Definition

In a category $\mathcal{C}$, an arrow $f: A \to B$ is called a monomorphism, if, given arrows $g, h: C \to A$, such that $f \circ g = f \circ h$, then $g = h$

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Rule 1

Let U:$\mathrm{Mon} \to \mathrm{Set}$ be the forgetful functor
Let $f: A \to B$ and $g: B \to C$ be arrows in Mon
Then $U(g \circ f) = U(g) \circ U(f)$, by the properties of a functor
In the following, U is denoted by $|\cdot|$
Thus $|g \circ f| = |g| \circ |f|$

Rule 2

Let U:$\mathrm{Mon} \to \mathrm{Set}$ be the forgetful functor
Let $f: A \to B$ and $g: A \to B$ be arrows in Mon and $f = g$
Then $U(f) = U(g)$, thus $|f| = |g|$

Definition

The UMP of the free monoid $M(1)$, where $1 = \{\star\}$, (Awodey p.19)
$|\cdot|$:$\mathrm{Mon} \to \mathrm{Set}$ is the forgetful functor
There is a set-function $i: 1 = \{\star\} \to |M(1)|$,
such that for every monoid L and every function $f: 1 \to |L|$, there is a unique $\overline{f}: M(1) \to L$ such that $|\overline{f}| \circ i = f$, where $|\overline{f}|: |M(1)| \to |L|$ (see diagram 1 below and the diagram on p.19)

Rule 3

If $f = g: 1 \to |L|$, then $\overline{f} = \overline{g}: M(1) \to L$, by the uniqueness of $\overline{f}: M(1) \to L$
Furthermore $|\overline{f}| = |\overline{g}|: |M(1)| \to |L|$, by Rule 2

Problem

Let M and N be monoids and $h: M \to N$ a monoid-homomorphism.
To prove:

"$h: M \to N$ is monic $\Leftrightarrow$ $|h|: |M| \to |N|$ is monic"

$\Rightarrow$) Suppose the monoid-homomorphism $h: M \to N$ is monic and let $x, y \in M$ such that $x \neq y$

This is equivalent with taking two different set-functions $x, y: 1 = \{\star\} \to |M|$
I guess $x(\star) = x \in |M|$ and $y(\star) = y \in |M|$ (sorry for the double notation)
We have to prove that $|h| \circ x \neq |h| \circ y$, then $|h|$ is a monomorphism (see the definition) and we are ready.

Now use the definition of the UMP of the free monoid $M(1)$ [[[not the same M]]] applied to $x: 1 \to |M|$:

[[[in the definition, replace L with M, and $f: 1 \to |L|$ with $x: 1 \to |M|$]]]
There is a unique $\overline{x}: M(1) \to M$ such that $|\overline{x}| \circ i = x$, where $|\overline{x}|: |M(1)| \to |M|$
(See diagram 2)

Similar, applied to $y: 1 \to |M|$:
[[[in the definition, replace L with M, and $f: 1 \to |L|$ with $y: 1 \to |M|$]]]
There is a unique $\overline{y}: M(1) \to M$ such that $|\overline{y}| \circ i = y$, where $|\overline{y}|: |M(1)| \to |M|$

Because $x \neq y$, we must have $\overline{x} \neq \overline{y}$
[[[because if $\overline{x} = \overline{y}$, then, by Rule 2, $|\overline{x}| = |\overline{y}|$ and $|\overline{x}|\big(i(\star)\big) = |\overline{y}|\big(i(\star)\big)$, thus $|\overline{x}| \circ i = |\overline{y}| \circ i$ and $x = y$]]]
Since $h: M \to N$ is monic, we must have $h \circ \overline{x} \neq h \circ \overline{y}$ *)
(recall the definition of monomorphisms)

Again use the definition of the UMP of the free monoid $M(1)$ applied to $|h| \circ x: 1 \to |M| \to |N|$:
[[[in the definition, replace L with N, and $f: 1 \to |L|$ with $|h| \circ x: 1 \to |N|$]]]
There is a unique $\overline{|h| \circ x}: M(1) \to N$ such that $\left|\overline{|h| \circ x}\right| \circ i = |h| \circ x$, where $\left|\overline{|h| \circ x}\right|: |M(1)| \to |N|$ (See diagram 3)

