13 Scattering amplitude at 1-loop, the massless limit and the $\overline{\text{MS}}$ scheme

13.1 $2 \to 2$ scattering amplitude at 1-loop accuracy

In lecture 7 we computed $\varphi\varphi \to \varphi\varphi$ scattering at tree level (see figure 7 and equation (120) there). The result takes the form

$$\langle k_1' k_2' | k_1 k_2 \rangle = (2\pi)^4 \, \delta^4(k_1' + k_2' - k_2 - k_1) \, \mathrm{i}\mathcal{T} \tag{203}$$

where

$$\mathrm{i}\mathcal{T} = (\mathrm{i}g)^2 \left[\widetilde{D}_F(s) + \widetilde{D}_F(t) + \widetilde{D}_F(u) \right] + \mathcal{O}(g^4) \tag{204}$$

where we defined the Mandelstam invariants:

$$s = (k_1 + k_2)^2 = (k_1' + k_2')^2 \,, \qquad t = (k_1 - k_1')^2 = (k_2' - k_2)^2 \,, \qquad u = (k_2 - k_1')^2 = (k_2' - k_1)^2 \,. \tag{205}$$

Now, having computed one-loop integrals, we have the means to determine this amplitude at one-loop accuracy, namely including all $\mathcal{O}(g^4)$ corrections. The most efficient way to do this is to use the calculations we have already done for the propagator, the three-leg vertex function and the four-leg vertex function at one-loop, as shown schematically in figure 17.

$$\mathrm{i}\mathcal{T} = (\mathrm{i}V_3(s))^2 \widetilde{\Delta}_F(s) + (\mathrm{i}V_3(t))^2 \widetilde{\Delta}_F(t) + (\mathrm{i}V_3(u))^2 \widetilde{\Delta}_F(u) + \mathrm{i}V_4(s,t,u) \tag{206}$$

We note that computed in this way, each of the components entering the scattering amplitude (which may be

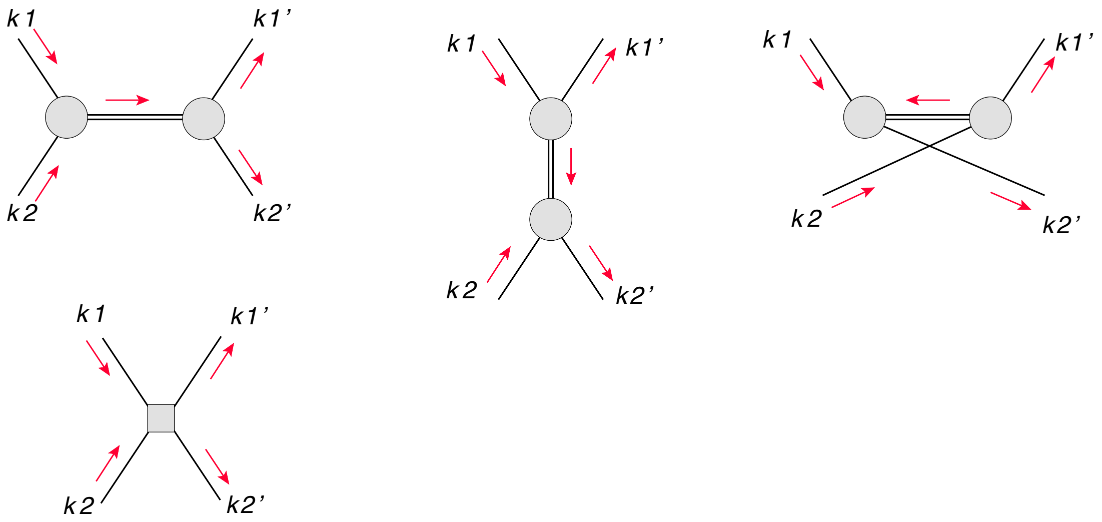

Figure 17: *Skeleton diagrams for $\varphi\varphi \to \varphi\varphi$ scattering in terms of the full propagator Δ_F (double line) the full 3-leg vertex V_3 (round 3-legged blob) and the full 4-leg vertex function V_4 (square blob).*

computed to any fixed order) is finite by itself. Let us review the results we got for the various ingredients in this expression at the one-loop order.

Let us start with the four-leg vertex function, shown to be finite in $d = 6$ space-time dimensions:

$$V_4(s,t,u) = g^2 \alpha \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \left[\frac{1}{M^2(s,t)} + \frac{1}{M^2(t,u)} + \frac{1}{M^2(u,s)} \right] , \tag{207}$$

where we defined

$$M^2(s,t) = M^2(m^2, m^2, m^2, m^2; s, t)$$

where the latter is the scale appearing in the box integral with all incoming momenta $p_1 + p_2 + p_3 + p_4 = 0$ (computed in lecture 11) defining $p_{ij} = (p_i + p_j)^2$ and $q_n = \sum_{i=1}^{n} p_i$, which reads

$$M^2(p_1^2, p_2^2, p_3^2, p_4^2; p_{12}^2, p_{13}^2) = m^2 + (q_1 x + q_2 y + q_3 z)^2 - q_1^2 x - q_2^2 y - q_3^2 z \tag{208}$$

Let us turn now to the three-leg vertices. Defining the one-loop renormalization constant for the coupling as

$$Z_g = 1 + \left[-\frac{1}{2\epsilon} + \frac{1}{2}\ln(m^2/\mu^2) - \kappa_g \right] \alpha + \mathcal{O}(\alpha^2)$$

we got the renormalized three-leg vertex where $p_1 + p_2 + p_3 = 0$,

$$\mathrm{i}V_3(p_1^2, p_2^2, p_3^2) = \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx \int_0^{1-x} dy \, \ln\left(\frac{m^2 + (xp_1 - yp_2)^2 - p_1^2 x - p_2^2 y}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right) \tag{209}$$

Note that κ_g is a constant which depend on the renormalization condition chosen, e.g. $V_3(0,0,0) = g$ is a valid condition. Based on this (209) we defined

$$\mathrm{i}V_3(p^2) \equiv \mathrm{i}V_3(m^2, m^2, p^2) \tag{210}$$

which enters (221) with three different assignments of the scale p^2, for the three different diagrams.

