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Theorem 1 (Nothing but relativity, 1+1 dimensions)**.** *Suppose that G is a subgroup of $\mathrm{GL}(\mathbb{R}^2)$ such that*

(1) There's a $V : G \to \mathbb{R}$ such that

(a) For all $\Lambda \in G$, $V(\Lambda) = \frac{(\Lambda^{-1})_{10}}{(\Lambda^{-1})_{00}}$.

(b) 0 is an interior point of $V(G)$.

(2) For all $\mu \in \{0,1\}$, we have $(\Lambda^{-1})_{\mu\mu} = \Lambda_{\mu\mu}$.

Then there's a $K \geq 0$ such that

$$G = \left\{ \frac{\sigma}{\sqrt{1-Kv^2}} \begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix} \middle|\, v \in \mathbb{R},\, 1-Kv^2 > 0,\, (\sigma,\rho) \in S \right\},$$

where S is one of the sets

$$\begin{aligned} S_r &= \{(1,1)\} \\ S_p &= \{(1,1),(-1,1)\} \\ S_o &= \{(1,1),(1,-1)\} \\ S_n &= \{(1,1),(-1,-1)\} \\ S_f &= \{-1,1\} \times \{-1,1\}. \end{aligned}$$

We will state and prove a number of lemmas that together imply that this statement is a theorem.

Definition 2 (Linear relativistic group)**.** A subgroup $G \subset \mathrm{GL}(\mathbb{R}^2)$ is said to be a *linear relativistic group* if

(1) There's a $V : G \to \mathbb{R}$ such that

(a) For all $\Lambda \in G$, $V(\Lambda) = \frac{(\Lambda^{-1})_{10}}{(\Lambda^{-1})_{00}}$.

(b) 0 is an interior point of $V(G)$.

(2) For all $\mu \in \{0,1\}$, we have $(\Lambda^{-1})_{\mu\mu} = \Lambda_{\mu\mu}$.

Lemma 3 (Members of a linear relativistic group)**.** *If G is a linear relativistic group, then there's a $K \in \mathbb{R}$ such that*

$$G \subset \left\{ \frac{\sigma}{\sqrt{1-Kv^2}} \begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix} \middle|\, (\sigma,\rho) \in \{-1,1\},\, v \in \mathbb{R},\, 1-Kv^2 > 0 \right\}.$$

Proof. Let $\Lambda \in G$ be arbitrary. Denote its components by a, b, c, d.

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \Lambda^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1}$$

Note that

$$V(\Lambda) = \frac{(\Lambda^{-1})_{10}}{(\Lambda^{-1})_{00}} = -\frac{c}{d}, \quad V(\Lambda^{-1}) = \frac{\Lambda_{10}}{\Lambda_{00}} = \frac{c}{a}, \tag{2}$$

and that this implies that $a \neq 0$, $d \neq 0$. Assumption 2 in the definition of "linear relativistic group" implies that

$$a = \frac{d}{\det \Lambda}, \quad d = \frac{a}{\det \Lambda} = \frac{d}{(\det \Lambda)^2}. \tag{3}$$

Since $d \neq 0$, this implies that $\det \Lambda = \pm 1$.

Define $\rho = \det \Lambda$, $\gamma = |a|$, $\sigma = \operatorname{sgn}(a)$, $\alpha = b/a$ and $v = -\rho c/a$. Note that since $d = \rho a$, this ensures that $v = -c/d = V(\Lambda)$. Also note that $\sigma, \rho \in \{-1, 1\}$.

$$\Lambda = \begin{pmatrix} a & b \\ c & \rho a \end{pmatrix} = a \begin{pmatrix} 1 & b/a \\ c/a & \rho \end{pmatrix} = \sigma\gamma \begin{pmatrix} 1 & \alpha \\ -\rho v & \rho \end{pmatrix}. \tag{4}$$

Let $\Lambda', \Lambda'' \in G$ be arbitrary.

$$\begin{aligned} G \ni \Lambda'\Lambda'' &= \sigma'\sigma''\gamma'\gamma'' \begin{pmatrix} 1 & \alpha' \\ -\rho' v' & \rho' \end{pmatrix} \begin{pmatrix} 1 & \alpha'' \\ -\rho'' v'' & \rho'' \end{pmatrix} \\ &= \sigma'\sigma''\gamma'\gamma'' \begin{pmatrix} 1 - \alpha'\rho'' v'' & \alpha'' + \alpha'\rho'' \\ -\rho' v' - \rho'\rho'' v'' & -\rho' v' \alpha'' + \rho'\rho'' \end{pmatrix} \end{aligned} \tag{5}$$

$$\begin{aligned} \rho'\rho'' = (\det \Lambda')(\det \Lambda'') = \det(\Lambda'\Lambda'') &= \frac{(\Lambda'\Lambda'')_{11}}{(\Lambda'\Lambda'')_{00}} \\ &= \frac{-\rho' v' \alpha'' + \rho'\rho''}{1 - \alpha'\rho'' v''}. \end{aligned} \tag{6}$$

If $\rho'\rho'' = 1$, we have $\rho' = \rho''$ and

$$1 = \frac{-\rho' v' \alpha'' + 1}{1 - \alpha'\rho'' v''}. \tag{7}$$

This is equivalent to $\alpha'\rho'' v'' = \rho' v' \alpha''$, and therefore also to $\alpha' v'' = v' \alpha''$. If $\rho'\rho'' = -1$, we have $\rho' = -\rho''$ and

$$-1 = \frac{-\rho' v' \alpha'' - 1}{1 - \alpha'\rho'' v''}. \tag{8}$$

This is equivalent to $\alpha'\rho''v'' = -\rho'v'\alpha''$, and therefore also to $\alpha'v'' = v'\alpha''$. So $\alpha'v'' = v'\alpha''$ for all $\Lambda', \Lambda'' \in G$. Since assumption 1b in the definition of "linear relativistic group" implies that there's a $\Lambda'' \in G$ such that $v'' \neq 0$, this result implies that both of the following statements are true.

(1) For all $\Lambda' \in G$, if $v' = 0$, then $\alpha' = 0$.

(2) For all $\Lambda', \Lambda'' \in G$ such that $v' \neq 0$ and $v'' \neq 0$, we have $\alpha''/v'' = \alpha'/v'$.

(2) implies that α'/v' has the same value for all $\Lambda' \in G$ such that $v' \neq 0$. Denote this value by $-K$. We have $\alpha' = -Kv'$ for all $\Lambda' \in G$ such that $v' \neq 0$. This result and (1) together imply that $\alpha = -Kv$.

The results we have obtained so far imply that

$$\Lambda = \sigma\gamma \begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix}, \quad \Lambda^{-1} = \frac{\sigma}{\gamma(1-Kv^2)} \begin{pmatrix} 1 & \rho K v \\ v & \rho \end{pmatrix}. \tag{9}$$

Assumption 2 in the definition of "linear relativistic group" implies that

$$\sigma\gamma = \frac{\sigma}{\gamma(1-Kv^2)}. \tag{10}$$

If $K > 0$, this implies that $1 - Kv^2 > 0$ (because $\gamma^2 > 0$). If $K \leq 0$, then $1 - Kv^2 > 0$ is obviously true. Since $\sigma \neq 0$ and $\gamma = |a| > 0$, the result above implies that

$$\gamma = \frac{1}{\sqrt{1-Kv^2}}. \tag{11}$$

Now we can write down the final result for Λ.

$$\Lambda = \frac{\sigma}{\sqrt{1-Kv^2}} \begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix}. \tag{12}$$

□ □

Note that if $K > 0$ and we define $c = 1/\sqrt{K}$, the inequality $1 - Kv^2 > 0$ is equivalent to $v \in (-c, c)$.

