1) \to M$ such that $|\overline{y}| \circ i = y$, where $|\overline{y}|: |M(1)| \to |M|$

Because $x \neq y$, we must have $\overline{x} \neq \overline{y}$
Because $h: M \to N$ is monic, we must have $h \circ \overline{x} \neq h \circ \overline{y}$ (recall the definition of mono-morphisms)

Now use the definition of the UMP of the free monoid $M(1)$ [[[not the same M]]] applied to $|h| \circ x: 1 \to |M| \to |N|$:
[[[in the definition, replace L with N, and $f: 1 \to |L|$ with $|h| \circ x: 1 \to |N|$]]]
There is a unique $\overline{|h| \circ x}: M(1) \to N$ such that $\left|\overline{|h| \circ x}\right| \circ i = |h| \circ x$, where $\left|\overline{|h| \circ x}\right|: |M(1)| \to |N|$ (See diagram 3)

Similar, applied to $|h| \circ y: 1 \to |M| \to |N|$:
[[[in the definition, replace L with N, and $f: 1 \to |L|$ with $|h| \circ y: 1 \to |N|$]]]
There is a unique $\overline{|h| \circ y}: M(1) \to N$ such that $\left|\overline{|h| \circ y}\right| \circ i = |h| \circ x$, where $\left|\overline{|h| \circ y}\right|: |M(1)| \to |N|$ (Make a diagram)

Now we have $|h| \circ |\overline{x}| \circ i = |h| \circ x$
And we have $\left|\overline{|h| \circ x}\right| \circ i = |h| \circ x$
By the uniqueness of $\overline{|h| \circ x}: M(1) \to N$ it follows that $\overline{|h| \circ x} = |h| \circ |\overline{x}|$
And $|h| \circ |\overline{x}| = |h \circ \overline{x}|$ [[[why ???]]]
Similar $\overline{|h| \circ y} = |h| \circ |\overline{y}| = |h \circ \overline{y}|$

Because $h \circ \overline{x} \neq h \circ \overline{y}$,
We have $|h \circ \overline{x}| \neq |h \circ \overline{y}|$
Therefore $\overline{|h| \circ x} \neq \overline{|h| \circ y}$
Thus $|h| \circ x \neq |h| \circ y$ [[[because, if $|h| \circ x = |h| \circ y$, then $\overline{|h| \circ x} = \overline{|h| \circ y}$]]] *)

We have elements $x, y \in M$, thus $hx, hy \in N$
Now, with the same double notation as above, this is equivalent with taking two different set-functions $hx, hy: 1 = \{\star\} \to |N|$
I guess $hx(\star) = hx$ and $hy(\star) = hy$

We have $hx = |h| \circ x: 1 \to |N|$ [[[why ???]]]
We have $hy = |h| \circ y: 1 \to |N|$ [[[why ???]]]
By *) above, we have $hx \neq hy$ and we may conclude that $|h|: |M| \to |N|$ is monic

Diagrams

$$\begin{tikzcd}
M(1) \arrow[r, "f^-"] & L \\
{|M(1)|} \arrow[r, "|f^-|"] & {|L|} \\
1 \arrow[u, "i"] \arrow[ur, "f"'] \arrow[ur, phantom, "1", near start, shift left=3ex] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M(1) \arrow[r, "x^-"] & M \\
{|M(1)|} \arrow[r, "|x^-|"] & {|M|} \\
1 \arrow[u, "i"] \arrow[ur, "x"'] \arrow[ur, phantom, "2", near start, shift left=3ex] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M(1) \arrow[r, "(|h|\circ x)^-"] & L \\
{|M(1)|} \arrow[r, "|(|h|\circ x)^-|"] & {|L|} \\
1 \arrow[u, "i"] \arrow[ur, "|h|\circ x"'] \arrow[ur, phantom, "3", near start, shift left=3ex] &
\end{tikzcd}$$

Sorry for slightly strange notation : f^- in stead of $\overline{f}$, I can't help it.