

Differential aging from acceleration, an explicit formula

E. Minguzzi

*Departamento de Matemáticas, Universidad de Salamanca,
Plaza de la Merced 1-4, E-37008 Salamanca, Spain
and INFN, Piazza dei Caprettari 70, I-00186 Roma, Italy**

We consider a clock “paradox” framework where an observer leaves an inertial frame, is accelerated and after an arbitrary trip comes back. We discuss a simple equation that gives, in the 1+1 dimensional case, an explicit relation between the time elapsed on the inertial frame and the acceleration measured by the accelerating observer during the trip.

A non-closed trip with respect to an inertial frame appears closed with respect to another suitable inertial frame. Using this observation we define the differential aging as a function of proper time and show that it is non-decreasing. The reconstruction problem of special relativity is also discussed showing that its, at least numerical, solution would allow the construction of an *inertial clock*.

I. INTRODUCTION

The differential aging implied by special relativity is surely one of the most astonishing results of modern physics (for an historical introduction see¹, for a bibliography with old papers see²). It has been largely debated, and in particular the relationship between the role of acceleration and the difference in proper times of inertial and accelerated observers has been discussed³. The old question as to whether acceleration could be considered responsible for differential aging receives a simple answer by noticing that proper and inertial time are related in the time dilation effect; since relative velocity enters there so does acceleration changing the velocity. The acceleration, however, is not the ultimate source of differential aging as the twin paradox in non-trivial space-time topologies can be reformulated without any need of accelerated observers⁴.

Here we give a simple equation that relates acceleration and differential aging in the case the accelerated observer undergoes an unidirectional, but otherwise arbitrary, motion. We shall prove that relation in the next section. Here we want to discuss and apply it to some cases previously investigated with more elementary methods.

Choose units such that $c = 1$. Let K be the inertial frame and choose coordinates in a way such that two of them can be suppressed. Let O be an accelerated observer with timelike worldline $x^\mu : [0, \bar{\tau}] \rightarrow M$ and $x^1(0) = x^1(\bar{\tau}) = 0$, where τ is the proper time parametrization and x^μ , $\mu = 0, 1$ are the coordinates of the inertial frame. Let, moreover, $a(\tau)$ be the acceleration of O with respect to the local inertial frame at $x(\tau)$. To be more precise, the quantity $-a$ is the apparent acceleration measured by O and so it has a positive or negative sign depending on the direction. Let $T = x^0(\bar{\tau}) - x^0(0)$ be the (positive) inertial time interval between the departure and arrival of O , we have¹¹

Theorem I.1. *The time dilation T is related to the acceleration $a(\tau)$ by (time dilation-acceleration equation)*

$$T^2 = \left[\int_0^{\bar{\tau}} e^{\int_0^\tau a(\tau') d\tau'} d\tau \right] \left[\int_0^{\bar{\tau}} e^{-\int_0^\tau a(\tau') d\tau'} d\tau \right]. \quad (1)$$

We have also

Theorem I.2. *The accelerated observer departs from K with zero velocity if and only if $\int_0^{\bar{\tau}} e^{\int_0^\tau a(\tau') d\tau'} d\tau = \int_0^{\bar{\tau}} e^{-\int_0^\tau a(\tau') d\tau'} d\tau$ and in this case*

$$T = \int_0^{\bar{\tau}} e^{\pm \int_0^\tau a(\tau') d\tau'} d\tau, \quad (2)$$

if moreover the final velocity of O with respect to K vanishes then $\int_0^{\bar{\tau}} a(\tau) d\tau = 0$.

Some comments are in order. In no place we need to specify the initial or final velocity of O with respect to K . Using the Cauchy-Schwarz inequality $(\int f g d\tau)^2 \leq (\int f^2 d\tau)(\int g^2 d\tau)$, with $f = g^{-1} = \exp(\int_0^\tau a(\tau') d\tau'/2)$ we find the expected relation $T \geq \bar{\tau}$ where the equality holds only if $f = kg$, with $k \in \mathbb{R}$, that is if and only if $a(\tau) = 0$. Thus $T > \bar{\tau}$ or the worldline of O coincides with that of the origin of K . This proves the differential aging effect. In section II A we shall give another proof that does not use the Cauchy-Schwarz inequality.

