$$I = \int (p_\mu dx^\mu - \mathcal{H} d\lambda)$$

$$\frac{\delta I}{\delta p_\mu} = \int \left(dx^\mu - \frac{\partial \mathcal{H}}{\partial p^\mu} d\lambda \right) = 0 \qquad \underline{\frac{dx^\mu}{d\lambda} - \frac{\partial \mathcal{H}}{\partial p_\mu} = 0}$$

$$\frac{\delta I}{\delta x^\mu} = \int \left(\frac{\partial p_\mu}{\partial x^\mu} dx^\mu - \frac{\partial \mathcal{H}}{\partial x^\mu} d\lambda \right) = 0 \qquad \underline{\frac{dp_\mu}{d\lambda} - \frac{\partial \mathcal{H}}{\partial x^\mu} = 0}$$

$$\frac{dx^\mu}{d\lambda} = p^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu}$$

$$g^{\mu\nu} p_\nu dp_\mu = d\mathcal{H}$$

$$\int g^{\mu\nu} p_\nu dp_\mu = \int d\mathcal{H} + C$$

$$g^{\mu\nu} p_\nu p_\mu = \mathcal{H} + \text{Constant}$$

$$2\mathcal{H} - \mathcal{H} = \text{Constant}$$

$$\underline{\mathcal{H} = \text{Constant}}$$

$$\mathcal{L} = p_\mu \frac{dx^\mu}{d\lambda} - \mathcal{H} = p_\mu p^\mu - \mathcal{H} = g_{\mu\nu} p^\nu p^\mu - \mathcal{H}$$

$$\frac{\delta \mathcal{L}}{\delta x^\sigma} = g_{\mu\nu,\sigma} p^\nu p^\mu - \frac{\partial \mathcal{H}}{\partial x^\sigma} = \underline{g_{\mu\nu,\sigma} p^\nu p^\mu - \frac{dp_\sigma}{d\lambda}}$$

$$\frac{\delta \mathcal{L}}{\delta p^\sigma} = g_{\mu\sigma} p^\mu + g_{\sigma\nu} p^\nu - \frac{\partial \mathcal{H}}{\partial p^\sigma}$$

$$\frac{\partial \mathcal{H}}{\partial p^\sigma} = \frac{\partial \mathcal{H}}{\partial p_\mu} \frac{\partial p_\mu}{\partial p^\sigma} \qquad p_\mu = g_{\mu\sigma} p^\sigma \qquad \frac{\partial p_\mu}{\partial p^\sigma} = g_{\mu\sigma}$$

$$\therefore \frac{\partial \mathcal{H}}{\partial p^\sigma} = g_{\mu\sigma} \frac{\partial \mathcal{H}}{\partial p_\mu} = g_{\mu\sigma} \frac{dx^\mu}{d\lambda}$$

$$\therefore \frac{\delta \mathcal{L}}{\delta p^\sigma} = 2 g_{\mu\sigma} p^\mu - g_{\mu\sigma} \frac{dx^\mu}{d\lambda}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial p^\sigma}\right) = 2\,\frac{d}{d\lambda}(g_{\mu\sigma})\,p^\mu + 2g_{\mu\sigma}\frac{dp^\mu}{d\lambda} - \frac{d}{d\lambda}(g_{\mu\sigma})\,p^\mu - g_{\mu\sigma}\frac{dp^\mu}{d\lambda}$$

$$= \frac{d}{d\lambda}(g_{\mu\sigma})\,p^\mu + g_{\mu\sigma}\frac{dp^\mu}{d\lambda}.$$

$$\left[\ \frac{d}{d\lambda}(g_{\mu\sigma}) = \frac{\partial g_{\mu\sigma}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} = g_{\mu\sigma,\alpha}\,p^\alpha\ \right].$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial p^\sigma}\right) = \underline{g_{\mu\sigma,\alpha}\,p^\alpha p^\mu + g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}}.$$

$$\frac{\partial L}{\partial x^\sigma} = g_{\mu\nu,\sigma}\,p^\nu p^\mu - \frac{dp_\sigma}{d\lambda}.$$

$$\left[\ \frac{dp_\sigma}{d\lambda} = \frac{d}{d\lambda}(g_{\sigma\alpha}p^\alpha) = \frac{d}{d\lambda}(g_{\sigma\alpha})\,p^\alpha + g_{\sigma\alpha}\frac{dp^\alpha}{d\lambda}\right.$$

$$\left. = g_{\sigma\alpha,\mu}\,p^\alpha p^\mu + g_{\sigma\alpha}\frac{d^2x^\alpha}{d\lambda^2}.\ \right]$$

