Exercise 10.17. ROTATION GROUP: GEODESICS AND CONNECTION COEFFICIENTS

[Continuation of exercises 9.13 and 9.14.] In discussing the rotation group, one must make a clear distinction between the *Euclidean space* (coordinates x, y, z; basis vectors $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) in which the rotation matrices act, and the *group manifold* $SO(3)$ (coordinates ψ, θ, ϕ; coordinate basis $\partial/\partial\psi$, $\partial/\partial\theta$, $\partial/\partial\phi$; basis of "generators" $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$), whose points $\mathcal{P}$ are rotation matrices.

(a) Pick a vector

$$\boldsymbol{n} = n^x\, \partial/\partial x + n^y\, \partial/\partial y + n^z\, \partial/\partial z$$

in Euclidean space. Show that

$$\mathcal{R}_{\boldsymbol{n}}(t) \equiv \exp[(n^x \mathcal{K}_1 + n^y \mathcal{K}_2 + n^z \mathcal{K}_3)t] \tag{10.29}$$

is a rotation matrix that rotates the axes of Euclidean space by an angle

$$t|\boldsymbol{n}| = t[(n^x)^2 + (n^y)^2 + (n^z)^2]^{1/2}$$

about the direction $\boldsymbol{n}$. ($\mathcal{K}_j$ are matrices defined in exercise 9.13.)

(b) In the group manifold $SO(3)$, pick a point (rotation matrix) $\mathcal{P}$, and pick a tangent vector $\boldsymbol{u} = u^\alpha \boldsymbol{e}_\alpha$ at $\mathcal{P}$. Let $\boldsymbol{u}$ be a vector in Euclidean space with the same components as $\boldsymbol{u}$ has in $SO(3)$:

$$\boldsymbol{u} = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3; \qquad \boldsymbol{u} = u^1\, \partial/\partial x + u^2\, \partial/\partial y + u^3\, \partial/\partial z. \tag{10.30}$$

Show that $\boldsymbol{u}$ is the tangent vector (at $t = 0$) to the curve

$$\mathcal{C}(t) = \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}. \tag{10.31}$$

The curve $\mathcal{C}(t)$ through the arbitrary point $\mathcal{P}$ with arbitrary tangent vector $\boldsymbol{u} = (d\mathcal{C}/dt)_{t=0}$ is a very special curve: every point on it differs from $\mathcal{P}$ by a rotation $\mathcal{R}_{\boldsymbol{u}}(t)$ about one and the same direction $\boldsymbol{u}$. No other curve in $SO(3)$ with "starting conditions" $\{\mathcal{P}, \boldsymbol{u}\}$ has such beautiful simplicity. Hence it is natural to decree that each such $\mathcal{C}(t)$ is a geodesic of the group manifold $SO(3)$. This decree adds new geometric structure to $SO(3)$; it converts $SO(3)$ from a differentiable manifold into something more special: an *affine manifold*.

One has no guarantee that an arbitrarily chosen family of curves in an arbitrary manifold *can* be decreed to be geodesics. Most families of curves simply do not possess the right geometric properties to function as geodesics. Most will lead to covariant derivatives that violate one or more of the fundamental conditions (10.2). To learn whether a given choice of geodesics is possible, one can try to derive connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ (for some given basis) corresponding to the chosen geodesics. If the derivation is successful, the choice of geodesics was a possible one. If the derivation produces inconsistencies, the chosen family of curves have the wrong geometric properties to function as geodesics.

(c) For the basis of generators $\{\boldsymbol{e}_\alpha\}$ derive connection coefficients corresponding to the chosen geodesics, $\mathcal{C}(t) = \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}$, of $SO(3)$. *Hint:* show that the components $u^\alpha = \langle \boldsymbol{\omega}^\alpha, \boldsymbol{u}\rangle$ of the tangent $\boldsymbol{u} = d\mathcal{C}/dt$ to a given geodesic are independent of position $\mathcal{C}(t)$ along the geodesic. Then use the geodesic equation $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0$, expanded in the basis $\{\boldsymbol{e}_\alpha\}$, to calculate the symmetric part of the connection $\Gamma^\alpha{}_{(\beta\gamma)}$. Finally use equation (10.23) to calculate $\Gamma^\alpha{}_{[\beta\gamma]}$. [*Answer:*

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\epsilon_{\alpha\beta\gamma}, \tag{10.32}$$

where $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric symbol with $\epsilon_{123} = +1$. This answer is independent of location $\mathcal{P}$ in $SO(3)$!]