Often⁵ the differential aging effect is proved in curved (and hence even in flat) spacetimes by noticing that the connecting geodesic, that is the trajectory of equation $x^1(\tau) = 0$ in our case, locally maximizes the proper time functional $I[\gamma] = \int_\gamma d\tau$. Theorem 1 implies the global maximization property in 1+1 Minkowski spacetime and has the advantage of giving an explicit formula for the inertial round-trip dilation.

A. The simplest example

The simplest example is that of uniform motion in two intervals $[0, \bar{\tau}/2]$ and $[\bar{\tau}/2, \bar{\tau}]$. In the first interval O moves with respect to K at velocity $v = dx^1/dx^0$, in the second interval at velocity $-v$. Although this is a quite elementary example it is interesting to look at the time dilation-acceleration equation and see how it predicts the same result. The first problem is that Eq. (1) holds for integrable acceleration functions. In this example, instead, the acceleration has a singularity at $\bar{\tau}/2$

(the initial and final singularities are not present if the motion of O is not forced to coincide with that of K 's origin for τ outside the interval). The reader can easily check (or see next section), that if $\theta(\tau) = \tanh^{-1} v(\tau)$ is the rapidity then $\frac{d\theta}{d\tau} = a$ (this follows from the additivity of the rapidity under boosts and the fact that a small increment in rapidity coincides with a small increment in velocity with respect to the local inertial frame) and so

$$\Delta\theta = \int a d\tau.$$

If the acceleration causes, in an arbitrary small interval centered at $\bar{\tau}$, a variation $\Delta\theta$ in rapidity then we must write $a = \Delta\theta\delta(\tau - \bar{\tau})$ and generalize the time dilation-acceleration equation with this interpretation. In presence of such singularities, however, it is no longer true that T does not depend on the initial and final velocities of K . Indeed, we need to use this information to find the coefficient $\Delta\theta$. In the case at hand we have

$$\Delta\theta = \tanh^{-1}(-v) - \tanh^{-1} v = -2 \tanh^{-1} v.$$

Inserting $a = -2 \tanh^{-1} v \delta(\tau - \bar{\tau})$ in Eq. (1) we find, after some work with hyperbolic functions that, $T = \bar{\tau}/\sqrt{1-v^2}$ as expected. The reader should not be surprised by the fact that this simple case needs so much work, as this is a rather pathological case. No real observer would survive an infinite acceleration. The advantage of the time dilation-acceleration equation turns out in more realistic cases.

B. The constant acceleration case

This case has also been treated extensively in the literature^{7,8}. The hypothesis is that in the interval $[0, \bar{\tau}]$ we have $a = -g$ with $g \in \mathbb{R}$. Equation (1) gives immediately

$$T^2 = \left[\int_0^{\bar{\tau}} e^{-g\tau} d\tau \right] \left[\int_0^{\bar{\tau}} e^{g\tau} d\tau \right] = \frac{2}{g^2} (\cosh g\bar{\tau} - 1)$$

$$\text{or } T = \frac{2}{g} \sinh \frac{g\bar{\tau}}{2}.$$

C. A more complicated example

This example was considered by Taylor and Wheeler⁹. It has the advantage that the acceleration has no Dirac's deltas and O departs from and arrives at K with zero velocity. The interval is divided into four equal parts of proper time duration $\bar{\tau}/4$. The acceleration in these intervals is successively $g, -g, -g$ and g .