$$\frac{\partial L}{\partial x^\sigma} = g_{\mu\nu,\sigma}\,p^\nu p^\mu - g_{\sigma\alpha,\mu}\,p^\alpha p^\mu - g_{\sigma\alpha}\frac{d^2x^\alpha}{d\lambda^2}$$

$$\frac{\partial L}{\partial x^\sigma} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial p^\sigma}\right) = 0 = g_{\mu\nu,\sigma}\,p^\mu p^\nu - g_{\sigma\alpha,\mu}\,p^\alpha p^\mu - g_{\sigma\alpha}\frac{d^2x^\alpha}{d\lambda^2}$$

$$- g_{\mu\sigma,\alpha}\,p^\alpha p^\mu - g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}$$

$$0 = g_{\mu\nu,\sigma}\,p^\mu p^\nu - g_{\sigma\nu,\mu}\,p^\nu p^\mu - g_{\mu\sigma,\nu}\,p^\nu p^\mu - 2g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}$$

$$= -2\Gamma_{\mu\nu\sigma}\,p^\mu p^\nu - 2g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}.$$

$$= g^{\alpha\sigma}\Gamma_{\mu\nu\sigma}\,p^\mu p^\nu + g^{\alpha\sigma}g_{\mu\sigma}\frac{d^2x^\mu}{d\lambda^2}$$

$$= \Gamma^\alpha_{\mu\nu}\,p^\mu p^\nu + \delta^\alpha_\mu\frac{d^2x^\mu}{d\lambda^2} = \underline{\Gamma^\alpha_{\mu\nu}\,p^\mu p^\nu + \frac{d^2x^\alpha}{d\lambda^2}}$$

To resolve the issue of what the various values of $\mathcal{H}$ represent, I have considered a simple example of motion along a single dimension 'x' on a space time diagram.

If $\frac{1}{2} g_{\mu\nu} p^\mu p^\nu = \mathcal{H}$.

Then when $\mathcal{H} = 0$ $\quad g_{00} p^0 p^0 + g_{11} p^1 p^1 = 0$

$$(-1)\frac{d\cancel{t}}{d\lambda}\frac{d\cancel{t}}{d\lambda} + (+1)\frac{d\cancel{x}}{d\lambda}\frac{d\cancel{x}}{d\lambda} = 0.$$

$$\therefore \frac{d\cancel{t}}{d\lambda}\frac{d\cancel{t}}{d\lambda} = \frac{d\cancel{x}}{d\lambda}\frac{d\cancel{x}}{d\lambda}. \qquad \frac{dx}{dt} = 1. \Rightarrow \text{photon}$$

When $\mathcal{H} = -\frac{1}{2}$

$$(-1)\frac{dt}{d\lambda}\frac{dt}{d\lambda} + \frac{dx}{d\lambda}\frac{dx}{d\lambda} = -1$$

$$\frac{dx}{d\lambda}\frac{dx}{d\lambda} = \frac{dt}{d\lambda}\frac{dt}{d\lambda} - 1.$$

$$\text{ie } \left(\frac{dx}{d\lambda}\right)\left(\frac{dx}{d\lambda}\right) < \frac{dt}{d\lambda}\frac{dt}{d\lambda} \qquad \frac{dx}{dt} < \pm 1 \Rightarrow \text{timelike}$$

When $\mathcal{H} = +\frac{1}{2}$

$$\left(\frac{dx}{d\lambda}\right)\left(\frac{dx}{d\lambda}\right) = \left(\frac{dt}{d\lambda}\right)\left(\frac{dt}{d\lambda}\right) + 1 \qquad \left(\frac{dx}{d\lambda}\frac{dx}{d\lambda}\right) > \left(\frac{dt}{d\lambda}\frac{dt}{d\lambda}\right); \ \frac{dx}{dt} > \pm 1 \Rightarrow \text{spacelike}$$

When $\mathcal{H} = -\frac{1}{2}\mu^2$

$$-\frac{dt}{d\lambda}\frac{dt}{d\lambda} + \frac{dx}{d\lambda}\frac{dx}{d\lambda} = -\mu^2.$$

$$-\frac{dt}{d\tau}\frac{dt}{d\tau} + \frac{dx}{d\tau}\frac{dx}{d\tau} = -1 \qquad \text{as } \frac{d}{d\lambda} = \mu\frac{d}{d\tau} \qquad \frac{dx}{dt} < \pm 1 \text{ spacelike}$$

mass independent.