# Math 865, Topics in Riemannian Geometry

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## Contents

# Introduction

We will cover the following topics:

- First few lectures will be a quick review of tensor calculus and Riemannian geometry: metrics, connections, curvature tensor, Bianchi identities, commuting covariant derivatives, etc.
- Decomposition of curvature tensor into irreducible summands.
- Bochner-Weitzenbock formulas: various curvature conditions yield topological restrictions on a manifold.
- Review of elliptic theory in Holder and Sobolev spaces. Theory of elliptic operators on Riemannian manifolds with basic Fredholm Theory, with applications to Hodge Theory.
- On non-compact manifolds we will consider Fredholm operators on weighted spaces, such as weighted Sobolev and Holder spaces. This has applications to the study of asymptotically locally Euclidean spaces (ALE) spaces, such as the Eguchi-Hanson metric.

Some basic references are [Bes87], [CLN06], [Lee97], [Pet06], [Poo81].

# 1 Lecture 1: September 6, 2011

## 1.1 Metrics, vectors, and one-forms

Let $(M, g)$ be a Riemannian manifold, with metric $g \in \Gamma(S^2(T^*M))$. In coordinates,

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j, \; g_{ij} = g_{ij}, \tag{1.1}$$

and $g_{ij} >> 0$ is a positive definite matrix. The symmetry condition is of course invariantly

$$g(X, Y) = g(Y, X). \tag{1.2}$$

A vector field is a section of the tangent bundle, $X \in \Gamma(TM)$. In coordinates,

$$X = X^i \partial_i, \quad X^i \in C^\infty(M), \tag{1.3}$$

where

$$\partial_i = \frac{\partial}{\partial x^i}, \tag{1.4}$$

is the coordinate partial. We will use the Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise noted.

A 1-form is a section of the cotangent bundle, $X \in \Gamma(T^*M)$. In coordinates,

$$\omega = \omega_i dx^i, \quad \omega_i \in C^\infty(M). \tag{1.5}$$

**Remark 1.1.** Note that components of vector fields have upper indices, while components of 1-forms have lower indices. However, a collection of vector fields will be indexed by lower indices, $\{Y_1, \ldots, Y_p\}$, and a collection of 1-forms will be indexed by upper indices $\{dx^1, \ldots, dx^n\}$. This is one reason why we write the coordinates with upper indices.

## 1.2 The musical isomorphisms

The metric gives an isomorphism between $TM$ and $T^*M$,

$$\flat : TM \to T^*M \tag{1.6}$$

defined by

$$\flat(X)(Y) = g(X, Y). \tag{1.7}$$

The inverse map is denoted by $\sharp : T^*M \to TM$. The cotangent bundle is endowed with the metric

$$\langle \omega_1, \omega_2 \rangle = g(\sharp\omega_1, \sharp\omega_2). \tag{1.8}$$

Note that if $g$ has components $g_{ij}$, then $\langle \cdot, \cdot \rangle$ has components $g^{ij}$, the inverse matrix of $g_{ij}$.

If $X \in \Gamma(TM)$, then

$$\flat(X) = X_i dx^i, \tag{1.9}$$

where

$$X_i = g_{ij} X^j, \tag{1.10}$$

so the flat operator "lowers" an index. If $\omega \in \Gamma(T^*M)$, then

$$\sharp(\omega) = \omega^i \partial_i, \tag{1.11}$$

where

$$\omega^i = g^{ij} \omega_j, \tag{1.12}$$

thus the sharp operator "raises" an index.

## 1.3 Inner product on tensor bundles

The metric induces a metric on $\Lambda^k(T^*M)$. We give 3 definitions, all of which are equivalent:

- Definition 1: If

$$\begin{aligned} \omega_1 &= \alpha_1 \wedge \cdots \wedge \alpha_k \\ \omega_2 &= \beta_1 \wedge \cdots \wedge \beta_k, \end{aligned} \tag{1.13}$$

then

$$\langle \omega_1, \omega_2 \rangle = \det(\langle \alpha_i, \beta_j \rangle), \tag{1.14}$$

and extend linearly. This is well-defined.

