It turns out that the isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ is independent of path if and only if the fundamental group is abelian. (See Exercise 3.) This is a stringent requirement on the space X.

Definition. A space X is said to be ***simply connected*** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma 52.3. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Proof. Let α and β be two paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0. Then

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta],$$

from which it follows that $[\alpha] = [\beta]$. ■

It is intuitively clear that the fundamental group is a topological invariant of the space X. A convenient way to prove this fact formally is to introduce the notion of the "homomorphism induced by a continuous map."

Suppose that $h : X \to Y$ is a continuous map that carries the point x_0 of X to the point y_0 of Y. We often denote this fact by writing

$$h : (X, x_0) \longrightarrow (Y, y_0).$$

If f is a loop in X based at x_0, then the composite $h \circ f : I \to Y$ is a loop in Y based at y_0. The correspondence $f \to h \circ f$ thus gives rise to a map carrying $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$. We define it formally as follows:

Definition. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the ***homomorphism induced by h***, relative to the base point x_0.

The map h_* is well-defined, for if F is a path homotopy between the paths f and f', then $h \circ F$ is a path homotopy between the paths $h \circ f$ and $h \circ f'$. The fact that h_* is a homomorphism follows from the equation

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

The homomorphism h_* depends not only on the map $h : X \to Y$ but also on the choice of the base point x_0. (Once x_0 is chosen, y_0 is determined by h.) So some notational difficulty will arise if we want to consider several different base points for X. If x_0 and x_1 are two different points of X, we cannot use the same symbol h_* to stand for two different homomorphisms, one having domain $\pi_1(X, x_0)$ and the other having domain $\pi_1(X, x_1)$. Even if X is path connected, so these groups are isomorphic, they are still not the same group. In such a case, we shall use the notation

$$(h_{x_0})_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

for the first homomorphism and $(h_{x_1})_*$ for the second. If there is only one base point under consideration, we shall omit mention of the base point and denote the induced homomorphism merely by h_*.

The induced homomorphism has two properties that are crucial in the applications. They are called its "functorial properties" and are given in the following theorem:

Theorem 52.4. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. The proof is a triviality. By definition,

$$(k \circ h)_*([f]) = [(k \circ h) \circ f],$$
$$(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)].$$

Similarly, $i_*([f]) = [i \circ f] = [f]$. ■

Corollary 52.5. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Proof. Let $k : (Y, y_0) \to (X, x_0)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, x_0); and $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, y_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is the inverse of h_*. ■

Exercises

1. A subset A of $\mathbb{R}^n$ is said to be ***star convex*** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.
   (a) Find a star convex set that is not convex.
   (b) Show that if A is star convex, A is simply connected.
2. Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.