We will continue to use the notation for components of members of G that we used in the proof above. For example, $V(\Lambda')$ will be denoted by v'. We will also use the notation $c = 1/\sqrt{|K|}$.

Definition 4 (Relativity)**.** If G is a linear relativistic group, then the value of $-\alpha/v$ for all $\Lambda \in G$ such that $v \neq 0$ is called the *relativity* of G.

Lemma 5 (A formula for the relativity)**.** *If G is a linear relativistic group, and K is its relativity, then for all $\Lambda \in G$ such that $v \neq 0$,*

$$K = \frac{1}{v^2}\left(1 - \frac{1}{(\Lambda_{00})^2}\right).$$

Proof. Let $\Lambda \in G$ be such that $v \neq 0$. (Assumption 1b in the definition of "linear relativistic group" implies that there's such a Λ). Lemma 3 tells us that

$$\gamma = \frac{1}{\sqrt{1 - Kv^2}}. \tag{13}$$

Since $\gamma = |\Lambda_{00}| > 0$, this is equivalent to

$$1 - Kv^2 = \frac{1}{\gamma^2} = \frac{1}{(\Lambda_{00})^2}, \tag{14}$$

which is clearly equivalent to the desired result. □

Our next goal is to prove that if K is the relativity of a linear relativistic group, then $K \geq 0$. Our strategy will be to prove that if G is a linear relativistic group with relativity $K < 0$, then the following statements are true.

- There's no $\Lambda \in G$ such that $\rho = 1$ and $v = c$.
- There's a $\Lambda \in G$ and a $n \in \mathbb{Z}^+$ such that $\det(\Lambda^n) = 1$ and $V(\Lambda^n) = c$.

This contradiction will allow us to rule out the possibility that $K < 0$.

Lemma 6 (If $K < 0$, there's no $\Lambda \in G$ such that $\rho = 1$ and $v = c$)**.** *Let G be a linear relativistic group, and denote its relativity by K. If $K < 0$, then there's no $\Lambda \in G$ such that $\rho = 1$ and $v = c$.*

Proof. Let $K < 0$ be arbitrary. For all $\Lambda, \Lambda' \in G$,

$$\begin{aligned}\Lambda\Lambda' &= \sigma\sigma'\gamma\gamma'\begin{pmatrix}1 & -Kv\\ -\rho v & \rho\end{pmatrix}\begin{pmatrix}1 & -Kv'\\ -\rho' v' & \rho'\end{pmatrix}\\ &= \sigma\sigma'\gamma\gamma'\begin{pmatrix}1 + Kvv'\rho' & -Kv' - Kv\rho'\\ -\rho v - \rho\rho' v' & Kvv'\rho + \rho\rho'\end{pmatrix}.\end{aligned} \tag{15}$$

This implies that for all $\Lambda \in G$ such that $\rho = 1$,

$$\Lambda^2 = \frac{1}{\sqrt{1 - Kv^2}}\begin{pmatrix}1 + Kv^2 & -Kv' - Kv\\ -v - v' & Kv^2 + 1\end{pmatrix}. \tag{16}$$

If there'a $\Lambda \in G$ such that $\rho = 1$ and $v = c$, we have $(\Lambda^2)_{00} = 0$ (because $1 + Kv^2 = 1 - |K|v^2 = 0$), and this contradicts the definition of "linear relativistic group". □

Lemma 7 (Addition of small velocities when $K < 0$)**.** *Let G be a linear relativistic group, and denote its relativity by K. Suppose that $K < 0$. For all $\Lambda, \Lambda' \in G$ such that $|vv'| < c^2$,*

$$V(\Lambda\Lambda') = \frac{\rho' v + v'}{1 - |K| vv'\rho'}.$$

Proof. Let Λ, Λ' be arbitrary members of G such that $|vv'| < c^2$. Note that

$$1 + Kvv'\rho' > 1 - |Kvv'\rho'| = 1 - |K||vv'| > 1 - 1 = 0. \tag{17}$$

This and (15) imply that

$$\begin{aligned}\Lambda\Lambda' &= \sigma\sigma'\gamma\gamma'(1 + Kvv'\rho') \begin{pmatrix} 1 & \frac{-Kv' - Kv\rho'}{1+Kvv'\rho'} \\ \frac{-\rho v - \rho\rho' v'}{1+Kvv'\rho'} & \frac{Kvv'\rho + \rho\rho'}{1+Kvv'\rho'} \end{pmatrix} \\ &= \sigma\sigma'\gamma\gamma'(1 + Kvv'\rho') \begin{pmatrix} 1 & -K\frac{v' + v\rho'}{1+Kvv'\rho'} \\ -\rho\rho' \frac{\rho' v + v'}{1+Kvv'\rho'} & \rho\rho' \end{pmatrix}.\end{aligned} \tag{18}$$

This implies that

$$V(\Lambda\Lambda') = \frac{((\Lambda\Lambda')^{-1})_{10}}{((\Lambda\Lambda')^{-1})_{00}} = -\frac{(\Lambda\Lambda')_{10}}{(\Lambda\Lambda')_{11}} = \frac{\rho' v + v'}{1 + Kvv'\rho'} = \frac{\rho' v + v}{1 - |K| vv'\rho'}. \tag{19}$$

□

Definition 8 (Rapidity when $K < 0$)**.** Let G be a linear relativistic group, and denote its relativity by K. Suppose that $K < 0$. Define $\theta_K : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by $\theta_K(v) \arctan(v/c)$ for all $v \in \mathbb{R}$. For all $\Lambda \in G$, we will call $\theta_K(V(\Lambda))$ the *rapidity* of Λ.

The point of this definition is that it simplifies the velocity addition law. It's especially simple when $\rho' = 1$. We have

$$\begin{aligned}V(\Lambda\Lambda') = \frac{v + v'}{1 - |K| vv'} &= \frac{c\tan\theta_K(v) + c\tan\theta_K(v')}{1 - \tan\theta_K(v)\tan\theta_K(v')} \\ &= c\tan(\theta_K(v) + \theta_K(v'))\end{aligned} \tag{20}$$

This implies that $\theta_K(V(\Lambda\Lambda')) = \theta_K(v) + \theta_K(v')$.