Next, the full propagator we used in (221) includes a geometric sum over any number of self-energy insertions, and it is given by

$$\widetilde{\Delta}_F(p^2) = \frac{\mathrm{i}}{p^2 - m^2 + \Pi(p^2) + \mathrm{i}\delta} \tag{211}$$

where $\Pi(p^2)$ is the renormalized self energy, for which we obtained the following result

$$\begin{aligned}\Pi(p^2) = \Bigg[& \left(\frac{1}{12}p^2 - \frac{1}{2}m^2\right)\left(\frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right)\right) + \frac{1}{2}\int_0^1 dx \Big(m^2 - p^2 x(1-x)\Big) \ln\left(\frac{m^2 - p^2 x(1-x)}{m^2}\right) \\ & + \; Z_\varphi^{(1)} p^2 - Z_m^{(1)} m^2 \Bigg] \alpha + \mathcal{O}(\alpha^2)\,. \end{aligned} \tag{212}$$

where

$$Z_\varphi^{(1)} p^2 - Z_m^{(1)} m^2 = -p^2 \frac{1}{12}\left(\frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right) + \kappa_\varphi\right) + \frac{1}{2}\left(\frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right) + \kappa_m\right) m^2 \tag{213}$$

were fixed based on the renormalization conditions (158a) and (158b) namely

$$\Pi(p^2)\big|_{p^2=m^2} = 0\,, \qquad \frac{d\Pi(p^2)}{dp^2}\bigg|_{p^2=m^2} = 0\,, \tag{214}$$

fixing the numerical constants κ_φ and κ_m, respectively, as in (176) and (177).

It is clear that the scattering amplitude, while finite, depends on the particular renormalization scheme chosen. Here we used the on-shell renormalization scheme, which is valid for any finite mass, but becomes ill-defined for $m^2 = 0$.

Let us consider for example the self energy function in this limit. Dropping all terms that vanish for $m^2 \to 0$ we get:

$$\Pi(p^2) = -p^2 \left[\frac{\kappa_\varphi}{12} + \frac{1}{2}\int_0^1 dx \, x(1-x) \, \ln\left(\frac{-p^2 x(1-x)}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\,. \tag{215}$$

Evidently, the result is logarithmically divergent for $m^2 \to 0$, and the strict limit cannot be taken. Moreover, we see that for $m^2 = 0$ the renormalization condition $\Pi(0) = 0$ is automatically satisfied, independently of the choice of Z_φ and Z_m, while the condition on the derivative $\Pi'(p^2)$ cannot be satisfied at all, since the derivative is ill-defined for $m^2 = 0$. We need to understand why this is the case, and whether this can be avoided, because we expect the massless theory should make sense!

The deep reason for the failure of the on-shell renormalization for $m^2 = 0$ is that it was based on the full propagator having an isolated pole corresponding to the physical single-particle state, $p^2 = m^2$. Considering however the Källen Lehman representation

$$\widetilde{\Delta}_F(p^2) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\delta} + \int_{\sim 4m^2}^{\infty} ds \, \rho(s) \, \frac{\mathrm{i}}{p^2 - s + \mathrm{i}\delta} \tag{216}$$

for $m^2 = 0$ we see that the spectral-density continuum is also starting at m^2: there is no more isolated pole, since any number of massless particles may still be consistent with total energy of exactly zero. The perturbative calculation indeed reveals that the branch point at $p^2 = 4m^2$ now coincides with the pole at $p^2 = 0$. In the next lecture we will present a different renormalization scheme, called minimal subtraction, which does not rely on the presence of an isolated single-particle state, and which therefore allows a consistent description of the massless theory.

13.2 Exercises

1. Derive an explicit expression for the function M^2 in (208) and computed the integral in (220).
Guidance:
Starting with (208) corresponding to the leftmost diagram in figure 15 and the expression in eq. (195) and using the definition of $q_i = \sum_{j=1}^{i} p_j$ show that

$$\begin{aligned} M^2 = m^2 &- (x+y+z)(1-x-y-z)p_1^2 - (y+z)(1-y-z)p_2^2 - z(1-z)p_3^2 \\ &- 2(1-x-y-z)(y+z)p_{12} - 2z(1-y-z)p_{23} - 2z(1-x-y-z)p_{13} \end{aligned} \tag{217}$$

where $p_{ij} = p_i \cdot p_j$.
Next specialise to the kinematics of the on-shell $2 \to 2$ scattering, where $p_i^2 = m^2$ from $i = 1..4$, and use momentum conservation:

$$p_{12} + p_{13} + p_{23} = \frac{1}{2}(p_4^2 - p_1^2 - p_2^2 - p_3^2),$$

to write

$$M^2 = m^2(1 + 2xy + 2yz - x(1-x) - 2y(1-y) - z(1-z)) - 2y(1-x-y-z)p_{12} - 2xzp_{23}\,. \tag{218}$$