- Definition 2: If $\{e_i\}$ is an ONB of $T_pM$, let $\{e^i\}$ denote the dual basis, defined by $e^i(e_j) = \delta^i_j$. Then declare that

$$e^{i_1} \wedge \cdots \wedge e^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n, \tag{1.15}$$

is an ONB of $\Lambda^k(T^*_pM)$.

- Definition 3: If $\omega \in \Lambda^k(T^*M)$, then in coordinates

$$\omega = \sum_{i_1 < \cdots < i_k = 1}^{n} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{1.16}$$

By skew-symmetry, extend the $\omega_{i_1 \dots i_k}$ to be defined for all indices. Then

$$\|\omega\|^2_{\Lambda^k} = \langle \omega, \omega \rangle = \frac{1}{k!} \sum_{i_1, \dots, i_k = 1}^{n} \omega^{i_1 \dots i_k} \omega_{i_1 \dots i_k}, \tag{1.17}$$

where

$$\omega^{i_1 \dots i_k} = g^{i_1 l_i} g^{i_2 l_2} \dots g^{i_k l_k} \omega_{l_1 \dots l_k}. \tag{1.18}$$

**Remark 1.2.** One has to choose an identification of $\Lambda(T^*M)$ with $\Lambda(TM)^*$, in order to view forms as multilinear alternating maps on the tangent space. We choose the identification as in [War83, page 59]: if $\omega = e^1 \wedge \cdots \wedge e^p \in \Lambda^p(T^*M)$, and $e = e_1 \wedge \cdots \wedge e_p \in \Lambda^p(TM)$, then

$$\omega(e) = \det[e^i(e_j)]. \tag{1.19}$$

This makes the wedge product defined as follows. If $\alpha \in \Omega^p$, and $\beta \in \Omega^q$, then

$$\alpha \wedge \beta(X_1, \dots, X_{p+q}) = \frac{1}{p!\ q!} \sum_{\sigma} \alpha(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \cdot \beta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}), \tag{1.20}$$

and the sum is over all permutations of length $p + q$.

To define an inner product on the full tensor bundle, we let

$$\Omega \in \Gamma\Big((TM)^{\otimes p} \otimes (T^*M)^{\otimes q}\Big). \tag{1.21}$$

We call such $\Omega$ a $(p,q)$-*tensor field*. As above, we can define a metric by declaring that

$$e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_q} \tag{1.22}$$

to be an ONB. If in coordinates,

$$\Omega = \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}, \tag{1.23}$$

then

$$\|\Omega\|^2 = \langle \omega, \omega \rangle = \Omega^{j_1 \dots j_q}_{i_1 \dots i_p} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q}, \tag{1.24}$$

where the term $\Omega^{j_1 \dots j_q}_{i_1 \dots i_p}$ is obtained by raising all of the lower indices and lowering all of the upper indices of $\Omega^{j_1 \dots j_q}_{i_1 \dots i_p}$, using the metric. By polarization, the inner product is given by

$$\langle \Omega_1, \Omega_2 \rangle = \frac{1}{2}\Big( \|\Omega_1 + \Omega_2\|^2 - \|\Omega_1\|^2 - \|\Omega_2\|^2 \Big). \tag{1.25}$$

**Remark 1.3.** We are using (1.19) to identify forms and alternating tensors. For example, as an element of $\Lambda^2(T^*M)$, $e^1 \wedge e^2$ has norm 1 if $e^1, e^2$ are orthonormal in $T^*M$. But under our identification with tensors, $e^1 \wedge e^2$ is identified with $e^1 \otimes e^2 - e^2 \otimes e^1$, which has norm $\sqrt{2}$ with respect to the tensor inner product. Thus our identification in (1.19) is *not* an isometry, but is a constant multiple of an isometry.