Similar, applied to $|h| \circ y: 1 \to |M| \to |N|$:
[[[in the definition, replace L with N, and $f: 1 \to |L|$ with $|h| \circ y: 1 \to |N|$]]]
There is a unique $\overline{|h| \circ y}: M(1) \to N$ such that $\left|\overline{|h| \circ y}\right| \circ i = |h| \circ y$, where $\left|\overline{|h| \circ y}\right|: |M(1)| \to |N|$

By Rule 1, $|h \circ \overline{x}| = |h| \circ |\overline{x}|$
Now we have $|h \circ \overline{x}| \circ i = |h| \circ |\overline{x}| \circ i = |h| \circ x$
and we have $\left|\overline{|h| \circ x}\right| \circ i = |h| \circ x$
By the uniqueness of $\overline{|h| \circ x}: M(1) \to N$ it follows that $\overline{|h| \circ x} = h \circ \overline{x}$
Similar $\overline{|h| \circ y} = h \circ \overline{y}$

By *), we have $h \circ \overline{x} \neq h \circ \overline{y}$,
therefore $\overline{|h| \circ x} \neq \overline{|h| \circ y}$
thus $|h| \circ x \neq |h| \circ y$ **)
[[[because, if $|h| \circ x = |h| \circ y$, then $\overline{|h| \circ x} = \overline{|h| \circ y}$, by the uniqueness of $\overline{|h| \circ x}$ (Rule 3)]]]
From this we may conclude that $|h|: |M| \to |N|$ is monic
By prop.2.2. (p.30), we can conclude that $|h|: |M| \to |N|$ is injective.

Some remarks:
Recall $|h| \circ x: 1 = \{\star\} \to |N|$ and $|h| \circ y: 1 = \{\star\} \to |N|$
By **) we have $|h| \circ x \neq |h| \circ y$,
thus $|h|\big(x(\star)\big) \neq |h|\big(y(\star)\big)$
thus $|h|(x) \neq |h|(y)$, where $x, y \in |M|$

And we may conclude that $|h|: |M| \to |N|$ is injective

Diagrams (sorry for slightly strange notation : f^{-} in stead of $\overline{f}$, I can't help it.)

$$\begin{tikzcd} M(1) \arrow[r, "f^{-}"] & L \\ |M(1)| \arrow[r, "|f^{-}|"] & |L| \\ 1 \arrow[u, "i"] \arrow[ur, "f"'] \arrow[u, phantom, shift right=4ex, "1"] & \end{tikzcd} \qquad \begin{tikzcd} M(1) \arrow[r, "x^{-}"] & M \\ |M(1)| \arrow[r, "|x^{-}|"] & |M| \\ 1 \arrow[u, "i"] \arrow[ur, "x"'] \arrow[u, phantom, shift right=4ex, "2"] & \end{tikzcd} \qquad \begin{tikzcd} M(1) \arrow[r, "(|h|\circ x)^{-}"] & N \\ |M(1)| \arrow[r, "|(|h|\circ x)^{-}|"] & |N| \\ 1 \arrow[u, "i"] \arrow[ur, "|h|\circ x"'] \arrow[u, phantom, shift right=4ex, "3"] & \end{tikzcd}$$

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Alternative proof:
Let M and N be monoids and let $f: M \to N$ be a monomorphism in Mon
We want to prove that f is a injective function
Suppose $f(a) = f(b)$ for $a, b \in M$
Consider the monoid-homomorphisms $\varphi_1, \varphi_2: \mathbb{N} \to M$ such that $\varphi_1(1) = a$ and $\varphi_2(1) = b$
($\mathbb{N}$ is a free monoid on a one-element set 1, see p.20 of Awodey)
It follows that $f \circ \varphi_1 = f \circ \varphi_2$. Hence $\varphi_1 = \varphi_2$, because f is a monomorphism, thus $\varphi_1(1) = \varphi_2(1)$, hence $a = b$, and f is injective ■

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