Recall that the momenta p_i were assigned cyclically to the 4 legs (see figure 15), and defined as incoming (meaning that positive energy p_i^0 corresponds to an incoming particle while negative energy to outgoing one). Making the identification $p_1 = k_1$, $p_2 = k_2$, $p_3 = -k_1'$, $p_4 = -k_2'$, and then $p_{12} = s/2$, $p_{23} = t/2$ and $p_{13} = u/2$, show that in the massless limit

$$M^2(s,t)\big|_{m^2=0} = -y(1-x-y-z)s - xzt$$

Next, performing the integral:

$$\begin{aligned} V_4^{\text{box}}(s,t)\big|_{m^2=0} &= g^2\alpha \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\, \frac{1}{M^2(s,t)\big|_{m^2=0}} \\ &= g^2\alpha \frac{1}{t}\int_0^\infty du \frac{\ln(u)}{1-u}\frac{1}{1+us/t} \\ &= g^2\alpha \frac{-1}{2(s+t)}\left(\pi^2 + \ln^2\left(\frac{s}{t}\right)\right) \end{aligned} \tag{219}$$

Finally, sum over all permutations corresponding to the three diagrams in figure 15 and the three terms in (220)

$$V_4(s,t,u) = g^2\alpha\left[\frac{1}{2u}\left(\pi^2 + \ln^2\left(\frac{s}{t}\right)\right) + \frac{1}{2s}\left(\pi^2 + \ln^2\left(\frac{t}{u}\right)\right) + \frac{1}{2t}\left(\pi^2 + \ln^2\left(\frac{u}{s}\right)\right)\right]. \tag{220}$$

2. Expand the self energy and the vertex functions for $m^2 \ll s,t,u$ dropping all terms that vanish for $m^2 = 0$. Derive the corresponding one-loop correction to $\mathcal{T}$ (keep log-enhanced terms and constant terms).

Guidance:
Start with the general expression (221)

$$\mathrm{i}\mathcal{T} = (\mathrm{i}V_3(s))^2\widetilde{\Delta}_F(s) + (\mathrm{i}V_3(t))^2\widetilde{\Delta}_F(t) + (\mathrm{i}V_3(u))^2\widetilde{\Delta}_F(u) \;+\; \mathrm{i}V_4(s,t,u) \tag{221}$$

and take the small mass limit of each component. For the self-energy in the on-shell scheme, where

$$Z_\varphi^{(1)}p^2 - Z_m^{(1)}m^2 = -p^2\frac{1}{12}\left(\frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right) + \kappa_\varphi\right) + \frac{1}{2}\left(\frac{1}{\epsilon} + 1 + \ln\left(\frac{\mu^2}{m^2}\right) + \kappa_m\right)m^2, \tag{222}$$

this was already done in (215), thus the propagator is

$$\widetilde{\Delta}_F(p^2) = \frac{\mathrm{i}}{p^2\left[1 - \left(\frac{\kappa_\varphi}{12} + \frac{1}{2}\int_0^1 dx\, x(1-x)\ln\left(\frac{-p^2x(1-x)}{m^2}\right)\right)\alpha + \mathcal{O}(\alpha^2)\right] + \mathrm{i}\delta}. \tag{223}$$

Performing the integration over x this yields:

$$\widetilde{\Delta}_F(p^2) = \frac{\mathrm{i}}{p^2\left[1 - \left(\frac{\kappa_\varphi}{12} - \frac{5}{36} + \frac{1}{12}\ln\left(\frac{-p^2}{m^2}\right)\right)\alpha + \mathcal{O}(\alpha^2)\right] + \mathrm{i}\delta}. \tag{224}$$

For the three-leg vertex function we start with the general expression (209)

$$\mathrm{i}V_3(p_1^2,p_2^2,p_3^2) = \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx \int_0^{1-x} dy\ \ln\left(\frac{m^2 + (xp_1 - yp_2)^2 - p_1^2 x - p_2^2 y}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right) \tag{225}$$

in the on-shell scheme, where

$$Z_g = 1 + \left[-\frac{1}{2\epsilon} + \frac{1}{2}\ln(m^2/\mu^2) - \kappa_g\right]\alpha + \mathcal{O}(\alpha^2). \tag{226}$$

Next specialize to the situation where two momenta are external, namely $p_1^2 = m^2$ and $p_2^2 = m^2$ while $p_3^2 = s = (p_1+p_2)^2 = 2m^2 + 2p_1\cdot p_2$, getting:

$$\begin{aligned}\mathrm{i}V_3(s) &= \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx \int_0^{1-x} dy\ \ln\left(\frac{m^2 - 2xyp_1\cdot p_2 - p_1^2 x(1-x) - p_2^2 y(1-y)}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\\ &= \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx \int_0^{1-x} dy\ \ln\left(\frac{m^2(1 - x(1-x) - y(1-y) + 2xy) - xys}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\end{aligned} \tag{227}$$

Next take the m^2 limit wherever possible, and integrate:

$$\begin{aligned}\mathrm{i}V_3(s)\big|_{m^2\to 0} &= \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx \int_0^{1-x} dy\ \ln\left(\frac{-xys}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\\ &= \mathrm{i}g\left(1 - \left[\kappa_g - \frac{3}{2} + \frac{1}{2}\ln\left(\frac{-s}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\end{aligned} \tag{228}$$