We remark that one may reduce a $(p,q)$-tensor field into a $(p-1,q-1)$-tensor field for $p \geq 1$ and $q \geq 1$. This is called a *contraction*, but one must specify which indices are contracted. For example, the contraction of $\Omega$ in the first contrvariant index and first covariant index is written invariantly as

$$Tr_{(1,1)} \Omega, \tag{1.26}$$

and in coordinates is given by

$$\delta^{j_1}_{i_1} \Omega^{i_1 \dots i_p}_{j_1 \dots j_q} = \Omega^{l i_2 \dots i_p}_{l j_2 \dots j_q}. \tag{1.27}$$

## 1.4 Connections on vector bundles

A connection is a mapping $\Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, with the properties

- $\nabla_X s \in \Gamma(E)$,
- $\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s$,

- $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

In coordinates, letting $s_i, i = 1 \dots p$, be a local basis of sections of $E$,

$$\nabla_{\partial_i} s_j = \Gamma^k_{ij} s_k. \tag{1.28}$$

If $E$ carries an inner product, then $\nabla$ is *compatible* if

$$X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle. \tag{1.29}$$

For a connection in $TM$, $\nabla$ is called *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(TM). \tag{1.30}$$

**Theorem 1.1.** *(Fundamental Theorem of Riemannian Geometry) There exists a unique symmetric, compatible connection in $TM$.*

Invariantly, the connection is defined by

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &-\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle\Big).\end{aligned} \tag{1.31}$$

Letting $X = \partial_i, Y = \partial_j, Z = \partial_k$, we obtain

$$\begin{aligned}\Gamma^l_{ij} g_{lk} &= \langle \Gamma^l_{ij}\partial_l, \partial_k\rangle = \langle \nabla_{\partial_i}\partial_j, \partial_k\rangle \\ &= \frac{1}{2}\Big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\Big),\end{aligned} \tag{1.32}$$

which yields the formula

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\Big)} \tag{1.33}$$

for the Riemannian Christoffel symbols.

## 1.5 Curvature in the tangent bundle

The curvature tensor is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.34}$$

for vector fields $X, Y$, and $Z$. We define

$$Rm(X, Y, Z, W) \equiv -\langle \mathcal{R}(X, Y)Z, W\rangle. \tag{1.35}$$

The algebraic symmetries are:

$$R(X, Y)Z = -R(Y, X)Z \tag{1.36}$$
$$0 = R(X, Y)Z + R(Y, Z)X + R(Z, X)Y \tag{1.37}$$
$$Rm(X, Y, Z, W) = -Rm(X, Y, W, Z) \tag{1.38}$$
$$Rm(X, Y, W, Z) = Rm(W, Z, X, Y). \tag{1.39}$$

In a coordinate system we define quantities $R_{ijk}{}^l$ by

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l, \tag{1.40}$$

or equivalently,

$$\mathcal{R} = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l. \tag{1.41}$$

Define quantities $R_{ijkl}$ by

$$R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l), \tag{1.42}$$

or equivalently,

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{1.43}$$

Then

$$R_{ijkl} = -\langle \mathcal{R}(\partial_i, \partial_j)\partial_k, \partial_l \rangle = -\langle R_{ijk}{}^m \partial_m, \partial_l \rangle = -R_{ijk}{}^m g_{ml}. \tag{1.44}$$

Equivalently,

$$R_{ijlk} = R_{ijk}{}^m g_{ml}, \tag{1.45}$$

that is, we lower the upper index to the *third* position.

**Remark 1.4.** Some authors choose to lower this index to a different position. One has to be very careful with this, or you might end up proving that $S^n$ has negative curvature!