One can easily convince him/herself that since the acceleration in the second half interval is opposite to the one in the first half interval the observer indeed returns

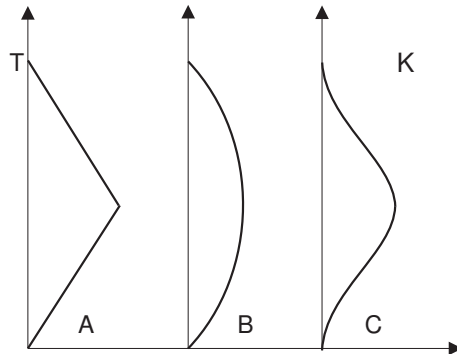


FIG. 1: The textbook round-trip examples.

to K 's worldline. Moreover, we know that O starts with zero velocity so we can apply equation (2). First we have

$$\int_0^{\bar{\tau}} a(\tau') d\tau' = \begin{cases} g\tau, & \tau \in [0, \frac{1}{4}\bar{\tau}], \\ -g\tau + g\bar{\tau}/2, & \tau \in [\frac{1}{4}\bar{\tau}, \frac{3}{4}\bar{\tau}], \\ g\tau - g\bar{\tau}, & \tau \in [\frac{3}{4}\bar{\tau}, \bar{\tau}]. \end{cases}$$

Integrating simple exponentials we arrive at $T = \frac{4}{g} \sinh \frac{g\bar{\tau}}{4}$.

II. THE RECONSTRUCTION PROBLEM IN SPECIAL RELATIVITY

In this section we consider the problem of reconstructing the motion in the inertial frame starting from the knowledge of the acceleration. Similar mechanical problems have been studied in¹⁰. It can be stated in full Minkowski spacetime as follows.

Consider a timelike worldline $x^\mu(\tau)$ on Minkowski spacetime and a Fermi-transported triad e_i . Let $a^i(\tau) = -(a(\tau) \cdot e_i)$ be the components of the acceleration vector with respect to the triad, $a = a^i e_i$. Determine, starting from the data $a^i(\tau)$, the original curve up to an affine transformation of Minkowski spacetime.

Here the Fermi-transported triad represents gyroscopes. The components of the acceleration with respect to this triad are therefore measurable by O using three orthogonal gyroscopes and an accelerometer. The solution to this problem may be relevant for future space travellers. Indeed, although the twin 'paradox' has been studied mainly assuming the possibility of some communication by light signals, it is more likely that when distances grow communication becomes impossible. Suppose the space traveller does not want to be lost but still wants the freedom to choose time by time its trajectory, then he/her should find some way to know its inertial coordinates. The only method, if no references in space are given, is to solve the reconstruction problem. Keeping track of the acceleration during the journey the observer would be able to reconstruct its inertial coordinates without looking outside the laboratory. In particular he would be able to construct (merging an ac-

celerometer, three gyroscopes, and an ordinary clock) an *inertial clock* i.e. a clock that displays $x^0(\tau)$.

The solution to the reconstruction problem gives also to O the advantage of knowing its own position even before K knows it. Indeed, O can know $x^\mu(\tau)$ immediately while K has to wait for light signals from O . In case of perturbations in the trajectory, O can immediately apply some corrections while, for great distances, a decision from K would take too much time.

In 1+1 dimensions the reconstruction problem can be solved easily. For higher dimensions it becomes much more complicated and numerical methods should be used. Let us give the solution to the 1+1 case. We use the timelike convention $\eta_{00} = 1$.

If $v^\mu = dx^\mu/dx^0$, $v = dx^1/dx^0$ and $u^\mu = dx^\mu/d\tau$, then we have

$$a^\mu = \frac{du^\mu}{d\tau} = \frac{d}{d\tau} \frac{v^\mu}{\sqrt{1-v^2}}.$$

Let $(0, a)$ be the components of the acceleration in the local inertial frame (the first component vanishes because $u \cdot a = 0$). Since the square of the acceleration is a Lorentz invariant we have $-a^2 = a^\mu a_\mu$ or

$$\begin{aligned} -a^2 &= \left(\frac{d}{d\tau} \frac{1}{\sqrt{1-v^2}} \right)^2 - \left(\frac{d}{d\tau} \frac{v}{\sqrt{1-v^2}} \right)^2 \\ &= -\frac{1}{(1-v^2)^2} \left(\frac{dv}{d\tau} \right)^2, \end{aligned}$$

but a has the same sign as $dv/d\tau$ and hence $a = \frac{d\theta}{d\tau}$, where $\theta = \tanh^{-1} v$ is the rapidity, or

$$v(\tau) = \tanh \left[\int_0^\tau a(\tau') d\tau' + \tanh^{-1} v(0) \right]. \quad (3)$$