Next consider the s-channel diagram in (221)

$$\begin{aligned}(\mathrm{i}V_3(s))^2\widetilde{\Delta}_F(s)\Big|_{m^2\to 0} &= \mathrm{i}(\mathrm{i}g)^2 \frac{1 - 2\left[\kappa_g - \frac{3}{2} + \frac{1}{2}\ln\left(\frac{-s}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)}{s\left[1 - \left(\frac{\kappa_\varphi}{12} - \frac{5}{36} + \frac{1}{12}\ln\left(\frac{-s}{m^2}\right)\right)\alpha + \mathcal{O}(\alpha^2)\right] + \mathrm{i}\delta}\\ &= \mathrm{i}(\mathrm{i}g)^2\frac{1}{s}\left(1 + \left[\underbrace{\frac{103}{36} - 2\kappa_g + \frac{\kappa_\varphi}{12}}_{K} - \frac{11}{12}\ln\left(\frac{-s}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\end{aligned} \tag{229}$$

Finally combine all contributions in (221) showing that

$$\begin{aligned}\mathcal{T} = -g^2\Bigg\{&\frac{1}{s}\left(1 + \left[K - \frac{11}{12}\ln\left(\frac{-s}{m^2}\right)\right]\alpha\right) + \frac{1}{t}\left(1 + \left[K - \frac{11}{12}\ln\left(\frac{-t}{m^2}\right)\right]\alpha\right) + \frac{1}{u}\left(1 + \left[K - \frac{11}{12}\ln\left(\frac{-u}{m^2}\right)\right]\alpha\right)\\ &- \alpha\left[\frac{1}{2u}\left(\pi^2 + \ln^2\left(\frac{s}{t}\right)\right) + \frac{1}{2s}\left(\pi^2 + \ln^2\left(\frac{t}{u}\right)\right) + \frac{1}{2t}\left(\pi^2 + \ln^2\left(\frac{u}{s}\right)\right)\right]\Bigg\}.\end{aligned} \tag{230}$$

14 The $\overline{\mathrm{MS}}$ scheme and the renormalization group

In the last lecture we encountered a problem in applying the on-shell renormalization scheme in the massless case. We analysed it and traced it back to the absence of an isolated single-particle state, corresponding to a simple pole in the full propagator.

Let us now present a solution, namely a renormalization method that does not rely on having an isolated pole. We start by recalling the calculation of the self-energy bubble in (153), yielding, after expansion in ϵ,

$$\Pi^{(b)}(p^2) = -\frac{\alpha}{2}\int_0^1 dx\,(m^2 - x(1-x)p^2)\left[\frac{1}{\epsilon} + 1 - \ln\left(\frac{m^2 - x(1-x)p^2}{4\pi\,\mathrm{e}^{-\gamma_E}\,\widetilde{\mu}^2}\right) + \mathcal{O}(\epsilon)\right] \tag{231}$$

We then defined $\mu^2 = 4\pi\,\mathrm{e}^{-\gamma_E}\,\widetilde{\mu}^2$, and together with the counter-terms we got the following one-loop result:

$$\Pi(p^2) = \alpha\left[p^2 Z_\varphi^{(1)} - m^2 Z_m^{(1)} - \frac{1}{2}\int_0^1 dx\,(m^2 - x(1-x)p^2)\left[\frac{1}{\epsilon} + 1 - \ln\left(\frac{m^2 - x(1-x)p^2}{\mu^2}\right)\right]\right] + \mathcal{O}(\alpha^2) \tag{232}$$

If we have no prejudice what $\Pi(p^2)$ should be, except that it should be finite, a natural choice is to simply to cancel the $1/\epsilon$ term in (232) by an appropriate choice of the renormalization parameters:

$$Z_\varphi^{(1)} = -\frac{1}{12}\frac{1}{\epsilon}\,, \qquad\qquad Z_m^{(1)} = -\frac{1}{2}\frac{1}{\epsilon} \tag{233}$$

which yields:

$$\Pi^{\overline{\mathrm{MS}}}(p^2) = -\frac{\alpha}{2}\left[\int_0^1 dx\,(m^2 - x(1-x)p^2)\left[1 - \ln\left(\frac{m^2 - x(1-x)p^2}{\mu^2}\right)\right]\right] + \mathcal{O}(\alpha^2) \tag{234}$$

This renormalization scheme is called the "modified minimal subtraction scheme", or the $\overline{\mathrm{MS}}$ scheme. The counter terms are minimal in the sense that they do not contain any finite terms, only negative powers of ϵ.

Recall that when working in the on-shell scheme we extracted the logarithm of μ^2/m^2 from the integral and absorbed it along with the $1/\epsilon$ singularity in the counterterms. Instead, now we leave μ^2 in the integral, so it remain in the finite expression for $\Pi(p^2)$. Thus the $\overline{\mathrm{MS}}$ scheme comes along with a reference scale μ^2. The reason for the name "modified" is that we have defined the reference scale by $\mu^2 = 4\pi\,\mathrm{e}^{-\gamma_E}\,\widetilde{\mu}^2$ (instead of e.g. using $\widetilde{\mu}^2$ as a reference scale - which is referred to as "minimal subtraction").

Now we come to the advantage of the $\overline{\mathrm{MS}}$ scheme compared to the on-shell scheme. In the case of the on-shell scheme we could not take the strict $m^2 \to 0$ limit: in (215) the renormalized $\Pi^{\overline{\mathrm{MS}}}(p^2)$ has a log-divergence in this limit. In contrast the $\overline{\mathrm{MS}}$ self energy (234) is well-defined for $m = 0$:

$$\Pi^{\overline{\mathrm{MS}}}(p^2) = -\frac{\alpha}{2}\left[\int_0^1 dx\,(-x(1-x)p^2)\left[1 - \ln\left(\frac{-x(1-x)p^2}{\mu^2}\right)\right]\right] + \mathcal{O}(\alpha^2) \tag{235}$$

Indeed the $\overline{\mathrm{MS}}$ scheme is the most popular one in gauge theories, where massless particles are essential.

