First of all, I assume that ϕ is a function of z

So.

$$I = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2} d\lambda.$$

Now create a path near to $z(\lambda)$. called $z'(\lambda)$ where $z'^\alpha = z^\alpha + \varepsilon^\alpha(\lambda)$

so that ε is a function of λ such that $\varepsilon = 0$ $\lambda_1 > \varepsilon > \lambda_2$ and $\varepsilon^\alpha(\lambda) \ll z^\alpha(\lambda)$

and $dz'^\alpha = dz^\alpha + d\varepsilon^\alpha$

Then,

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda} + \frac{d\varepsilon^\alpha}{d\lambda}\right)\left(\frac{dz^\beta}{d\lambda} + \frac{d\varepsilon^\beta}{d\lambda}\right)\right)^{1/2} d\lambda.$$

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda} + \frac{dz^\alpha}{d\lambda}\frac{d\varepsilon^\beta}{d\lambda} + \frac{dz^\beta}{d\lambda}\frac{d\varepsilon^\alpha}{d\lambda} + \frac{d\varepsilon^\alpha}{d\lambda}\frac{d\varepsilon^\beta}{d\lambda}\right)\right)^{1/2} d\lambda.$$

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + \frac{\frac{dz^\alpha}{d\lambda}\frac{d\varepsilon^\beta}{d\lambda}}{\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}} + \frac{\frac{dz^\beta}{d\lambda}\frac{d\varepsilon^\alpha}{d\lambda}}{\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}} + \frac{\frac{d\varepsilon^\alpha}{d\lambda}\frac{d\varepsilon^\beta}{d\lambda}}{\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}}\right)\right)^{1/2} d\lambda$$

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + \frac{d\varepsilon^\beta}{dz^\beta} + \frac{d\varepsilon^\alpha}{dz^\alpha} + \frac{d\varepsilon^\alpha}{dz^\alpha}\frac{d\varepsilon^\beta}{dz^\beta}\right)\right)^{1/2} d\lambda.$$

If $\varepsilon^\alpha \ll z^\alpha$ we can ignore $\frac{d\varepsilon^\alpha}{dz^\alpha}\frac{d\varepsilon^\beta}{dz^\beta}$ and we can switch indices on $\frac{d\varepsilon^\alpha}{dz^\beta}$ to give

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + 2\frac{d\varepsilon^\alpha}{dz^\beta}\right)\right)^{1/2} d\lambda.$$

Then we can expand the square root expression to give

$$I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z')}\left(-\eta_{\alpha\beta}\cdot\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + \frac{d\varepsilon^\alpha}{dz^\beta}\right)\right) d\lambda.$$

Turning to $e^{\phi(z')}$; $\quad \phi(z') = \phi(z) + \varepsilon\cdot\nabla\phi$

$$\therefore\ e^{\phi(z')} = e^{\phi(z) + \varepsilon\cdot\nabla\phi} = e^{\phi(z)} \times e^{\varepsilon\cdot\nabla\phi} = e^{\phi(z)}(1 + \varepsilon\cdot\nabla\phi)$$

$$\therefore\ I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left((1 + \varepsilon\cdot\nabla\phi)\left(-\eta_{\alpha\beta}\left(\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + \frac{d\varepsilon^\alpha}{dz^\beta}\right)\right)\right) d\lambda.$$

$$\therefore \quad I' = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(1 + \frac{d\varepsilon^\alpha}{dz^\beta} + \varepsilon\cdot\nabla\phi + \varepsilon\cdot\nabla\phi\,\frac{d\varepsilon^\alpha}{dz^\beta}\right)d\lambda.$$

$$\text{So} \quad I' - I = -m\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}\left(\frac{d\varepsilon^\alpha}{dz^\beta} + \varepsilon\cdot\nabla\phi + \frac{d\varepsilon^\alpha}{dz^\beta}\,\varepsilon\cdot\nabla\phi\right)d\lambda$$

$$\therefore \quad \delta I = -m\int_{\lambda_1}^{\lambda_2}\frac{d\varepsilon^\alpha}{dz^\beta}\, e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda$$

$$-m\int_{\lambda_1}^{\lambda_2}\varepsilon\cdot\nabla\phi\; e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda$$

$$-m\int_{\lambda_1}^{\lambda_2}\frac{d\varepsilon^\alpha}{dz^\beta}\,\varepsilon\cdot\nabla\phi\; e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda.$$

Each of these ~~equa~~ integrals is of the form $-m\int_{\lambda_1}^{\lambda_2} f(\lambda)\, e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda.$

so integrating by parts we get:

$$-m\left[f(\lambda)\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda\right]_{\lambda_1}^{\lambda_2} + m\int_{\lambda_1}^{\lambda_2}\frac{d}{d\lambda}f(\lambda)\cdot\int_{\lambda_1}^{\lambda_2} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda$$

But $\displaystyle\int_{\lambda_1}^{\lambda} e^{\phi(z)}\left(-\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2}d\lambda = I$

So the integral becomes.

$$-m\Big[f(\lambda)\, I\Big]_{\lambda_1}^{\lambda_2} + m\int_{\lambda_1}^{\lambda_2}\frac{d}{d\lambda}\left(f(\lambda)\right) I\, d\lambda.$$

$$= -m\Big[f(\lambda)\, I\Big]_{\lambda_1}^{\lambda_2} + m\Big[f(\lambda)\, I\Big]_{\lambda_1}^{\lambda_2} = 0.$$

So ~~I~~ have proved that $\delta I = 0$, but it hasn't produced any differential equations governing the particles' motion.