From $dx^0 = d\tau/\sqrt{1-v^2}$ and $dx^1 = d\tau v/\sqrt{1-v^2}$ we have

$$x^0(\tau) - x^0(0) = \int_0^\tau \cosh \left[\int_0^{\tau'} a(\tau'') d\tau'' + \tanh^{-1} v(0) \right] d\tau', \quad (4)$$

$$x^1(\tau) - x^1(0) = \int_0^\tau \sinh \left[\int_0^{\tau'} a(\tau'') d\tau'' + \tanh^{-1} v(0) \right] d\tau', \quad (5)$$

Note that $v(0)$ is also easily measurable by O since at $\tau = 0$, K and O are crossing each other. Without knowing $v(0)$ the inertial coordinates are determined only up to a global affine transformation. Indeed, we may say that the knowledge of $v(0)$ specifies, up to translations, the inertial coordinates and frame with respect to which we describe O 's motion.

Now, consider the invariant under affine transformations

$$T^2(\tau) = [x^0(\tau) - x^0(0)]^2 - [x^1(\tau) - x^1(0)]^2. \quad (6)$$

Since $x(\tau)$ is in the chronological future of $x(0)$ there is a timelike geodesic passing through them. The inertial

observer $K(\tau)$ moving along that geodesic sees the motion of the accelerated observer as a round trip. If $x_{K(\tau)}^\mu$ are its coordinates $x_{K(\tau)}^1(\tau) = x_{K(\tau)}^1(0) = 0$, thus the previous invariant reads

$$T(\tau) = x_{K(\tau)}^0(\tau) - x_{K(\tau)}^0(0),$$

that is, $T(\tau)$ is the travel duration with respect to an inertial observer that sees the motion of the accelerated observer as a round trip that ends at τ . Using the relation $a^2 - b^2 = (a-b)(a+b)$ we have from Eq. (6)

$$T^2(\tau) = \left[\int_0^\tau e^{\int_0^{\tau'} a(\tau'') d\tau''} d\tau' \right] \left[\int_0^\tau e^{-\int_0^{\tau'} a(\tau'') d\tau''} d\tau' \right].$$

Remarkably the dependence on $v(0)$ disappears. This follows from the fact that contrary to $x^0(\tau)$ and $x^1(\tau)$, the quantity $T(\tau)$ is a Lorentz invariant and as such should not depend on the choice of frame (i.e. the choice of $v(0)$).

In order to prove the second theorem note that if, with respect to K , O departs with zero velocity then from (5), after imposing the round-trip condition $x^1(\bar{\tau}) = x^1(0)$, we have

$$\int_0^{\bar{\tau}} \sinh \left[\int_0^{\tau'} a(\tau'') d\tau'' \right] d\tau = 0.$$

that is, the two factors in the formula for T^2 coincide. Finally, if O departs and returns with zero velocity we have $\int_0^{\bar{\tau}} a d\tau = 0$ as it follows from the already derived relation $a = d\theta/d\tau$.

A. Differential aging

We give now a different proof that $T(\bar{\tau}) > \bar{\tau}$ unless $a(\tau) = 0$ for all $\tau \in [0, \bar{\tau}]$ in which case $T(\bar{\tau}) = \bar{\tau}$ and O is at rest in K .