Lemma 9 (Repeated small-velocity $\rho = 1$ transformations when $K < 0$)**.** *Let G be a linear relativistic group, and denote its relativity by K. Suppose that $K < 0$. For all $n \in \mathbb{Z}^+$, if $\Lambda \in G$ is such that $\rho = 1$ and $|n\theta_K(v)| \le \theta_K(c)$, then*

$$V(\Lambda^n) = c\tan(n\theta_K(v)).$$

Proof. We will prove this by induction. The $n = 1$ statement is obviously true. We will prove the $n = 2$ statement as well. Let Λ be an arbitrary member of G such that $\rho = 1$ and $|2\theta_K(v)| \leq \theta_K(c)$. Since tan is strictly increasing and odd, we have

$$|v| = |c \tan \theta_K(v)| = c \tan |\theta_K(v)| \leq c \tan(\theta_K(c)/2) < c \tan \theta_K(c) = c. \quad (21)$$

This means that we can use the velocity addition law to compute $V(\Lambda^2)$. Now (20) tells us that $V(\Lambda^2) = c \tan(2\theta_K(v))$.

Let p be an arbitrary integer such that $p \geq 2$, and suppose that the $n = p$ statement is true. Let Λ be an arbitrary member of G such that $\rho = 1$ and $|p\theta_K(v)| \leq \theta_K(c)$.

$$\begin{aligned} |\theta_K(V(\Lambda^p))| &= |\theta_K(c \tan(p\theta_K(v)))| = |p\theta_K(v)| \leq \theta_K(c) \\ |\theta_K(V(\Lambda))| &= |\theta_K(v)| < |p\theta_K(v)| \leq \theta_K(c) \end{aligned} \quad (22)$$

This implies that $|V(\Lambda^p)| \leq c$ and $|V(\Lambda)| < c$. So lemma 7 tells us that

$$\begin{aligned} V(\Lambda^{p+1}) = V(\Lambda^p \Lambda) &= \frac{V(\Lambda^p) + V(\Lambda)}{1 - |K| V(\Lambda^p) V(\Lambda)} = \frac{c \tan(p\theta_K(v)) + c \tan \theta_K(v)}{1 - \tan(p\theta_K(v)) \tan \theta_K(v)} \\ &= c \tan(p\theta_K(v) + \theta_K(v)) = c \tan((p+1)\theta_K(v)). \end{aligned} \quad (23)$$

So the $n = p + 1$ statement is true as well. □

Definition 10 (Ugly velocity)**.** Let G be a linear relativistic group, and denote its relativity by K. A real number r is said to be an *ugly velocity* if $r \in V(G)$, and there's no $\Lambda \in G$ such that $\rho = 1$ and $v = r$.

Recall that assumption 1b in the definition of "linear relativistic group" tells us that if G is a linear relativistic group, there's an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \in V(G)$.

Lemma 11 (There are lots of $\rho = 1$ transformations when $K < 0$)**.** *Let G be a linear relativistic group, and denote its relativity by K. Suppose that $K < 0$. Let $\varepsilon \in (0, c)$ be such that $(-\varepsilon, \varepsilon) \subset V(G)$. For each $r \in (-\varepsilon, \varepsilon)$, there's a $\Lambda \in G$ such that $\rho = 1$ and $v = r$.*

Proof. Our goal is to prove is that that there are no ugly velocities in $(-\varepsilon, \varepsilon)$.

Let $r \in (-\varepsilon, \varepsilon)$ be arbitrary. We will prove that r is not ugly by deriving a contradiction from the assumption that it is. So supoose that r is ugly. Let $\Lambda \in G$ be such that $v = r$. Since r is ugly, we have $\rho = -1$. $c \tan(\theta_K(r)/2)$ is either ugly or it's not. We will see that both options lead to a contradiction.

Suppose that $c\tan(\theta_K(r)/2))$ is *not* ugly. Since tan and θ_K are both strictly increasing and odd, we have

$$|c\tan(\theta_K(r)/2)| = c\tan(\theta_K(|r|)/2) < c\tan(\theta_K(|r|)) = |r| < \varepsilon. \tag{24}$$

Let $\Lambda' \in G$ be such that $\rho' = 1$ and $v' = c\tan(\theta_K(r)/2)$. (Such a Λ' exists because $c\tan(\theta_K(r)/2) \in (-\varepsilon, \varepsilon)$). Since $|v'| < \varepsilon < c$, we can use the velocity addition law to compute $V(\Lambda'^2)$. We have $\det(\Lambda'^2) = (\det\Lambda')^2 = 1$, and

$$\begin{aligned} V_K(\Lambda'^2) &= \frac{\rho' v' + v'}{1 + Kv'^2\rho'} = \frac{2v'}{1 - |K|v'^2} = \frac{2c\tan\theta_K(v')}{1 - \tan^2\theta_K(v')} = c\tan(2\theta_K(v')) \\ &= c\tan\theta_K(r) = r. \end{aligned} \tag{25}$$

These results contradict that r is ugly.

Suppose that $c\tan(\theta_K(r)/2))$ is ugly. Let $\Lambda' \in G$ be such that $v' = c\tan(\theta_K(r)/2))$. Since $c\tan(\theta_K(r)/2))$ is ugly, we have $\rho = -1$. Note that $|v'| < |v| = r < \varepsilon < c$. This means that we can use the velocity addition law to compute $V(\Lambda'\Lambda)$. We have $\det(\Lambda'\Lambda) = (\det\Lambda')(\det\Lambda) = 1$, and

$$\begin{aligned} V(\Lambda'\Lambda) &= \frac{\rho v' + v}{1 + Kv'v\rho} = \frac{-v' + v}{1 + |K|vv'} = \frac{c\tan\theta_K(v) - c\tan\theta_K(v')}{1 + \tan\theta_K(v)\tan\theta_K(v')} \\ &= c\frac{\tan\theta_K(v) + \tan(-\theta_K(v'))}{1 - \tan\theta_K(v)\tan(-\theta_K(v))} = c\tan(\theta_K(v) - \theta_K(v')) \\ &= c\tan(\theta_K(r) - \theta_K(r)/2) = c\tan(\theta_K(r)/2). \end{aligned} \tag{26}$$

These results contradict that $c\tan(\theta_K(r)/2)$ is ugly. $\square$

Lemma 12 (No linear relativistic group has a negative relativity)**.** *If K is the relativity of a linear relativistic group, then $K \geq 0$.*

Proof. Let G be a linear relativistic group, and let K be its relativity. We will prove that $K \geq 0$ by deriving a contradiction from the assumption that $K < 0$. So suppose that $K < 0$. Let $\varepsilon \in (0, c)$ be such that $(-\varepsilon, \varepsilon) \subset V(G)$. (Assumption 1b in the definition of "linear relativistic group" ensures that such a Λ exists). Let $n \in \mathbb{Z}^+$ be such that $\theta_K(c)/n < \theta_K(\varepsilon)$. Note that since tan is strictly increasing, this implies that

$$c\tan(\theta_K(c)/n) < c\tan\theta_K(\varepsilon) = \varepsilon < c \tag{27}$$

Let $\Lambda \in G$ be such that $\rho = 1$ and $v = c\tan(\theta_K(c)/n)$. (Since $c\tan(\theta_K(c)/n) \in (-\varepsilon, \varepsilon)$, lemma 11 ensures that such a Λ exists). Then $|v| < c$, and
$(\det\Lambda^n) = (\det\Lambda)^n = 1$. Since $|n\theta_K(v)| = \theta_K(c)$, lemma 9 tells us that

$$V(\Lambda^n) = c\tan(n\theta_K(v)) = c\tan(\theta_K(c)) = c. \tag{28}$$

These results contradict lemma 6, which says that there's no $\Lambda \in G$ such that $\rho = 1$ and $|v| = c$. $\square$

Our final goal is to prove that for each $K \geq 0$, there are exactly five linear relativistic groups with relativity K.

