The Michelson – Morley Experiment

Let's consider the Earth, with the Michelson-Morley setup as a frame S' in motion with the speed v with respect to another frame S, the Sun. An observer at rest in S views the Michelson-Morley as follows: the light travels with the speed c along the rod of length l_2, parallel with v. Because the mirror of the end of the rod recedes with the speed +v in one direction of light propagation and with speed –v in the other direction, the observer in S views the roundtrip time as:

$$t_2 = \frac{l_2}{c+v} + \frac{l_2}{c-v} = \gamma^2 \frac{2l_2}{c} \qquad (1.1)$$

The rod perpendicular on v has the length l_1. In S' $l_1=l_2$. From S, the observer sees $(ct)^2=l_1^2+(vt)^2$ from where we obtain:

$$t = \gamma \frac{l_1}{c}$$

The roundtrip time as seen from S is:

$$t_1 = 2t = \gamma \frac{2l_1}{c} = \gamma \frac{2l'_1}{c} \qquad (1.2)$$

As seen from S, $l_1=l_1'$ and $l_2=l_2'/\gamma$

Therefore:

$$t_2 = \gamma \frac{2l'_2}{c} \qquad (1.3)$$

From the perspective of S,

$$t_2 - t_1 = \gamma \frac{2(l_2' - l_1')}{c} = 0 \qquad (1.4)$$

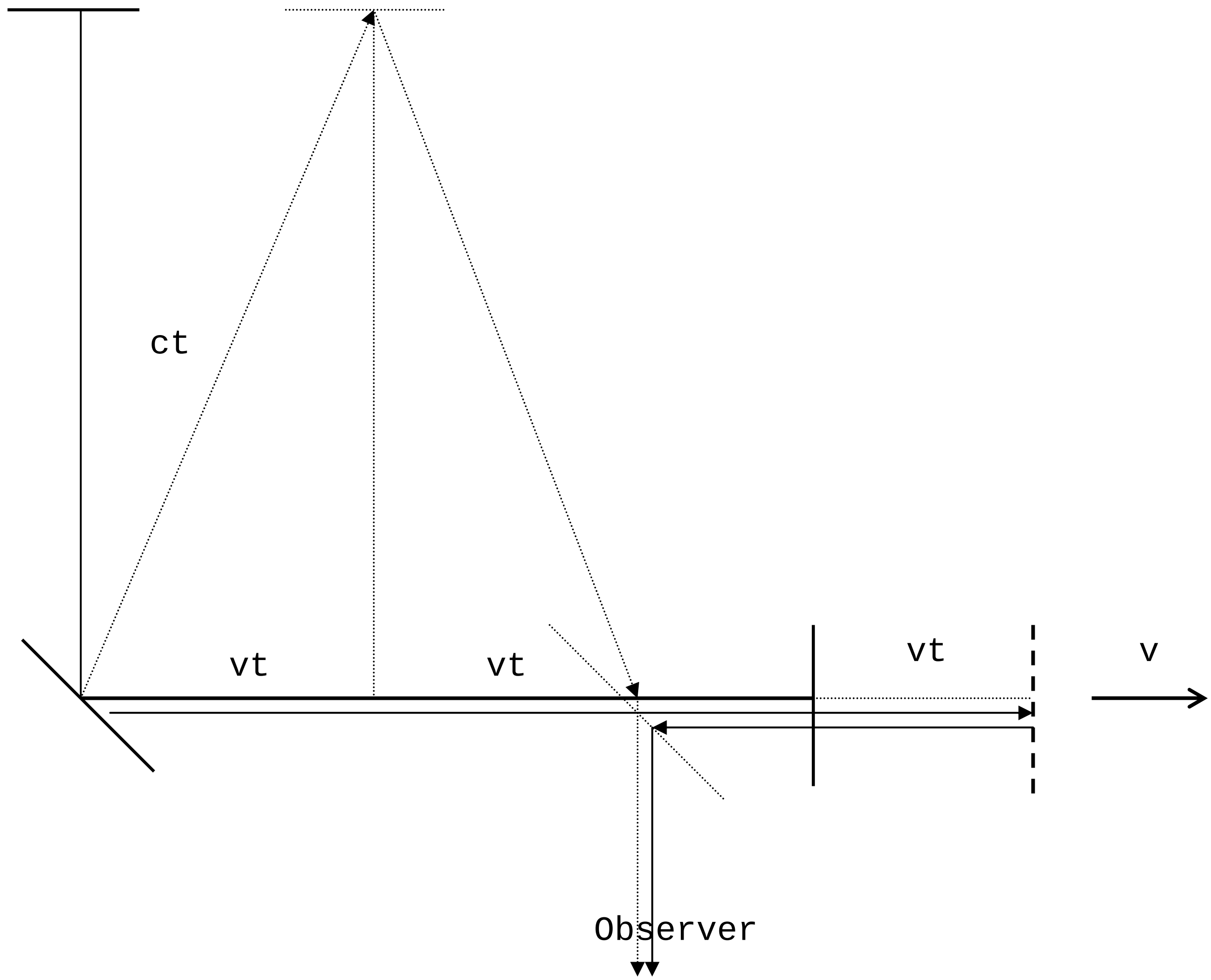
(1.4) explains the null result of the Michelson-Morley experiment as viewed from the frame S, the Sun.