Next let us look at the disadvantages of the $\overline{\mathrm{MS}}$ scheme compared to the on-shell scheme. Using (234) in the propagator we have:

$$\widetilde{\Delta}_F^{\overline{\mathrm{MS}}}(p^2) = \frac{\mathrm{i}}{p^2 - m^2 + \Pi^{\overline{\mathrm{MS}}}(p^2) + \mathrm{i}\delta} \tag{236}$$

Note that now neither $\Pi^{\overline{\mathrm{MS}}}(p^2)$ nor its derivative vanish for $p^2 = m^2$. This has two immediate consequences with far-reaching implications:

- The pole of the one-loop $\overline{\mathrm{MS}}$ propagator *does not* occur at $p^2 = m^2$. Thus we cannot identify the mass parameter m, which appears in the Lagrangian

$$\mathcal{L} = \frac{1}{2}Z_\varphi \partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}Z_m m^2\varphi^2 + \frac{1}{6}Z_g g\varphi^3 \tag{237}$$

  as the physical mass m_{pole} corresponding to the pole of the single particle state in the Källen Lehmann representation. The latter can of course be computed in terms of the former by solving the equation namely solving

$$\left(\widetilde{\Delta}_F^{\overline{\mathrm{MS}}}(p^2)\right)^{-1}\Big|_{p^2 = m_{\mathrm{pole}}^2} = 0 \qquad \Longrightarrow \qquad m_{\mathrm{pole}}^2 - m^2 + \Pi^{\overline{\mathrm{MS}}}(m_{\mathrm{pole}}^2) = 0 \tag{238}$$

  for $m_{\mathrm{pole}}^2(m^2)$.

- The residue $R_{\overline{\rm MS}}$ of the pole in (236)

$$R^{-1}_{\overline{\rm MS}} \equiv \frac{d}{dp^2}\left(\frac{1}{\rm i}\widetilde{\Delta}^{\overline{\rm MS}}_F(p^2)\right)^{-1}\Bigg|_{p^2=m^2_{\rm pole}} = 1+\frac{d\Pi^{\overline{\rm MS}}(p^2)}{dp^2}\Bigg|_{p^2=m^2_{\rm pole}}, \tag{239}$$

is not one. This means that the expansion of the propagator about the physical pole looks like

$$\begin{aligned}\widetilde{\Delta}^{\overline{\rm MS}}_F(p^2) &= \frac{\rm i}{p^2-m^2+\Pi^{\overline{\rm MS}}(p^2)+{\rm i}\delta}\\ &= \frac{{\rm i}R_{\overline{\rm MS}}(\alpha,\mu^2/m^2)}{p^2-m^2_{\rm pole}(\alpha,m^2)+{\rm i}\delta} + \text{terms that are finite for } p^2\to m^2_{\rm pole}\end{aligned} \tag{240}$$

Returning to the definition of the propagator and the derivation of the Källen Lehmann representation (130)

$$\begin{aligned}\widetilde{\Delta}_F(p^2) &= \int d^4x{\rm e}^{{\rm i}p\cdot x}\, \langle 0|\,{\rm T}\,(\varphi(x)\varphi(0))\,|0\rangle\\ &= \sum_\lambda \frac{\rm i}{p^2-m^2_\lambda+{\rm i}\delta}\,\Big|\langle 0\,|\varphi(0)|\,\lambda_0\rangle\Big|^2,\end{aligned} \tag{241}$$

we are forced to desert our previous assertion that the field φ generates from the vacuum a single particle state with normalization one. Instead, comparing (241) to (240) we deduce that in $\overline{\rm MS}$, the single-particle state is normalized as:

$$\Big|\langle 0\,|\varphi(0)|\,\lambda_0\rangle\Big|^2 = R_{\overline{\rm MS}}\,.$$

Of course, in the free theory $\Big|\langle 0\,|\varphi(0)|\,\lambda_0\rangle\Big|^2 = 1$. So far we have tried to preserve this property in the interacting theory, but we now see that it is incompatible with renormalization in $\overline{\rm MS}$.

Further consequences of this are that the LSZ formula and the (momentum space) Feynman rules needs to be corrected when working in $\overline{\rm MS}$. To explain this it is useful to first reinterpret the normalization of the $\overline{\rm MS}$ propagator in eq. (240) in terms of the renormalization factor for the field Z_φ. The kinetic term in the Lagrangian has been written as $\frac{1}{2}Z_\varphi\partial_\mu\varphi\partial^\mu\varphi$. This is true in any scheme, but both the bilinear field operator and the corresponding Z_φ vary. Thus we have:

$$\mathcal{L}_{\rm Kin.} = \frac{1}{2}Z^{\rm OS}_\varphi\left(\partial_\mu\varphi\partial^\mu\varphi\right)_{\rm OS} = \frac{1}{2}Z^{\overline{\rm MS}}_\varphi\left(\partial_\mu\varphi\partial^\mu\varphi\right)_{\overline{\rm MS}} = \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 \tag{242}$$

where in the first expression the field is renormalized on-shell (OS), in the second it is renormalized in $\overline{\rm MS}$ and in the third we have used the bare field (non-renormalized field) a concept we shall heavily use in the next lecture. It is now clear that the normalization found in (240) for the $\overline{\rm MS}$ propagator

$$\begin{aligned}\widetilde{\Delta}^{\overline{\rm MS}}_F(p^2) &= \int d^4x{\rm e}^{{\rm i}p\cdot x}\, \langle 0|\,{\rm T}\,(\varphi(x)\varphi(0))_{\overline{\rm MS}}\,|0\rangle\\ &= \frac{{\rm i}R_{\overline{\rm MS}}(\alpha,\mu^2/m^2)}{p^2-m^2_{\rm pole}(\alpha,m^2)+{\rm i}\delta} + \text{terms that are finite for } p^2\to m^2_{\rm pole}\end{aligned} \tag{243}$$