Definition 13 (Useful functions)**.** Let $K \geq \mathbb{R}$ be arbitrary. Define $\mathcal{D}_K$ by $\mathcal{D}_K = \{v \in \mathbb{R} | 1 - Kv^2 > 0\}$. Define $\Lambda_K : \{-1, 1\} \times \{-1, 1\} \times \mathcal{D}_K \to M_2(\mathbb{R})$ by

$$\Lambda_K(\sigma, \rho, v) = \frac{\sigma}{\sqrt{1 - Kv^2}} \begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix}$$

for all $\sigma, \rho \in \{-1, 1\}$ and all $v \in \mathcal{D}_K$. Denote the range of Λ_K by $\mathcal{R}_K$ and define $V_K : \mathcal{R}_K \to \mathbb{R}$ by

$$V_K(\Lambda) = \frac{(\Lambda^{-1})_{10}}{(\Lambda^{-1})_{00}}$$

for all $\Lambda \in \mathcal{R}_K$.

Note that lemmas 3 and 12 are saying that if G is a linear relativistic group, then there's a $K \geq 0$ such that $G \subset \mathcal{R}_K$. Also note that $\mathcal{D}_K = \mathbb{R}$ if $K = 0$, and $\mathcal{D}_K = (-c, c)$ if $K > 0$.

Definition 14 (Components of $\mathcal{R}_K$)**.** Let $K \geq 0$ be arbitrary. Define $G_K{}_+^\uparrow$, $G_K{}_+^\downarrow$, $G_K{}_-^\uparrow$, $G_K{}_-^\downarrow$ by

$$\begin{aligned} G_K{}_+^\uparrow &= \{\Lambda_K(1, 1, v) | v \in \mathcal{D}_K\} \\ G_K{}_+^\downarrow &= \{\Lambda_K(-1, 1, v) | v \in \mathcal{D}_K\} \\ G_K{}_-^\uparrow &= \{\Lambda_K(1, -1, v) | v \in \mathcal{D}_K\} \\ G_K{}_-^\downarrow &= \{\Lambda_K(-1, -1, v) | v \in \mathcal{D}_K\} \end{aligned} \tag{29}$$

These sets are called the *components* of $\mathcal{R}_K$.

Note that the components are mutually disjoint, and that their union is $\mathcal{R}_K$.

We are going to prove that $\mathcal{R}_K$ is a group. Then we are going to prove that if G is a linear relativistic group with relativity K, $G_K{}_+^\uparrow$ is a subgroup of G. This involves a few steps that are very similar to what we went through to rule out $K < 0$. In particular, we need to find a velocity addition formula and rule out the possibility of "ugly velocities".

Lemma 15 (σ, ρ and v)**.** *Let $K \geq 0$ be arbitrary. For all $\sigma, \rho \in \{-1, 1\}$ and all $v \in \mathcal{D}_K$, we have* $\operatorname{sgn}(\Lambda_K(\sigma, \rho, v))_{00} = \sigma$*,* $\det \Lambda_K(\sigma, \rho, v) = \rho$ *and* $V_K(\Lambda) = v$*.*

Proof.

$$\begin{aligned}
&\operatorname{sgn}(\Lambda_K(\sigma,\rho,v))_{00} = \operatorname{sgn}\sigma = \sigma. \\
&\det\Lambda_K(\sigma,\rho,v) = \frac{\sigma^2}{1-Kv^2}\det\begin{pmatrix}1 & -Kv\\ -\rho v & \rho\end{pmatrix} = \frac{\rho-\rho Kv^2}{1-Kv^2} = \rho. \\
&V_K(\Lambda_K(\sigma,\rho,v)) = \frac{(\Lambda_K(\sigma,\rho,v)^{-1})_{10}}{(\Lambda_K(\sigma,\rho,v)^{-1})_{00}} = -\frac{\Lambda_K(\sigma,\rho,v)_{10}}{\Lambda_K(\sigma,\rho,v)_{11}} = -\frac{-\rho v}{\rho} = v. \qquad (30)
\end{aligned}$$

□

Lemma 16 (Injectivity)**.** *For all $K \geq 0$ and all $\sigma,\rho \in \{-1,1\}$, the map $\Lambda_K(\sigma,\rho,\cdot) : \mathcal{D}_K \to M_2(\mathbb{R})$ is injective.*

Proof. Suppose that $\Lambda_K(\sigma,\rho,v) = \Lambda(\sigma,\rho,v')$. Then

$$\frac{\sigma}{\sqrt{1-Kv^2}}\begin{pmatrix}1 & -Kv\\ -\rho v & \rho\end{pmatrix} = \frac{\sigma}{\sqrt{1-Kv'^2}}\begin{pmatrix}1 & -Kv'\\ -\rho v' & \rho\end{pmatrix}. \qquad (31)$$

The 00 component of this equality tells us that $v^2 = v'^2$. This result and the 10 component of the equality tell us that $v = v'$. □

Definition 17 (The rapidity of a member of $\mathcal{R}_K$, when $K > 0$)**.** For each $K > 0$ define $\theta_K : \mathcal{D}_K \to (-\frac{\pi}{2},\frac{\pi}{2})$ by $\theta_K(v) = \arctan(v/c)$ for all $v \in \mathcal{D}_K$. For all $\Lambda \in \mathcal{R}_K$, we will call $\theta_K(V_K(\Lambda))$ the *rapidity* of Λ.

Lemma 18 ($\mathcal{R}_K$ is closed under matrix multiplication)**.** *Let $K \geq 0$ be arbitrary. For all $\sigma,\sigma',\rho,\rho' \in \{-1,1\}$ and all $v,v' \in \mathcal{D}_K$,*

$$\Lambda_K(\sigma,\rho,v)\Lambda_K(\sigma',\rho',v') = \Lambda_K\left(\sigma\sigma',\rho\rho',\frac{\rho' v + v'}{1+Kvv'\rho'}\right).$$

Proof. Let $\sigma,\sigma',\rho,\rho' \in \{-1,1\}$ and $v,v' \in \mathcal{D}_K$ be arbitrary. First note that

$$1 + Kvv'\rho' > 1 - |Kvv'\rho'| = 1 - K|v||v'| > 1 - 1 = 0.$$

This implies that

$$\begin{aligned}
\Lambda_K(\sigma,\rho,v)\Lambda_K(\sigma',\rho',v') &= \frac{\sigma}{\sqrt{1-Kv^2}}\frac{\sigma'}{\sqrt{1-Kv'^2}}\begin{pmatrix}1 & -Kv\\ -\rho v & \rho\end{pmatrix}\begin{pmatrix}1 & -Kv'\\ -\rho' v' & \rho'\end{pmatrix} \\
&= \frac{\sigma}{\sqrt{1-Kv^2}}\frac{\sigma'}{\sqrt{1-Kv'^2}}\begin{pmatrix}1+Kvv'\rho' & -Kv' - Kv\rho'\\ -\rho v - \rho\rho' v' & \rho Kvv' + \rho\rho'\end{pmatrix} \\
&= \frac{\sigma\sigma'(1+Kvv'\rho')}{\sqrt{1-Kv^2}\sqrt{1-Kv'^2}}\begin{pmatrix}1 & -K\frac{\rho' v + v'}{1+Kvv'\rho'}\\ -\rho\rho'\frac{\rho' v + v'}{1+Kvv'\rho'} & \rho\rho'\end{pmatrix}
\end{aligned} \qquad (32)$$