In coordinates, the algebraic symmetries of the curvature tensor are

$$\begin{align} R_{ijk}{}^l &= -R_{jik}{}^l \tag{1.46}\\ 0 &= R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l \tag{1.47}\\ R_{ijkl} &= -R_{ijlk} \tag{1.48}\\ R_{ijkl} &= R_{klij}. \tag{1.49} \end{align}$$

Of course, we can write the first 2 symmetries as a $(0,4)$ tensor,

$$\begin{align} R_{ijkl} &= -R_{jikl} \tag{1.50}\\ 0 &= R_{ijkl} + R_{jkil} + R_{kijl}. \tag{1.51} \end{align}$$

Note that using (1.49), the algebraic Bianchi identity (1.51) may be written as

$$0 = R_{ijkl} + R_{iklj} + R_{iljk}. \tag{1.52}$$

We next compute the curvature tensor in coordinates.

$$
\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^{l}\partial_l \\
&= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \partial_i(\Gamma^l_{jk})\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - \partial_j(\Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \Big(\partial_i(\Gamma^l_{jk}) + \Gamma^m_{jk}\Gamma^l_{im} - \partial_j(\Gamma^l_{ik}) - \Gamma^m_{ik}\Gamma^l_{jm}\Big)\partial_l,
\end{aligned}
\tag{1.53}
$$

which is the formula

$$
\boxed{R_{ijk}{}^{l} = \partial_i(\Gamma^l_{jk}) - \partial_j(\Gamma^l_{ik}) + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}}
\tag{1.54}
$$

Fix a point $p$. Exponential coordinates around $p$ form a normal coordinate system at $p$. That is $g_{ij}(p) = \delta_{ij}$, and $\partial_k g_{ij}(p) = 0$, which is equivalent to $\Gamma^k_{ij}(p) = 0$. The Christoffel symbols are

$$
\Gamma^l_{jk} = \frac{1}{2}g^{lm}\Big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\Big).
\tag{1.55}
$$

In normal coordinates at the point $p$,

$$
\partial_i\Gamma^l_{jk} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} + \partial_i\partial_j g_{km} - \partial_i\partial_m g_{jk}\Big).
\tag{1.56}
$$

We then have at $p$

$$
R_{ijk}{}^{l} = \frac{1}{2}\delta^{lm}\Big(\partial_i\partial_k g_{jm} - \partial_i\partial_m g_{jk} - \partial_j\partial_k g_{im} + \partial_j\partial_m g_{ik}\Big).
\tag{1.57}
$$

Lowering an index, we have at $p$

$$
\begin{aligned}
R_{ijkl} &= -\frac{1}{2}\Big(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}\Big) \\
&= -\frac{1}{2}\Big(\partial^2 \owedge g\Big).
\end{aligned}
\tag{1.58}
$$

The $\owedge$ symbol is the Kulkarni-Nomizu product, which takes 2 symmetric $(0,2)$ tensors and gives a $(0,4)$ tensor with the same algebraic symmetries of the curvature tensor, and is defined by

$$
\begin{aligned}
A \owedge B(X,Y,Z,W) =& A(X,Z)B(Y,W) - A(Y,Z)B(X,W) \\
&- A(X,W)B(Y,Z) + A(Y,W)B(X,Z).
\end{aligned}
$$

To remember: the first term is $A(X,Z)B(Y,W)$, skew symmetrize in $X$ and $Y$ to get the second term. Then skew-symmetrize both of these in $Z$ and $W$.

## 1.6 Sectional curvature, Ricci tensor, and scalar curvature

Let $\Pi \subset T_pM$ be a 2-plane, and let $X_p, Y_p \in T_pM$ span $\Pi$. Then

$$K(\Pi) = \frac{Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{1.59}$$

is independent of the particular chosen basis for $\Pi$, and is called the *sectional curvature* of the 2-plane $\Pi$. The sectional curvatures in fact determine the full curvature tensor:

**Proposition 1.1.** *Let $Rm$ and $Rm'$ be two $(0,4)$-curvature tensors which satisfy $K(\Pi) = K'(\Pi)$ for all 2-planes $\Pi$, then $Rm = Rm'$.*

From this proposition, if $K(\Pi) = k_0$ is constant for all 2-planes $\Pi$, then we must have

$$Rm(X,Y,Z,W) = k_0\Big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\Big), \tag{1.60}$$