must be related to the normalization of the on-shell propagator

$$\begin{aligned}\widetilde{\Delta}^{\rm OS}_F(p^2) &= \int d^4x{\rm e}^{{\rm i}p\cdot x}\, \langle 0|\,{\rm T}\,(\varphi(x)\varphi(0))_{\rm OS}\,|0\rangle\\ &= \frac{\rm i}{p^2-m^2_{\rm pole}+{\rm i}\delta} + \text{terms that are finite for } p^2\to m^2_{\rm pole}\end{aligned} \tag{244}$$

through the ratio of the two wave-function renormalization constants:

$$\langle 0|\,{\rm T}\,(\varphi(x)\varphi(0))_{\overline{\rm MS}}\,|0\rangle = R_{\overline{\rm MS}}\,\langle 0|\,{\rm T}\,(\varphi(x)\varphi(0))_{\rm OS}\,|0\rangle \qquad\Longrightarrow\qquad R_{\overline{\rm MS}} = Z^{\rm OS}_\varphi/Z^{\overline{\rm MS}}_\varphi \tag{245}$$

Let us now turn to the LSZ formula, which was derived in (33) assuming that single-particle states in the far past and in the far future are generated by the field with the same unit normalization as in the free theory – see eq. (42) above. This may be contrasted with the $\overline{\rm MS}$ scheme where $\Big|\langle 0\,|\varphi(0)|\,\lambda_0\rangle\Big| = (R_{\overline{\rm MS}})^{1/2}$. The conclusion is that the LSZ formula as written in (33) is correct only in the on-shell scheme. In the $\overline{\rm MS}$ scheme a single-particle state with unity normalization is generated by $(R_{\overline{\rm MS}})^{-\frac{1}{2}}\,\varphi$, so the LSZ formula in $\overline{\rm MS}$ takes the form:

$$\begin{aligned}\langle f|i\rangle = \left(\mathrm{i}R_{\overline{\rm MS}}^{-\frac{1}{2}}\right)^4 &\int d^dx_1'\mathrm{e}^{\mathrm{i}k_1'\cdot x_1'}(\partial^2_{x_1'}+m^2)\int d^dx_2'\mathrm{e}^{\mathrm{i}k_2'\cdot x_2'}(\partial^2_{x_2'}+m^2)\\ &\int d^dx_1\mathrm{e}^{-\mathrm{i}k_1\cdot x_1}(\partial^2_{x_1}+m^2)\int d^dx_2\mathrm{e}^{-\mathrm{i}k_2\cdot x_2}(\partial^2_{x_2}+m^2)\ \langle 0|\ \mathrm{T}\Big(\varphi(x_1')\,\varphi(x_2')\varphi(x_1)\varphi(x_2)\Big)\,|0\rangle\end{aligned}\tag{246}$$

instead of (33).

To go further we wish to deduce the momentum-space Feynam rules when computing a scattering amplitude in $\overline{\rm MS}$. Here another factor ot $R_{\overline{\rm MS}}$ pops up as follows. Recall that when computing the correlator in configuration space (e.g. $\langle 0|\ \mathrm{T}\Big(\varphi(x_1')\,\varphi(x_2')\varphi(x_1)\varphi(x_2)\Big)\,|0\rangle$ above) one obtains propagators also between the external points, say x_i and the internal vertices z. At leading order this propagator is just $D_F(x_i-z)$, but of course, at higher orders these will be decorated by self-energy insertions thus promoted to be full propagators $\Delta_F(x_i-z)$. When using the LSZ formula to deduce the (momentum-space) Feynman rules for the scattering amplitude we have noticed that the Klein-Gordon operators $(\partial^2_{x_i}+m^2)$ acting on each field remove the corresponding propagator, producing a d-dimentional δ function as follows:

$$(\partial^2_{x_i}+m^2)D_F(x_i-z) = -\mathrm{i}\delta^{(d)}(x_i-z)\,,$$

which was then used to perform the integration over the corresponding position x_i. Consequently, our Feynman rules for the scattering amplitude did not include a propagator for the external leg. This is obviously correct at leading order, but beyond leading order one needs to account for the self-energy corrections on that leg. According to (243), when the Klein-Gordon operator acts on the full $\overline{\rm MS}$ propagator it yields $-\mathrm{i}R_{\overline{\rm MS}}\,\delta^{(d)}(x_i-z)$ instead of $-\mathrm{i}\,\delta^{(d)}(x_i-z)$. Thus for each external leg there is an extra factor of $R_{\overline{\rm MS}}$ accounting for self-energy diagrams on that leg.

This factor comes on top of the $R_{\overline{\rm MS}}^{-\frac{1}{2}}$ which we explicitly written as a prefactor in (246), resulting in a total of $R_{\overline{\rm MS}}^{+\frac{1}{2}}$ for each external leg. Therefore the calculation of a scattering amplitude in the $\overline{\rm MS}$ scheme proceeds similarly to the one in the on-shell scheme (meaning in particular that one needs not include propagators on external legs) except that at the end a factor of $R_{\overline{\rm MS}}^{+\frac{1}{2}}$ needs to be included for each external leg.