Define ρ'' and v'' by

$$\rho'' = \rho\rho', \quad v'' = \frac{\rho' v + v'}{1 + Kvv'\rho'}. \tag{33}$$

Note that $\rho \in \{-1, 1\}$. Since

$$\begin{aligned} |v''| &= \left| \frac{\rho' v + v'}{1 + Kvv'\rho'} \right| = \left| \frac{\rho' c \tanh\theta_K(v) + c\tanh\theta_K(v')}{1 + \rho' \tanh\theta_K(v)\tanh\theta_K(v')} \right| \\ &= c \left| \frac{\tanh(\rho'\theta_K(v)) + \tanh\theta_K(v')}{1 + \tanh(\rho'\theta_K(v))\tanh\theta_K(v')} \right| = c|\tanh(\rho'\theta_K(v) + \theta_K(v'))| < c, \end{aligned} \tag{34}$$

we also have $v'' \in \mathcal{D}_K$. Since $|v''| < c$, we have $1 - Kv''^2 > 0$. Define σ'' by

$$\frac{\sigma''}{\sqrt{1 - Kv''^2}} = \frac{\sigma\sigma'(1 + Kvv'\rho')}{\sqrt{1 - Kv^2}\sqrt{1 - Kv'^2}}. \tag{35}$$

Note that

$$\begin{aligned} 1 - Kv''^2 &= 1 - K\left(\frac{\rho' v + v'}{1 - Kvv'\rho'}\right)^2 = 1 - \frac{Kv^2 + Kv'^2 + 2Kvv'\rho'}{(1 + Kvv'\rho')^2} \\ &= \frac{1 + K^2v^2v'^2 + 2Kvv'\rho' - Kv^2 - Kv'^2 - 2Kvv'\rho'}{(1 + Kvv'\rho')^2} \\ &= \frac{1 + K^2v^2v'^2 - Kv^2 - Kv'^2}{(1 + Kvv'\rho')^2} = \frac{(1 - Kv^2)(1 - Kv'^2)}{(1 + Kvv'\rho')^2}. \end{aligned} \tag{36}$$

Since $1 - Kv''^2 > 0$ and $1 + Kvv'\rho' > 0$, this implies that $\sigma'' = \sigma\sigma' \in \{-1, 1\}$. $\square$

Corollary 19. *For all $K \geq 0$, $\mathcal{R}_K$ is a group.*

Proof. Lemma 18 implies that for all $\sigma, \rho \in \{-1, 1\}$ and all $v \in \mathcal{D}_K$,

$$\Lambda_K(\sigma, \rho, v)^{-1} = \Lambda(\sigma, \rho, -\rho v). \tag{37}$$

We also have $I = \Lambda_K(1, 1, 0) \in \mathcal{R}_K$. $\square$

Corollary 20 (Relativistic velocity addition)**.** *Let $K \geq 0$. For all $\sigma, \sigma', \rho, \rho' \in \{-1, 1\}$ and all $v, v' \in \mathcal{D}_K$,*

$$V_K\big(\Lambda_K(\sigma, \rho, v)\Lambda_K(\sigma', \rho', v')\big) = \frac{\rho' v + v'}{1 + Kvv'\rho'}.$$

Proof. Lemmas 18 and 15 tell us that

$$V_K\big(\Lambda_K(\sigma,\rho,v)\Lambda_K(\sigma',\rho',v')\big) = V_K\left(\Lambda_K\left(\sigma\sigma',\rho\rho',\frac{\rho' v+v'}{1+Kvv'\rho'}\right)\right) = \frac{\rho' v+v'}{1+Kvv'\rho'}. \tag{38}$$

□

Corollary 21 (Relativistic velocity addition, again)**.** *Let G be a linear relativistic group, and denote its relativity by K. For all $\Lambda,\Lambda' \in G$,*

$$V(\Lambda\Lambda') = \frac{\rho' v+v'}{1+Kvv'\rho'}.$$

Proof. Let $\Lambda,\Lambda' \in G$ be arbitrary. Lemmas 3 and 18 tell us that G is a subset of $\mathcal{R}_K$. So $\Lambda,\Lambda' \in \mathcal{R}_K$. Now lemma 15 implies that $\Lambda = \Lambda_K(\sigma,\rho,v)$ and $\Lambda' = \Lambda_K(\sigma',\rho',v')$. So corollary 20 tells us that

$$V(\Lambda\Lambda') = V_K\big(\Lambda_K(\sigma,\rho,v)\Lambda_K(\sigma',\rho',v')\big) = \frac{\rho' v+v'}{1+Kvv'\rho'}. \tag{39}$$

□

Definition 22 (Proper, orthochronous, orthochorous)**.** Let $K \geq 0$, $\sigma,\rho \in \{-1,1\}$ and $v \in \mathcal{D}_K$ be arbitrary. $\Lambda_K(\sigma,\rho,v)$ is said to be

(a) *proper* if $\rho = 1$.

(b) *orthochronous* if $\sigma = 1$.

(c) *orthochorous* if $\sigma\rho = 1$.

These definitions can also be stated without using the variables σ and ρ. $\Lambda \in \mathcal{R}_K$ is said to be *proper* if $\det\Lambda = 1$, *orthochronous* if $\Lambda_{00} > 0$ and *orthochorous* if $\operatorname{sgn}(\Lambda_{00})\det\Lambda = 1$.

Lemma 23 (Products)**.** *Let $K \geq 0$ be arbitrary. Let $\Lambda,\Lambda' \in \mathcal{R}_K$ be arbitrary.*

(a) If Λ and Λ' are proper, then $\Lambda\Lambda'$ is proper.

(b) If Λ and Λ' are orthochronous, then $\Lambda\Lambda'$ is orthochronous.

(c) If Λ is proper, then Λ^2 is proper and orthochronous.

Proof. (a): If $\rho = \rho' = 1$, then $\det(\Lambda\Lambda') = (\det\Lambda)(\det\Lambda') = \rho\rho' = 1$.

(b): If $\sigma = \sigma' = 1$, then $(\Lambda\Lambda')_{00} = \sigma\sigma' = 1$.

(c): If $\rho = 1$, then $\det\Lambda^2 = (\det\Lambda)^2 = 1$ and $(\Lambda^2)_{00} = \sigma^2 = 1$. $\square$

Lemma 24 (There are lots of $\rho = 1$ transformations)**.** *Let G be a linear relativistic group, and denote its relativity by K. Let $\varepsilon > 0$ be such that $(-\varepsilon, \varepsilon) \subset V(G)$. For each $r \in (-\varepsilon, \varepsilon)$, there's a $\Lambda \in G$ such that $v = r$.*

Proof. Our goal is to prove is that that there are no ugly velocities in $(-\varepsilon, \varepsilon)$. We will deal with the possibilities $K = 0$ and $K > 0$ separately. Suppose that $K = 0$.