That is

$$Rm = \frac{k_0}{2} g \owedge g. \tag{1.61}$$

In coordinates, this is

$$R_{ijkl} = k_0(g_{ik}g_{jl} - g_{jk}g_{il}). \tag{1.62}$$

We define the *Ricci tensor* as the $(0,2)$-tensor

$$Ric(X,Y) = tr(U \to \mathcal{R}(U,X)Y). \tag{1.63}$$

We clearly have

$$Ric(X,Y) = R(Y,X), \tag{1.64}$$

so $Ric \in \Gamma(S^2(T^*M))$. We let $R_{ij}$ denote the components of the Ricci tensor,

$$Ric = R_{ij} dx^i \otimes dx^i, \tag{1.65}$$

where $R_{ij} = R_{ji}$. From the definition,

$$R_{ij} = R_{lij}{}^{l} = g^{lm} R_{limj}. \tag{1.66}$$

Notice for a space of constant curvature, we have

$$R_{jl} = g^{ik} R_{ijkl} = k_0 g^{ik}(g_{ik}g_{jl} - g_{jk}g_{il}) = (n-1)k_0 g_{jl}, \tag{1.67}$$

or invariantly

$$Ric = (n-1)k_0 g. \tag{1.68}$$

The *Ricci endomorphism* is defined by

$$Rc(X) \equiv \sharp\Big(Ric(X,\cdot)\Big). \tag{1.69}$$

The *scalar curvature* is defined as the trace of the Ricci endomorphism

$$R \equiv tr(X \to Rc(X)). \tag{1.70}$$

In coordinates,

$$R = g^{pq}R_{pq} = g^{pq}g^{lm}R_{lpmq}. \tag{1.71}$$

Note for a space of constant curvature $k_0$,

$$R = n(n-1)k_0. \tag{1.72}$$

# 2 Lecture 2: September 8, 2011

## 2.1 Covariant derivatives of tensor fields

Let $E$ and $E'$ be vector bundles over $M$, with covariant derivative operators $\nabla$, and $\nabla'$, respectively. The covariant derivative operators in $E \otimes E'$ and $Hom(E, E')$ are

$$\nabla_X(s \otimes s') = (\nabla_X s) \otimes s' + s \otimes (\nabla'_X s') \tag{2.1}$$

$$(\nabla_X L)(s) = \nabla'_X(L(s)) - L(\nabla_X s), \tag{2.2}$$

for $s \in \Gamma(E), s' \in \Gamma(E')$, and $L \in \Gamma(Hom(E, E'))$. Note also that the covariant derivative operator in $\Lambda(E)$ is given by

$$\nabla_X(s_1 \wedge \cdots \wedge s_r) = \sum_{i=1}^{r} s_1 \wedge \cdots \wedge (\nabla_X s_i) \wedge \cdots \wedge s_r, \tag{2.3}$$

for $s_i \in \Gamma(E)$.

These rules imply that if $T$ is an $(r, s)$ tensor, then the covariant derivative $\nabla T$ is an $(r, s+1)$ tensor given by

$$\nabla T(X, Y_1, \ldots, Y_s) = \nabla_X(T(Y_1, \ldots Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s). \tag{2.4}$$

We next consider the above definitions in components for $(r, s)$-tensors. For the case of a vector field $X \in \Gamma(TM)$, $\nabla X$ is a $(1, 1)$ tensor field. By the definition of a connection, we have

$$\nabla_m X = \nabla_m(X^j \partial_j) = (\partial_m X^j)\partial_j + X^j \Gamma^l_{mj}\partial_l = (\nabla_m X^i + X^l \Gamma^i_{ml})\partial_i. \tag{2.5}$$