An alternative explanation is the following: due to light speed isotropy in S', we have

$$t'_2 = \frac{2l'_2}{c} = \frac{2l'_1}{c} = t'_1 \qquad (1.5)$$

Then, applying time dilation:

$$t_2 - t_1 = \gamma (t'_2 - t'_1) = 0 \qquad (1.6)$$



ct
vt
vt
vt
v
Observer

: “How about length contraction for frames moving inertially but along an arbitrary direction with respect to each other?”. To construct the answer to this question start with the Lorentz transform for an arbitrary direction of motion between inertial frames S and S’ (see fig.1). The axis of S and S’ are presumed parallel:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+(\gamma-1)\dfrac{\beta_x^{\ 2}}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y^{\ 2}}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z^{\ 2}}{\beta^2} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{1}$$

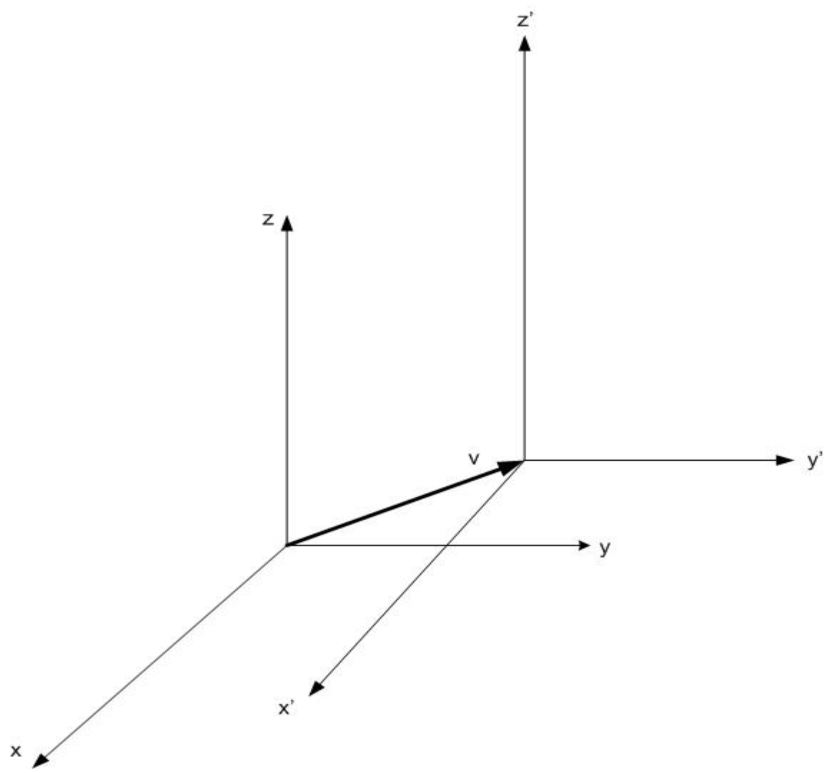


Fig.1 Arbitrary frames with aligned axes

To make a length measurement in frame S’ we need to mark both ends of the rod simultaneously, so $\Delta t' = 0$. Additionally, we can consider for simplicity that the rod is aligned with the x axis in frame S so $\Delta y = \Delta z = 0$

$$\begin{bmatrix} 0 \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+(\gamma-1)\frac{\beta_x^{\ 2}}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^{\ 2}}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^{\ 2}}{\beta^2} \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

where $\beta_x = \frac{v_x}{c}, \beta_y = \frac{v_y}{c}, \beta_z = \frac{v_z}{c}$

$$\sum \Delta x'^2 = (1 + \frac{1-\gamma^2}{\gamma^2}\frac{\beta_x^{\ 2}}{\beta^2})\Delta x^2 \tag{3}$$

Since $-1 < \frac{1-\gamma^2}{\gamma^2}\frac{\beta_x^{\ 2}}{\beta^2} < 0$ formula (3) represents indeed a length contraction that can be rewritten as:

$$L' = L\sqrt{1 + \frac{1-\gamma^2}{\gamma^2}\frac{\beta_x^{\ 2}}{\beta^2}} \tag{4}$$

We can quickly verify the correctness by setting $\beta_x = \beta$ (S and S' relative movement aligned with the x axis):

$$L' = \frac{L}{\gamma} \tag{5}$$

Time dilation

In this case, we are measuring a time interval Δt at the fixed location (x,y,z) in frame S. This translates in frame S' into:

$$\begin{bmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+(\gamma-1)\frac{\beta_x^{\ 2}}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\beta_y\gamma & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^{\ 2}}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\beta_z\gamma & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^{\ 2}}{\beta^2} \end{bmatrix} \begin{bmatrix} c\Delta t \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$\Delta t' = \gamma \Delta t \tag{7}$$

Thus, we can conclude that the formulas for length contraction, time dilation for the general case retain the simplicity of the more elementary case when S and S' move along their common x-axis.