Let $r \in (-\varepsilon, \varepsilon)$ be arbitrary. We will prove that r is not ugly by deriving a contradiction from the assumption that it is. So suppose that r is ugly. Let $\Lambda \in G$ be such that $v = r$. Since r is ugly, we have $\rho = -1$. $r/2$ is either ugly or it's not. We will see that both options lead to a contradiction.

Suppose that $r/2$ is not ugly. Let $\Lambda' \in G$ be such that $\rho' = 1$ and $v' = r/2$. Then $\det(\Lambda'^2) = (\det\Lambda)^2 = 1$, and $V(\Lambda\Lambda') = \rho' v' + v' = 2v' = r$. These results contradict that r is ugly.

Suppose that $r/2$ is ugly. Let $\Lambda' \in G$ be such that $v' = r/2$. Since $r/2$ is ugly, we have $\rho' = -1$. This implies that $\det(\Lambda'\Lambda) = (\det\Lambda')(\det\Lambda) = 1$, and also that $V(\Lambda'\Lambda) = \rho v' + v = v - v' = r - r/2 = r/2$. These results contradict that $r/2$ is ugly.

Suppose that $K > 0$. The proof is essentially the same as for the case $K < 0$, so we won't repeat it here. Since rapidity is now defined using tanh instead of tan, we have to use the identity

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x\ \tanh y} \tag{40}$$

instead of

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\ \tan y}. \tag{41}$$

$\square$

Lemma 25 (Repeated $\rho = 1$ transformations)**.** *Let G be a linear relativistic group, and let K be its relativity.*

(a) If $K = 0$, then for all $n \in \mathbb{Z}^+$ and all $\Lambda \in G$ such that $\rho = 1$,

$$V(\Lambda^n) = nV(\Lambda).$$

(b) If $K > 0$, then for all $n \in \mathbb{Z}^+$ and all $\Lambda \in G$ such that $\rho = 1$,

$$V(\Lambda^n) = c\tanh(n\theta_K(v)).$$

Proof. (a): We will prove this by induction. The $n = 1$ statement is obviously true. Let $p \in \mathbb{Z}^+$ be arbitrary and suppose that the $n = p$ statement is true. Corollary 21 tells us that

$$V(\Lambda^{p+1}) = V(\Lambda^p\Lambda) = V(\Lambda^p) + V(\Lambda) = pV(\Lambda) + V(\Lambda) = (p+1)V(\Lambda). \tag{42}$$

So the $n = p + 1$ statement is true as well.

(b): The $n = 1$ statement is obviously true. Let $p \in \mathbb{Z}^+$ be arbitrary, and suppose that the $n = p$ statement is true. Corollary 20 tells us that

$$\begin{aligned} V(\Lambda^{p+1}) = V(\Lambda^p\Lambda) = \frac{V(\Lambda^p) + V(\Lambda)}{1 + KV(\Lambda^p)V(\Lambda)} &= \frac{c\tanh(p\theta_K(v)) + c\tanh\theta_K(v)}{1 + \tanh(p\theta_K(v))\tanh\theta_K(v)} \\ = c\tanh(p\theta_K(v) + \theta_K(v)) &= c\tanh((p+1)\theta_K(v)). \end{aligned} \tag{43}$$

So the $n = p + 1$ statement is true as well. □

Lemma 26 (There are lots of $\sigma = \rho = 1$ transformations)**.** *Let G be a linear relativistic group, and denote its relativity by K. For each $r \in \mathcal{D}_K$, there's a $\Lambda \in G$ such that $\sigma = \rho = 1$ and $v = r$.*

Proof. Let $r \in \mathcal{D}_K$ be arbitrary. Let $\varepsilon > 0$ be such that $(-\varepsilon, \varepsilon) \subset V(G)$. Assumption 1b in the definition of "linear relativistic group" tells us that such an ε exists.

Suppose that $K = 0$. Let $n \in \mathbb{Z}^+$ be such that $r/n \in (-\varepsilon, \varepsilon)$. Let Λ be an arbitrary member of G such that $\rho = 1$ and $v = r/(2n)$. (Lemma 24 ensures that such a Λ exists). Lemma 23 tells us that Λ^2 is proper and orthochronous. It also implies that $\Lambda^{2n} = (\Lambda^2)^n$ is proper and orthochronous. Lemma 20 tells us that $V(\Lambda^{2n}) = 2nV(\Lambda) = r$.

Suppose that $K > 0$. Let $n \in \mathbb{R}^+$ be such that $\theta_K(r)/(2n) \in (-\theta_K(\varepsilon), \theta_K(\varepsilon))$. Since tanh is strictly increasing and odd, we have

$$|c\tanh(\theta_K(r)/(2n))| = c\tanh|\theta_K(r)/(2n)| < c\tanh\theta_K(\varepsilon) = \varepsilon. \tag{44}$$

Let Λ be an arbitrary member of G such that $\rho = 1$ and $v = c\tanh(\theta_K(r)/(2n))$. (Lemma 24 ensures that such a Λ exists). Note that $\theta_K(v) = \theta_K(r)/(2n)$. Lemma 23 tells us that Λ^2 is proper and orthochronous, and implies that $\Lambda^{2n} = (\Lambda^2)^n$ is proper and orthochronous. Corollary 21 tells us that

$$V(\Lambda^{2n}) = c\tanh(2n\theta_K(v)) = c\tanh\theta_K(r) = r. \tag{45}$$

□

Lemma 27 (The restricted subgroup)**.** *If G is a linear relativistic group, and K is its relativity, then $G_K{}^{\uparrow}_{+}$ is a subgroup of G.*

Proof. First we prove that $G_{K+}^{\uparrow} \subset G$. Let $\Lambda \in G_{K+}^{\uparrow}$ be arbitrary. Let $r \in \mathcal{D}_K$ be such that $\Lambda_K(1,1,r) = \Lambda$. Let $\Lambda' \in G$ be such that $\sigma' = \rho' = 1$ and $v' = r$. (Lemma 26 ensures that such a Λ' exists). Lemma 3 tells us that $\Lambda' = \Lambda_K(1,1,r)$. So $\Lambda = \Lambda_K(1,1,r) = \Lambda' \in G$.

Let $\Lambda, \Lambda' \in G_{K+}^{\uparrow}$ be arbitrary. Since $G_{K+}^{\uparrow} \subset G$, we have $\Lambda, \Lambda' \in G$. Since G is a group, this implies that $\Lambda\Lambda' \in G$. Since G is a linear relativistic group, and Λ, Λ' are proper and orthochronous members of G, lemma 23 tells us that $\Lambda\Lambda'$ is proper and orthochronous. Now lemma 3 tells us that there's a $u \in \mathcal{D}_K$ such that $\Lambda\Lambda' = \Lambda_K(1,1,u) \in G_{K+}^{\uparrow}$. So $G_{K+}^{\uparrow}$ is closed under matrix multiplication. Since $I \in G_{K+}^{\uparrow}$, this implies that $G_{K+}^{\uparrow}$ is a subgroup of G. □

This group is called the *restricted* subgroup of $\mathcal{R}_K$. Note that what we've done so far is to prove that if G is a linear relativistic group, then there's a $K \geq 0$ such that

$$G_{K+}^{\uparrow} \subset G \subset G_{K+}^{\uparrow} \cup G_{K+}^{\downarrow} \cup G_{K-}^{\uparrow} \cup G_{K-}^{\downarrow}. \tag{46}$$

Definition 28 (Inversion matrices)**.** The matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called the *space inversion* (or *parity*) matrix, the *time inversion* (or *time reversal*) matrix and the *spacetime inversion* matrix respectively.

