

- (b) Find the probability current for the wave function in the previous problem. (This is not a very pithy example, I'm afraid; we'll encounter some more substantial ones in due course.)

****Problem 1.10** Suppose you wanted to describe an **unstable particle** that spontaneously disintegrates with a "lifetime" τ . In that case the total probability of finding the particle somewhere should *not* be constant, but should decrease at (say) an exponential rate:

$$P(t) \equiv \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = e^{-t/\tau}.$$

A crude way of achieving this result is as follows. In Equation 1.24 we tacitly assumed that V (the potential energy) is *real*. That is certainly reasonable, but it leads to the **conservation of probability** enshrined in Equation 1.27. What if we assign to V an imaginary part:

$$V = V_0 - i\Gamma,$$

where V_0 is the true potential energy and Γ is a positive real constant?

- (a) Show that (in place of Equation 1.27) we now get

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P.$$

- (b) Solve for $P(t)$, and find the lifetime of the particle in terms of Γ .
-

1.5 MOMENTUM

For a particle in state Ψ , the expectation value of x is

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\Psi(x, t)|^2 dx.$$

[1.28]

What exactly does this mean? It emphatically does *not* mean that if you measure the position of one particle over and over again, $\int x |\Psi|^2 dx$ is the average of the results you'll get. On the contrary, the first measurement (whose outcome is indeterminate) will collapse the wave function to a spike at the value actually obtained, and the subsequent measurements (if they're performed quickly) will simply repeat that same result. Rather, $\langle x \rangle$ is the average of measurements performed on particles *all in the state* Ψ , which means that either you must find some way of returning the particle to its original state after each measurement, or else you prepare a whole ensemble of particles, each in the same state Ψ , and measure the positions of all of them: $\langle x \rangle$ is the

average of *these* results. [I like to picture a row of bottles on a shelf, each containing a particle in the state Ψ (relative to the center of the bottle). A graduate student with a ruler is assigned to each bottle, and at a signal they all measure the positions of their respective particles. We then construct a histogram of the results, which should match $|\Psi|^2$, and compute the average, which should agree with $\langle x \rangle$. (Of course, since we're only using a finite sample, we can't expect perfect agreement, but the more bottles we use, the closer we ought to come.)] In short, the expectation value is the average of repeated measurements on an ensemble of identically prepared systems, *not* the average of repeated measurements on one and the same system.

Now, as time goes on, $\langle x \rangle$ will change (because of the time dependence of Ψ), and we might be interested in knowing how fast it moves. Referring to Equations 1.25 and 1.28, we see that⁹

$$\frac{d\langle x \rangle}{dt} = \int x \frac{\partial}{\partial t} |\Psi|^2 dx = \frac{i\hbar}{2m} \int x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx. \quad [1.29]$$

This expression can be simplified using integration by parts¹⁰:

$$\frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{2m} \int \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx. \quad [1.30]$$

[I used the fact that $\partial x / \partial x = 1$, and threw away the boundary term, on the ground that Ψ goes to zero at (\pm) infinity.] Performing another integration by parts on the second term, we conclude that

$$\frac{d\langle x \rangle}{dt} = -\frac{i\hbar}{m} \int \Psi^* \frac{\partial \Psi}{\partial x} dx. \quad [1.31]$$

What are we to make of this result? Note that we're talking about the “velocity” of the *expectation* value of x , which is not the same thing as the velocity of the *particle*. Nothing we have seen so far would enable us to calculate the velocity of a particle—it's not even clear what velocity *means* in quantum mechanics. If the particle doesn't have a determinate position (prior to measurement), neither does it have a well-defined velocity. All we could reasonably ask for is the *probability* of getting a particular value. We'll see in Chapter 3 how to construct the probability density for velocity,

⁹To keep things from getting too cluttered, I suppress the limits of integration when they are $\pm\infty$.

¹⁰The product rule says that

$$\frac{d}{dx} (fg) = f \frac{dg}{dx} + \frac{df}{dx} g,$$

from which it follows that

$$\int_a^b f \frac{dg}{dx} dx = - \int_a^b \frac{df}{dx} g dx + fg \Big|_a^b.$$

Under the integral sign, then, you can peel a derivative off one factor in a product and slap it onto the other one—it'll cost you a minus sign, and you'll pick up a boundary term.