If $K = \varnothing$, then trivially $K \bigcup B = B$ is a basis for *V*. Let $B = \{b_i \mid i \in I\}$, where *I* is an index set (not necessarily finite or countable). By the Well-Ordering Principle, there exists an order relation $<_V$ on *V* that is a well-ordering. Let *U* be the set of all vectors in *V* such that for any linearly independent subset *K* of *V* containing vectors only from *U*, there exists a subset *S* of *B* so that $K \bigcup S$ is a basis for *V*. Note that we have at least $U = B$ since if $K \subset B$, which is a linearly independent set, then $K \bigcup (B - K) = B$ is a basis for *V*.

Suppose that $S_v \subset U$, where S_v is the section

$$S_v = \{\alpha \in V \mid \alpha <_V v\}.$$

Let *K* be any linearly independent subset of *V* containing *v*, with all other vectors in *K* only from *U*. Now $K - \{v\}$ is a linearly independent subset of *V* since it is a subset of the linearly independent set *K*, and since $K - \{v\}$ contains only vectors from *U*, then by definition of *U* there exists a subset *S* of *B* such that $(K - \{v\}) \bigcup S$ is a basis for *V*. In particular, *v* itself can be expressed (uniquely) as a linear combination of vectors from $(K - \{v\}) \bigcup S$. Consequently, writing $K - \{v\} = \{v_l \mid l \in L\}$ and $S = \{v_p \mid p \in P\}$, we have

$$v = \sum_{l \in L} a_l v_l + \sum_{p \in P} a_p v_p, \quad a_l, a_p \in F,$$

where only finitely many of the a_l and a_p are nonzero. Now at least one of the a_p is nonzero (or else we would have $v = \sum_{l \in L} a_l v_l$, contradicting the assumption that *K*, which contains *v*, is linearly independent), say a_{p_0}. Hence, since $a_{p_0} \in F$ so that ${a_{p_0}}^{-1}$ exists, $s_{p_0} = {a_{p_0}}^{-1} v - \sum_{l \in L} {a_{p_0}}^{-1} a_l v_l - \sum_{p \in P, p \neq p_0} {a_{p_0}}^{-1} a_p v_p$ is a linear combination of vectors from $K \bigcup (S - \{v_{p_0}\})$. It follows that

$$V = \operatorname{span}\left((K - \{v\}) \bigcup S\right) = \operatorname{span}\left(K \bigcup (S - \{v_{p_0}\})\right).$$

Now $S - \{v_{p_0}\} \subset B$ since $S \subset B$. Thus we have found a subset of *B*, namely $S - \{v_{p_0}\}$, such that its union with *K* spans *V*.

To show that $K \bigcup (S - \{v_{p_0}\})$ is actually a basis for *V*, we must now show that $K \bigcup (S - \{v_{p_0}\})$ is linearly independent. Suppose that

$$av + \sum_{l \in L} a_l v_l + \sum_{p \in P, p \neq p_0} a_p v_p = 0.$$

If $a = 0$, then all the vectors remaining in the above equation are from $(K - \{v\}) \bigcup S$ and so all the a_l and a_p are zero by the linear independence of $(K - \{v\}) \bigcup S$. If on the other hand $a \neq 0$, then we have $v = -\sum_{l \in L} a^{-1} a_l v_l - \sum_{p \in P, p \neq p_0} a^{-1} a_p v_p$. This linear combination uses only vectors from $(K - \{v\}) \bigcup S$, which we know is a basis for *V*. Such a representation is thus unique. However, we know from before that $v = \sum_{l \in L} a_l v_l + \sum_{p \in P} a_p v_p$, which is another representation of *v* using vectors from the same basis $(K - \{v\}) \bigcup S$, but this representation is different since this it uses v_{p_0} (recall that $a_{p_0} \neq 0$) while the previous representation of *v* doesn't. This contradiction means that *a* must be zero. Thus all the a_l and a_p are zero, and so $K \bigcup (S - \{v_p\})$ is linearly independent.

Thus $K \bigcup (S - \{s_p\})$ is a basis for *V*, with $S - \{v_{p_0}\} \subset B$, and so by definition of *U*, we have $v \in U$. Consequently, *U* is an inductive subset of *V*. By the principle of transfinite induction, we thus have $U = V$. Hence we have shown that for any linearly independent subset *K* of *V*, there exists a subset *S* of *B* such that $K \bigcup S$ is a basis for *V*.