In other words,

$$\nabla X = \nabla_m X^i (dx^m \otimes \partial_i), \tag{2.6}$$

where

$$\nabla_m X^i = \partial_m X^i + X^l \Gamma^i_{ml}. \tag{2.7}$$

However, for a 1-form $\omega$, (2.2) implies that

$$\nabla \omega = (\nabla_m \omega_i) dx^m \otimes dx^i, \tag{2.8}$$

with

$$\nabla_m \omega_i = \partial_m \omega_i - \omega_l \Gamma^l_{im}. \tag{2.9}$$

The definition (2.1) then implies that for a general $(r,s)$-tensor field $S$,

$$\begin{aligned} \nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} \equiv \partial_m S^{i_1 \dots i_r}_{j_1 \dots j_s} &+ S^{l i_2 \dots i_r}_{j_1 \dots j_s} \Gamma^{i_1}_{ml} + \dots + S^{i_1 \dots i_{r-1} l}_{j_1 \dots j_s} \Gamma^{i_r}_{ml} \\ &- S^{i_1 \dots i_r}_{l j_2 \dots j_s} \Gamma^l_{m j_1} - \dots - S^{i_1 \dots i_r}_{j_1 \dots j_{s-1} l} \Gamma^l_{m j_s}. \end{aligned} \tag{2.10}$$

**Remark 2.1.** Some authors instead write covariant derivatives with a semi-colon

$$\nabla_m S^{i_1 \dots i_r}_{j_1 \dots j_s} = S^{i_1 \dots i_r}_{j_1 \dots j_s ; m}. \tag{2.11}$$

However, the $\nabla$ notation fits nicer with our conventions, since the *first* index is the direction of covariant differentiation.

Notice the following calculation,

$$(\nabla g)(X, Y, Z) = X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \tag{2.12}$$

so the metric is parallel. Next, let $I : TM \to TM$ denote the identity map, which is naturally a $(1,1)$ tensor. We have

$$(\nabla I)(X, Y) = \nabla_X (I(Y)) - I(\nabla_X Y) = \nabla_X Y - \nabla_X Y = 0, \tag{2.13}$$

so the identity map is also parallel.

Also note that covariant differentiation commutes with contraction,

$$\nabla_m \left( \delta^{j_1}_{i_1} X^{i_1 i_2 \dots}_{j_1 j_2 \dots} \right) = \delta^{j_1}_{i_1} \nabla_m X^{i_1 i_2 \dots}_{j_1 j_2 \dots} \tag{2.14}$$

## 2.2 Double covariant derivatives

For an $(r,s)$ tensor field $T$, we will write the double covariant derivative as

$$\nabla^2 T = \nabla \nabla T, \tag{2.15}$$

which is an $(r, s+2)$ tensor.

**Proposition 2.1.** *If $T$ is an $(r,s)$-tensor field, then the double covariant derivative satisfies*

$$\nabla^2 T(X, Y, Z_1, \dots, Z_s) = \nabla_X (\nabla_Y T)(Z_1, \dots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \dots Z_s). \tag{2.16}$$

*Proof.* We compute

$$\begin{aligned}\nabla^2 T(X, Y, Z_1, \ldots, Z_s) &= \nabla(\nabla T)(X, Y, Z_1, \ldots, Z_s) \\ &= \nabla_X(\nabla T(Y, Z_1, \ldots, Z_s)) - \nabla T(\nabla_X Y, Z_1, \ldots, Z_s) \\ &\quad - \sum_{i=i}^{s} \nabla T(Y, \ldots, \nabla_X Z_i, \ldots Z_s).\end{aligned} \tag{2.17}$$

The right hand side of (2.16) is

$$\begin{aligned}&\nabla_X(\nabla_Y T)(Z_1, \ldots, Z_s) - (\nabla_{\nabla_X Y} T)(Z_1, \ldots Z_s) \\ &= \nabla_X(\nabla_Y T(Z_1, \ldots, Z_s)) - \sum_{i=1}^{s} (\nabla_Y T)(Z_1, \ldots, \nabla_X Z_i, \ldots, Z_s) \\ &\quad - \nabla_{\nabla_X Y}(T(Z_1, \ldots, Z_s)) + \sum_{i=1}^{s} T(Z_1, \ldots, \nabla_{\nabla_X Y} Z_i, \ldots, Z_s).\end{aligned} \tag{2.18}$$