The idea is to define the differential aging even for proper times $\tau < \bar{\tau}$ as the differential aging between $K(\tau)$ and O . The differential aging at τ is therefore by definition $\Delta(\tau) = T(\tau) - \tau$, that is the difference between the proper time elapsed for an inertial observer that reach $x(\tau)$ from $x(0)$ and that elapsed in the accelerating frame. Roughly speaking if at proper time τ the accelerating observer asks ‘‘What is the differential aging now?’’ the answer using this idea would be: it is the differential aging between you and an imaginary twin who reached the same event where you are now, but moving along a geodesic. This definition has the advantage of avoiding conventions for distant simultaneity.

Theorem II.1. *The differential aging $\Delta(\tau)$ is a non-decreasing function*

$$\frac{d\Delta}{d\tau} \geq 0, \quad (7)$$

where the equality holds for all $\tau' \in [0, \tau]$ iff $a(\tau') = 0$ for all $\tau' \in [0, \tau]$.

Proof. Let $\Theta(\tau) = \int_0^\tau a(\tau')d\tau'$. The derivative of $T(\tau)$ is

$$\frac{dT}{d\tau} = \cosh A(\tau),$$

where

$$A(\tau) = \Theta(\tau) + \frac{1}{2} \ln \left\{ \frac{\int_0^\tau e^{-\Theta(\tau')} d\tau'}{\int_0^\tau e^{\Theta(\tau')} d\tau'} \right\}.$$

Since $\cosh A \geq 1$ this proves Eq. (7). Now, suppose $\frac{d\Delta}{d\tau}(\tau) = 0$ then $A(\tau) = 0$ or

$$e^{-2\Theta} = \frac{\int_0^\tau e^{-\Theta(\tau')} d\tau'}{\int_0^\tau e^{\Theta(\tau')} d\tau'}. \quad (8)$$

Assume $\frac{d\Delta}{d\tau}(\tau') = 0$ for all $\tau' \in [0, \tau]$ then the previous equation holds for all $\tau' \in [0, \tau]$. Differentiating we obtain

$$-2a(\tau)e^{-2\Theta} = \frac{e^{-\Theta} \int_0^\tau e^{\Theta(\tau')} d\tau' - e^{\Theta} \int_0^\tau e^{-\Theta(\tau')} d\tau'}{(\int_0^\tau e^{\Theta(\tau')} d\tau')^2} = 0,$$

that is $a(\tau) = 0$ for all $\tau' \in [0, \tau]$. \square

Since $\Delta(0) = 0$ this theorem implies that $\Delta(\tau) > 0$ for $\tau > 0$ unless $a(\tau') = 0$ for all $\tau' \leq \tau$. This proves again the differential aging effect. However, the theorem says something more. It proves that the definition of differential aging we have given is particularly well behaved. It allows us to say that, as proper time passes, the imaginary twin is getting older and older with respect to the accelerating observer.

III. CONCLUSIONS

We have discussed the reconstruction problem in special relativity showing its relevance for the construction of *inertial clocks* and in general for the positioning of the space traveller. We have given a simple formula that relates the round-trip inertial time dilation with the acceleration measured by the non-inertial observer and have applied it to some well know cases to show how it works even in the presence of singularities. We believe that it could be useful in order to explain clearly the relationship between acceleration and differential aging $T(\tau) - \tau$. Indeed, the differential aging effect is obtained easily by applying the Cauchy-Schwarz inequality.

Although there is a section on the twin paradox in almost every textbook on special relativity, examples with singularities are not always completely satisfactory, while more refined examples require a lot of work. On the contrary the derivation of the time dilation-acceleration formula is quite elementary needing only some concepts from calculus. Its derivation as a classroom exercise would probably convince students of the reality of the differential aging effect.

Acknowledgments

I would like to acknowledge useful conversations with: D. Amadei, A. L pez Almorox, C. Rodrigo and C. Tejero Prieto. I am also grateful to A. Macdonald for his comments and suggestions. This work has been supported by INFN, grant n  9503/02.

* Electronic address: minguzzi@usal.es

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- ¹¹ These theorems hold also in a non-topologically trivial Minkowski spacetime but there O should move along a worldline which is homotopic to K ’s origin worldline; this follows from their validity in the covering Minkowski spacetime.