Question: Let V be an infinite-dimensional vector space. Prove that there exists a bijection between any two bases of V, and so all bases of a vector space have the same cardinality. Hint: Use transfinite induction.

My Solution:

Let G be a generator of V and D a linearly independent subset of V. By the Well-Ordering Theorem, G and D are well-ordered by some order relations $<_D$ and $<_G$, respectively. So let us write $G = \{v_1, v_2, ...\}$, $D = \{w_1, w_2, ...\}$, where v_1 is the smallest element in G, v_2 the smallest element in $G - \{v_1\}$, v_3 the smallest element in $G - \{v_1, v_2\}$, etc… and similarly for D. Now the set

$$B_1 = \{w_1, v_1, v_2, ...\}$$

is linearly dependent (since w_1 is a linear combination of the v_i) and also spans V. Consequently, one of the vectors in B_1 is a linear combination of the preceding vectors (if no such vector exists, then B_1 would be linearly independent, a contradiction). This vector cannot be w_1 so it must be from G. By continuing to well-order G (and listing the elements of the G in order in B_1), there will be $v_{m_1} \in G$ that is a linear combination of all the preceding vectors. Consequently, $B_1' = B_1 - \{v_{m_1}\}$ spans V. Note that $|B_1'| = |G|$ since we added one element to G (namely w_1) and took one out (namely v_{m_1}).

We repeat the argument with w_2. That is the set

$$B_2 = \{w_1, w_2, v_1, v_2, ...\}$$

is linearly dependent and also spans V. Again, one of the vectors in B_2 is a linear combination of the preceding vectors. This vector cannot be w_1 or w_2, since D is linearly independent; hence it must be from G, say v_{m_2}. Consequently, $B_2' = B_2 - \{v_{m_2}\}$ spans V. Note that $|B_2'| = |G|$.

We repeat this argument with w_3 and so forth. On the n^{th} step, we have

$$B_{n-1}' = \{w_1, ..., w_{n-1}, v_1, v_2, ...\},$$

a linearly dependent subset that spans V, with $|B_{n-1}'| = |G|$. Then

$$B_n = \{w_1, ..., w_{n-1}, w_n, v_1, v_2, ...\}$$

is linearly dependent and spans V. So one of the vectors in B_n is a linear combination of the preceding vectors. This vector cannot be from any of $w_1, ..., w_n$ since D is linearly independent; hence it must be from G, say v_{m_n}. Consequently, $B_n' = B_n - \{v_{m_n}\}$ spans V, with $|B_n'| = |G|$.

Let $D_0 \subset D$ be set of all elements of D such that $D_0 \cup G'$, where $G' \subset G$, spans V and $|D_0 \cup G'| = |G|$. We have just shown above that if, for any $w_n \in D$, the section $S_{w_n} = \{ w \in D | w <_D w_n \}$ is a subset of D_0, then $w_n \in D_0$. Thus D_0 is inductive. Since D is a well-ordered set, then by the Principle of Transfinite Induction, we have $D_0 = D$. Thus $D \cup G'$ spans V (for some $G' \subset G$), with $|D \cup G'| = |G|$. Thus we have $|D| \leq |G|$.

Suppose now that A and B are two bases for V. Since A is linearly independent and B generates V, then we must have $|A| \leq |B|$. Similarly, since B is linearly independent and A generates V, we must have $|B| \leq |A|$. Then by the Schroeder-Bernstein Theorem, we must have $|A| = |B|$. Thus there is a bijection between any two bases of V.