The first term on the RHS of (2.18) is the same as first term on the RHS of (2.17). The second term on the RHS of (2.18) is the same as third term on the RHS of (2.17) Finally, the last two terms on the RHS of (2.18) are the same as the second term on the RHS of (2.17). □

**Remark 2.2.** When we write

$$\nabla_i \nabla_j T_{i_i \ldots i_s}^{j_1 \ldots j_r} \tag{2.19}$$

we mean the components of the double covariant derivative of $T$ as a $(r, s+2)$ tensor. This does NOT mean to take one covariant derivative $\nabla T$, plug in $\partial_j$ to get an $(r, s)$ tensor, and then take a covariant derivative in the $\partial_i$ direction; this would yield only the first term on the right hand side of (2.16).

## 2.3 Commuting covariant derivatives

Let $X, Y, Z \in \Gamma(TM)$, and compute using Proposition 2.1

$$\begin{aligned}\nabla^2 Z(X, Y) - \nabla^2 Z(Y, X) &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{\nabla_X Y - \nabla_Y X} Z \\ &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \\ &= \mathcal{R}(X, Y) Z,\end{aligned} \tag{2.20}$$

which is just the definition of the curvature tensor. In coordinates,

$$\boxed{\nabla_i \nabla_j Z^k = \nabla_j \nabla_i Z^k + R_{ijm}{}^k Z^m.} \tag{2.21}$$

We extend this to $(p,0)$-tensor fields:

$$
\begin{aligned}
&\nabla^2(Z_1 \otimes \cdots \otimes Z_p)(X,Y) - \nabla^2(Z_1 \otimes \cdots \otimes Z_p)(Y,X)\\
&= \nabla_X(\nabla_Y(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_X Y}(Z_1 \otimes \cdots \otimes Z_p)\\
&\quad - \nabla_Y(\nabla_X(Z_1 \otimes \cdots \otimes Z_p)) - \nabla_{\nabla_Y X}(Z_1 \otimes \ldots \otimes Z_p\\
&= \nabla_X\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\Big) - \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_X Y} Z_i \otimes \cdots \otimes Z_p\\
&- \nabla_Y\Big(\sum_{i=1}^p Z_1 \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\Big) + \sum_{i=1}^p Z_1 \otimes \cdots \nabla_{\nabla_Y X} Z_i \otimes \cdots \otimes Z_p\\
&= \sum_{j=1}^p \sum_{i=1, i\neq j}^p Z_1 \otimes \nabla_X Z_j \otimes \cdots \nabla_Y Z_i \otimes \cdots \otimes Z_p\\
&- \sum_{j=1}^p \sum_{i=1, i\neq j}^p Z_1 \otimes \nabla_Y Z_j \otimes \cdots \nabla_X Z_i \otimes \cdots \otimes Z_p\\
&+ \sum_{i=1}^p Z_1 \otimes \cdots \otimes (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z_i \otimes \cdots \otimes Z_p\\
&= \sum_{i=1}^p Z_1 \otimes \cdots \otimes \mathcal{R}(X,Y) Z_i \otimes \cdots \otimes Z_p.
\end{aligned}
\tag{2.22}
$$

In coordinates, this is

$$
\boxed{\nabla_i \nabla_j Z^{i_1 \ldots i_p} = \nabla_j \nabla_i Z^{i_i \ldots i_p} + \sum_{k=1}^p R_{ijm}{}^{i_k} Z^{i_1 \ldots i_{k-1} m i_{k+1} \ldots i_p}.}
\tag{2.23}
$$

**Proposition 2.2.** *For a 1-form $\omega$, we have*

$$
\nabla^2 \omega(X,Y,Z) - \nabla^2 \omega(Y,X,Z) = \omega(\mathcal{R}(Y,X)Z).
\tag{2.24}
$$