Lemma 29 (Zero-velocity transformations)**.** *For all $K \geq 0$,*

$$\begin{aligned} I &= \Lambda_K(1,1,0) \in G_{K+}^{\uparrow} \\ -I &= \Lambda_K(-1,1,0) \in G_{K+}^{\downarrow} \\ P &= \Lambda_K(1,-1,0) \in G_{K-}^{\uparrow} \\ T &= \Lambda_K(-1,-1,0) \in G_{K-}^{\downarrow}. \end{aligned} \tag{47}$$

Proof. The definition of Λ_K implies that for each $\sigma, \rho \in \{-1, 1\}$,

$$\Lambda_K(\sigma,\rho,0) = \sigma \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}. \tag{48}$$

□

Lemma 30 (Each component is either disjoint from G or a subset of G)**.** *Let G be a linear relativistic group, and let K be its relativity.*

(a) If G contains one member of $G_{K}{}^{\downarrow}_{+}$, then it contains all of them.

(b) If G contains one member of $G_{K}{}^{\uparrow}_{-}$, then it contains all of them.

(c) If G contains one member of $G_{K}{}^{\downarrow}_{-}$, then it contains all of them.

Proof. Lemma 18 tells us that for all $\sigma, \sigma', \rho, \rho' \in \{-1, 1\}$ and all $v, v' \in \mathcal{D}_K$,

$$\Lambda_K(\sigma, \rho, v)\Lambda_K(\sigma', \rho', v') = \Lambda_K\left(\sigma\sigma', \rho\rho', \frac{\rho' v + v'}{1 + Kvv'\rho}\right). \tag{49}$$

This implies that for all $\sigma, \rho \in \{-1, 1\}$ and all $v \in \mathcal{D}_K$,

$$\Lambda_K(\sigma, \rho, v)\Lambda_K(1, 1, -v) = \Lambda_K\left(\sigma, \rho, \frac{v + (-v)}{1 + Kv(-v)}\right) = \Lambda_K(\sigma, \rho, 0). \tag{50}$$

Let $v \in \mathcal{D}_K$ be arbitrary.

(a): Suppose that $\Lambda_K(-1, 1, v) \in G$. Since $\Lambda_K(1, 1, -v) \in G_{K}{}^{\uparrow}_{+} \subset G$ we have

$$G \ni \Lambda_K(-1, 1, v)\Lambda_K(1, 1, -v) = \Lambda_K(-1, 1, 0) = -I. \tag{51}$$

This implies that for all $v' \in \mathcal{D}_K$,

$$\Lambda_K(-1, 1, v') = -I\Lambda(1, 1, v') \in G. \tag{52}$$

(b): Suppose that $\Lambda_K(1, -1, v) \in G$. Since $\Lambda_K(1, 1, v) \in G_{K}{}^{\uparrow}_{+} \subset G$ we have

$$G \ni \Lambda_K(1, -1, v)\Lambda_K(1, 1, -v) = \Lambda_K(1, -1, 0) = P. \tag{53}$$

This implies that for all $v' \in \mathcal{D}_K$,

$$\Lambda_K(1, -1, v') = P\Lambda(1, 1, v') \in G. \tag{54}$$

(c): Suppose that $\Lambda_K(-1, -1, v) \in G$. Since $\Lambda_K(1, 1, v) \in G_{K}{}^{\uparrow}_{+} \subset G$ we have

$$G \ni \Lambda_K(-1, -1, v)\Lambda_K(1, 1, -v) = \Lambda_K(-1, -1, 0) = T. \tag{55}$$

This implies that for all $v' \in \mathcal{D}_K$,

$$\Lambda_K(-1, -1, v') = T\Lambda(1, 1, v') \in G. \tag{56}$$

$\square$

Corollary 31 (If an inversion matrix is in G)**.** *Let G be a linear relativistic group, and denote its relativity by K. If $-I \in G$, then ${G_K}_+^\downarrow \subset G$. If $P \in G$, then ${G_K}_-^\uparrow \subset G$. If $T \in G$, then ${G_K}_-^\downarrow \subset G$.*

Corollary 32 (If an inversion matrix isn't in G)**.** *Let G be a linear relativistic group, and denote its relativity by K. If $-I \notin G$, then $G \cap {G_K}_+^\downarrow = \emptyset$. If $P \notin G$, then $G \cap {G_K}_-^\uparrow = \emptyset$. If $T \notin G$, then $G \cap {G_K}_-^\downarrow = \emptyset$.*

The results we have obtained so far imply that a linear relativistic group is completely determined by its relativity and its intersection with the set $\{-I, P, T\}$. (The proof of the next corollary will make that perfectly clear). This raises the question of what subsets of $\{-I, P, T\}$ can be a subset of G. Clearly, G will contain 0, 1, 2 or 3 members of $\{-I, P, T\}$. There's 1 way to choose zero members from that set. There are 3 ways to choose one, 3 ways to choose two, and 1 way to choose three. But we have $-IP = T$, $PT = -I$, $T(-I) = P$, so if two inversion matrices are in G, the third one is too. This implies that the intersection can't be a set with cardinality 2. This leaves us with five candidates for $G \cap \{-I, P, T\}$ that we haven't ruled out: $\emptyset, \{-I\}, \{P\}, \{T\}, \{-I, P, T\}$.

Corollary 33 (Five candidates for each $K \geq 0$)**.** *Let G be a linear relativistic group, and let K be its relativity.*

(a) If none of $-I, P, T$ are in G, then $G = {G_K}_+^\uparrow$.

(b) If $-I \in G$ and $P, T \notin G$, then $G = {G_K}_+^\uparrow \cup {G_K}_+^\downarrow$.

(c) If $P \in G$ and $T, -I \notin G$, then $G = {G_K}_+^\uparrow \cup {G_K}_-^\uparrow$.

(d) If $T \in T$ and $-I, P \notin G$, then $G = {G_K}_+^\uparrow \cup {G_K}_-^\downarrow$.

(e) If two of $-I, P, T$ are in G, then the third one is too, and $G = {G_K}_+^\uparrow \cup {G_K}_+^\downarrow \cup {G_K}_-^\uparrow \cup {G_K}_-^\downarrow$

Proof. The proofs are very similar, so we will only do one.