14.1 One-loop relation between the on-shell mass and $\overline{\rm MS}$ mass

Starting with (238) and using (234) we get:

$$m^2_{\rm pole}-m^2-\frac{\alpha}{2}\left[\int_0^1 dx\,(m^2-x(1-x)m^2_{\rm pole})\left[1-\ln\left(\frac{m^2-x(1-x)m^2_{\rm pole}}{\mu^2}\right)\right]\right]+\mathcal{O}(\alpha^2)=0\tag{247}$$

We observe that at leading order $m^2_{\rm pole}=m^2+\mathcal{O}(\alpha)$. Therefore, inside the integral of the $\mathcal{O}(\alpha)$ term we may substitute $m^2_{\rm pole}=m^2$ getting

$$m^2_{\rm pole}=m^2\left(1+\frac{\alpha}{2}\left[\frac{17}{9}-\frac{\sqrt{3}\pi}{6}+\frac{5}{6}\ln\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha^2)\right)\tag{248}$$

Now we observe that the lhs is a physical mass, which cannot depend on the renormalization scheme or scale. In contrast the mass on the rhs, m^2, has a different status. It is a parameter in the Lagrangian, but not directly physical. For the relation (248) to be satisfied, it is necessary to acknowledge the fact that depending in the renormalization scale μ^2 chosen, m^2 will take different numerical values. We therefore refer to the renormalized $\overline{\rm MS}$ mass as a running mass: it depends on the scale: $m(\mu)$. This dependence can be read off (248). The easiest way to extract it is to first take the log:

$$\begin{aligned}\ln(m^2_{\rm pole})&=\ln(m^2)+\ln\left(1+\frac{\alpha}{2}\left[\frac{17}{9}-\frac{\sqrt{3}\pi}{6}+\frac{5}{6}\ln\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha^2)\right)\\ &=\ln(m^2)+\frac{\alpha}{2}\left[\frac{17}{9}-\frac{\sqrt{3}\pi}{6}+\frac{5}{6}\ln\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha^2)\end{aligned}\tag{249}$$

and then differentiate:

$$\frac{d\ln(m^2_{\rm pole})}{\ln\mu^2}=\frac{d\ln(m^2)}{d\ln\mu^2}+\frac{5}{12}\alpha+\mathcal{O}(\alpha^2)\tag{250}$$

The lhs must be zero, as explained above, so we deduce:

$$\gamma_m \equiv \frac{1}{m}\frac{dm}{d\ln\mu} = -\frac{5}{12}\alpha + \mathcal{O}(\alpha^2) \tag{251}$$

This equation defines the mass anomalous dimension γ_m. Note that in differentiating (249) we assumed that the coupling may only depend on the scale such that $d\alpha/d\ln\mu^2 = \mathcal{O}(\alpha^2)$ (or higher powers). We shall verify this below.

Before that, let's explain the name anomalous dimension. To this end consider the case where the coupling does not depend on the scale, and solve the equation (251) we get:

$$\ln\frac{m(\mu)}{m(\mu_0)} = \gamma_m \ln\frac{\mu}{\mu_0} \qquad \Longrightarrow \qquad \frac{m(\mu)}{m(\mu_0)} = \left(\frac{\mu}{\mu_0}\right)^{\gamma_m}$$

namely, γ_m determines the scaling of the mass with the scale.

14.2 The residue of the propagator pole in $\overline{\mathrm{MS}}$ at one loop

Next consider the value of the residue given by (239)

$$R_{\overline{\mathrm{MS}}}^{-1} \equiv \frac{d}{dp^2}\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_F^{\overline{\mathrm{MS}}}(p^2)\right)^{-1}\Bigg|_{p^2=m^2_{\mathrm{pole}}} = 1 + \frac{d\Pi^{\overline{\mathrm{MS}}}(p^2)}{dp^2}\Bigg|_{p^2=m^2_{\mathrm{pole}}}, \tag{252}$$

The derivative of $\Pi^{\overline{\mathrm{MS}}}(p^2)$ is

$$\begin{aligned}\frac{d\Pi^{\overline{\mathrm{MS}}}}{dp^2}(p^2) &= -\frac{\alpha}{2}\left[\int_0^1 dx\left[(-x(1-x))\left[1-\ln\left(\frac{m^2-x(1-x)p^2}{\mu^2}\right)\right]\right] - (-x(1-x))\right] + \mathcal{O}(\alpha^2)\\ &= -\frac{\alpha}{2}\left[\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2-x(1-x)p^2}{\mu^2}\right)\right] + \mathcal{O}(\alpha^2)\end{aligned} \tag{253}$$

So at $p^2 = m^2_{\mathrm{pole}}$ we get

$$\begin{aligned}\frac{d\Pi^{\overline{\mathrm{MS}}}}{dp^2}(m^2_{\mathrm{pole}}) &= -\frac{\alpha}{2}\left[\frac{1}{6}\ln\frac{m^2}{\mu^2} + \int_0^1 dx\, x(1-x)\ln\left(1-x(1-x)\right)\right] + \mathcal{O}(\alpha^2)\\ &= -\frac{\alpha}{2}\left[\frac{1}{6}\ln\frac{m^2}{\mu^2} + \frac{\sqrt{3}\pi}{6} - \frac{17}{18}\right] + \mathcal{O}(\alpha^2)\end{aligned} \tag{254}$$

Thus

$$R_{\overline{\mathrm{MS}}}^{-1} = 1 - \alpha\left[\frac{1}{12}\ln\frac{m^2}{\mu^2} + \frac{\sqrt{3}\pi}{12} - \frac{17}{36}\right] + \mathcal{O}(\alpha^2) \tag{255}$$

14.3 Exercises

1. Start from (238) and derive (248).
2. solve
$$\frac{1}{m}\frac{dm}{d\ln\mu} = \gamma_m^{(0)}\alpha \tag{256}$$
for the running mass $m(\mu)$ in two cases:
   (a) Assuming α is constant.
   (b) Assuming α admits the equation:
   $$\frac{d\alpha}{d\ln\mu} = \beta_0\alpha^2$$
   Distinguish the 4 cases corresponding to $\gamma_m^{(0)}$ and β_0 positive and negative.
3. Using the $\overline{\mathrm{MS}}$ for all components (propagator and vertices) derive the one-loop correction to $\mathcal{T}$ corresponding to the $2 \to 2$ scattering process in (203) $\langle k_1' k_2' | k_1 k_2\rangle$ in the small mass limit (keep log-enhanced terms and constant terms).