(b): Suppose that $P \in G$ and $T, -I \notin G$. Then corollary 31 tells us that ${G_K}_-^\uparrow \subset G$, and corollary 32 tells us that G is disjoint from both ${G_K}_-^\downarrow$ and ${G_K}_+^\downarrow$. Lemma 27 tells us that ${G_K}_+^\uparrow \subset G$. So

$$ {G_K}_+^\uparrow \cup {G_K}_-^\uparrow \subset G \subset \mathcal{R}_K - ({G_K}_+^\downarrow \cup {G_K}_-^\downarrow) = {G_K}_+^\uparrow \cup {G_K}_-^\uparrow. \tag{57} $$

□

Nothing we have done so far implies that there's a $K \geq 0$ such that the five sets mentioned in this corollary are linear relativistic groups. In fact, we still haven't proved that there *are* any linear relativistic groups. We ruled out the possibility that $K < 0$, but we still don't know for sure that no other values of K can be ruled out. Also, we still don't know for sure that none of the statements $-I \in G$, $P \in G$, $T \in G$, $-I \notin G$, $P \notin G$, $T \notin G$ follow from the assumption that G is a linear relativistic group. The following lemma solves all of these problems, and completes the proof of theorem 1

Lemma 34 (Those sets are linear relativistic groups)**.** *Let $K \geq 0$ be arbitrary. The following sets are linear relativistic groups with relativity K.*

(a) ${G_K}^{\uparrow}_{+}$.

(b) ${G_K}^{\uparrow}_{+} \cup {G_K}^{\downarrow}_{+}$.

(c) ${G_K}^{\uparrow}_{+} \cup {G_K}^{\uparrow}_{-}$.

(d) ${G_K}^{\uparrow}_{+} \cup {G_K}^{\downarrow}_{-}$.

(e) ${G_K}^{\uparrow}_{+} \cup {G_K}^{\downarrow}_{+} \cup {G_K}^{\uparrow}_{-} \cup {G_K}^{\downarrow}_{-}$

Proof. Let $K \geq 0$ be arbitrary and let H_K be any of the five sets listed in (a)-(e). We will prove that H_K is a group. First note that $I = \Lambda_K(1,1,0) \in H_K$. Let $\Lambda, \Lambda' \in H_K$ be arbitrary. Let $\sigma, \sigma'\rho, \rho' \in \{-1,1\}$ and $v, v' \in \mathcal{D}_K$ be such that $\Lambda_K(\sigma,\rho,v) = \Lambda$ and $\Lambda_K(\sigma',\rho',v') = \Lambda'$. Lemma (49) tells us that

$$\Lambda\Lambda' = \Lambda_K(\sigma,\rho,v)\Lambda_K(\sigma',\rho',v') = \Lambda_K\left(\sigma\sigma', \rho\rho', \frac{\rho' v + v'}{1 + Kvv'\rho}\right). \qquad (58)$$

If we can prove that the right-hand side is in H_K, we can conclude that

$$\Lambda_K(\sigma,\rho,v)^{-1} = \Lambda_K(\sigma,\rho,-\rho' v) \in H_K, \qquad (59)$$

and this will complete the proof that H_K is a group. Since the right-hand side of (58) is in the same component as $\Lambda_K(\sigma\sigma', \rho\rho', 0)$, it's sufficient to prove that $\Lambda_K(\sigma\sigma', \rho\rho', 0) \in H_K$. Define a binary operation $\star$ on $\{-1,1\} \times \{-1,1\}$ by $(\sigma,\rho) \star (\sigma',\rho') = (\sigma\sigma', \rho\rho')$ for all $\sigma, \sigma'\rho, \rho' \in \{-1,1\}$. In each of the cases (a)-(e), define

$$S = \big\{(\sigma,\rho) \in \{-1,1\} \times \{-1,1\} \,|\, \Lambda_K(\sigma,\rho,0) \in H_K\big\}.$$

It's sufficient to prove that $(\sigma,\rho) \star (\sigma',\rho') \in S$ for all $(\sigma,\rho), (\sigma',\rho') \in S$. Since the $\star$ operation is commutative, we won't have to check all possible products.

$$\begin{aligned} \text{(a):} \quad & S = \{(1,1)\} \\ & (1,1) \star (1,1) = (1,1) \in S \end{aligned}$$

(b): $S = \{(1,1),(-1,1)\}$

$$(1,1)\star(1,1) = (1,1) \in S$$
$$(1,1)\star(-1,1) = (-1,1) \in S$$
$$(-1,1)\star(-1,1) = (1,1) \in S$$

(c): $S = \{(1,1),(1,-1)\}$

$$(1,1)\star(1,1) = (1,1) \in S$$
$$(1,-1)\star(1,1) = (1,-1)$$
$$(1,-1)\star(1,-1) = (1,1)$$

(d): $S = \{(1,1),(-1,-1)\}$

$$(1,1)\star(1,1) = (1,1) \in S$$
$$(-1,-1)\star(1,1) = (-1,-1) \in S$$
$$(-1,-1)\star(-1,-1) = (1,1) \in S$$

(e): $S = \{-1,1\}\times\{-1,1\}$

In this case, every product is obviously in S. (60)

We have proved that the five sets are groups. We're going to prove that they are linear relativistic groups. They are obviously all subsets of $\mathrm{GL}(\mathbb{R}^2)$. Again, let H_K be any of those five sets. Let $\Lambda \in H_K$ be arbitrary. Let $\sigma,\rho \in \{-1,1\}$ and $v \in \mathcal{D}_K$ be such that $\Lambda_K(\sigma,\rho,v) = \Lambda$. Let $V : H_K \to \mathbb{R}$ be the restriction of the function V_K (definition 13) to H_K.

$$V(\Lambda) = V_K(\Lambda_K(1,1,v)) = \frac{(\Lambda_K(1,1,v)^{-1})_{10}}{(\Lambda_K(1,1,v)^{-1})_{00}} = \frac{(\Lambda^{-1})_{10}}{(\Lambda^{-1})_{00}}. \tag{61}$$

Lemma 15 tells us that $v = V_K(\sigma,\rho,v) = V(\Lambda) \in V(H_K)$. Since v is an arbitrary member of $\mathcal{D}_K$, this implies that $\mathcal{D}_K \subset V(H_K)$. Since $\mathcal{D}_K$ is an open set that contains 0, this implies that 0 is an interior point of $V(H_K)$. Since

$$\Lambda = \Lambda_K(\sigma,\rho,v) = \frac{\sigma}{\sqrt{1-Kv^2}}\begin{pmatrix} 1 & -Kv \\ -\rho v & \rho \end{pmatrix} \tag{62}$$

and $\Lambda^{-1} = \Lambda_K(\sigma,\rho,v)^{-1} = \Lambda_K(\sigma,\rho,-\rho v)$, we have $(\Lambda^{-1})_{\mu\mu} = \Lambda_{\mu\mu}$ for all $\mu \in \{0,1\}$.

Finally, we're going to prove that the relativity of H_K is K. Let v be an arbitrary member of $\mathcal{D}_K$ such that $v \neq 0$. Define $\Lambda = \Lambda_K(1,1,v)$. Lemma 5 tells us that the relativity of H_K is

$$\frac{1}{V(\Lambda)^2}\left(1 - \frac{1}{(\Lambda_{00})^2}\right) = \frac{1}{v^2}(1-(1-Kv^2)) = K. \tag{63}$$

□