Guidance:
You may read off the values of the constants κ_φ and κ_g, respectively, in the $\overline{\text{MS}}$ using the results in question 2 of the previous lecture, which dealt with the same scattering process in the on-shell scheme. According to (222) we have:

$$\kappa_\varphi^{\overline{\text{MS}}} = -1 - \ln\frac{\mu^2}{m^2}$$

so the propagator is

$$\widetilde{\Delta}_F^{\overline{\text{MS}}}(p^2)\Big|_{m^2\to 0} = \frac{\mathrm{i}}{p^2\left[1 - \left(-\frac{2}{9} + \frac{1}{12}\ln\left(\frac{-p^2}{\mu^2}\right)\right)\alpha + \mathcal{O}(\alpha^2)\right] + \mathrm{i}\delta}. \tag{257}$$

Similarly, according to (226), we have

$$Z_g = 1 - \alpha\frac{1}{2\epsilon} \tag{258}$$

with

$$\kappa_g^{\overline{\text{MS}}} = -\frac{1}{2}\ln\frac{\mu^2}{m^2}$$

so the one-loop vertex (259) becomes

$$\begin{aligned}\mathrm{i}V_3^{\overline{\text{MS}}}(s)\big|_{m^2\to 0} &= \mathrm{i}g\left(1 - \left[\kappa_g + \int_0^1 dx\int_0^{1-x} dy\ \ln\left(\frac{-xys}{m^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\\ &= \mathrm{i}g\left(1 - \left[-\frac{3}{2} + \frac{1}{2}\ln\left(\frac{-s}{\mu^2}\right)\right]\alpha + \mathcal{O}(\alpha^2)\right)\end{aligned} \tag{259}$$

Recalling the modification of the Feynman rules discussed above, we need to multiply my an extra factor of $R_{\overline{\text{MS}}}^{\frac{1}{2}}$ for each external field, so we deduce that the scattering amplitude (230) becomes:

$$\begin{aligned}\mathcal{T} = -g^2 R_{\overline{\text{MS}}}^2 \Bigg\{&\frac{1}{s}\left(1 + \left[\frac{25}{9} - \frac{11}{12}\ln\left(\frac{-s}{\mu^2}\right)\right]\alpha\right) + \frac{1}{t}\left(1 + \left[\frac{25}{9} - \frac{11}{12}\ln\left(\frac{-t}{\mu^2}\right)\right]\alpha\right) + \frac{1}{u}\left(1 + \left[\frac{25}{9} - \frac{11}{12}\ln\left(\frac{-u}{\mu^2}\right)\right]\alpha\right)\\ &- \alpha\left[\frac{1}{2u}\left(\pi^2 + \ln^2\left(\frac{s}{t}\right)\right) + \frac{1}{2s}\left(\pi^2 + \ln^2\left(\frac{t}{u}\right)\right) + \frac{1}{2t}\left(\pi^2 + \ln^2\left(\frac{u}{s}\right)\right)\right]\Bigg\}.\end{aligned} \tag{260}$$

where $R_{\overline{\text{MS}}}$ is given by (261):

$$R_{\overline{\text{MS}}} = 1 + \alpha\left[\frac{1}{12}\ln\frac{m^2}{\mu^2} + \frac{\sqrt{3}\pi}{12} - \frac{17}{36}\right] + \mathcal{O}(\alpha^2) \tag{261}$$

Comments and additional exercises:

- The fact that the strict zero mass limit cannot be taken – $R_{\overline{\text{MS}}}$ has a logarithmic divergence, and therefore so does $\mathcal{T}$ – reflects an *infrared* divergence. This singularity will only cancel in the corresponding cross section upon including soft real emission, a $2 \to 3$ process, in addition to the $2 \to 2$ one computed here. Additional reading and exercise: read sections 26-27 in Srednicki and show that the observable cross section corresponding to the sum of the $2 \to 2$ and $2 \to 3$ process with a given angular resolution of the detector is indeed finite.
- Note that $\mathcal{T}$ computed in (260) must be independent on μ; this means that the explicit dependence on μ^2 in (260) must cancel with the dependence on μ^2 via the coupling. Therefore, show that the result for $\mathcal{T}$ may be recast as:

  $$\mathcal{T} = \mathcal{T}_{\text{LO}}\alpha + \left(\frac{3}{4}\mathcal{T}_{\text{LO}}\ln(\mu^2) + \mathcal{T}_{\text{NLO}}\right)\alpha^2 + \ldots$$

  and then used

  $$\frac{\partial\mathcal{T}}{\partial\alpha}\frac{d\alpha}{d\ln\mu^2} + \frac{\partial\mathcal{T}}{\partial\ln\mu^2} = 0$$

  to show that

  $$\beta(\alpha) \equiv \frac{d\alpha}{d\ln\mu} = -\frac{3}{2}\alpha^2 + \mathcal{O}(\alpha^3)\,. \tag{262}$$