*Proof.* Using Proposition 2.1, we compute

$$
\begin{aligned}
&\nabla^2 \omega(X,Y,Z) - \nabla^2 \omega(Y,X,Z)\\
&= \nabla_X(\nabla_Y \omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z) - \nabla_Y(\nabla_X \omega)(Z) - (\nabla_{\nabla_Y X}\omega)(Z)\\
&= X(\nabla_Y \omega(Z)) - \nabla_Y \omega(\nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z)\\
&\quad - Y(\nabla_X \omega(Z)) + \nabla_X \omega(\nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z)\\
&= X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) + \omega(\nabla_Y \nabla_X Z) - \nabla_X Y(\omega(Z)) + \omega(\nabla_{\nabla_X Y} Z)\\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) - \omega(\nabla_X \nabla_Y Z) + \nabla_Y X(\omega(Z)) - \omega(\nabla_{\nabla_Y X} Z)\\
&= \omega\Big(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\Big) + X(\nabla_Y \omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z))\\
&\quad - Y(\nabla_X \omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z)).
\end{aligned}
\tag{2.25}
$$

The last six terms are

$$
\begin{aligned}
&X(\nabla_Y\omega(Z)) - Y(\omega(\nabla_X Z)) - \nabla_X Y(\omega(Z))\\
&- Y(\nabla_X\omega(Z)) + X(\omega(\nabla_Y Z)) + \nabla_Y X(\omega(Z))\\
&= X\Big(Y(\omega(Z)) - \omega(\nabla_Y Z)\Big) - Y(\omega(\nabla_X Z)) - [X,Y](\omega(Z))\\
&- Y\Big(X(\omega(Z)) - \omega(\nabla_X Z)\Big) + X(\omega(\nabla_Y Z))\\
&= 0.
\end{aligned}
\tag{2.26}
$$

□

**Remark 2.3.** It would have been a bit easier to assume we were in normal coordinates, and assume terms with $\nabla_X Y$ vanished, but we did the above for illustration.

In coordinates, this formula becomes

$$\boxed{\nabla_i\nabla_j\omega_k = \nabla_j\nabla_i\omega_k - R_{ijk}{}^p\omega_p.} \tag{2.27}$$

As above, we can extend this to $(0,s)$ tensors using the tensor product, in an almost identical calculation to the $(r,0)$ tensor case. Finally, putting everything together, the analogous formula in coordinates for a general $(r,s)$-tensor $T$ is

$$\boxed{\nabla_i\nabla_j T^{i_1\dots i_r}_{j_1\dots j_s} = \nabla_j\nabla_i T^{i_1\dots i_r}_{j_1\dots j_s} + \sum_{k=1}^{r} R_{ijm}{}^{i_k} T^{i_1\dots i_{k-1} m i_{k+1}\dots i_r}_{j_1\dots j_s} - \sum_{k=1}^{s} R_{ijj_k}{}^{m} T^{i_1\dots i_r}_{j_1\dots j_{k-1} m j_{k+1}\dots j_s}.} \tag{2.28}$$

## 2.4 Gradient and Hessian

As an example of the above, we consider the Hessian of a function. For $f \in C^1(M,\mathbb{R})$, the *gradient* is defined as

$$\nabla f = \sharp(df), \tag{2.29}$$

which is a vector field. This is standard notation, although in our notation above, $\nabla f = df$, where this $\nabla$ denotes the covariant derivative. The *Hessian* is the $(0,2)$-tensor defined by the double covariant derivative of a function, which by Proposition 2.1 is given by

$$\nabla^2 f(X,Y) = \nabla df(X,Y) = X(df(Y)) - df(\nabla_X Y) = X(Yf) - (\nabla_X Y)f. \tag{2.30}$$

In components, this formula is

$$\nabla^2 f(\partial_i,\partial_j) = \nabla_i\nabla_j f = \partial_i\partial_j f - \Gamma^k_{ij}(\partial_k f). \tag{2.31}$$

The symmetry of the Hessian

$$\nabla^2 f(X,Y) = \nabla^2 f(Y,X), \tag{2.32}$$

then follows easily from the symmetry of the Riemannian connection. No curvature terms appear!

## 2.5 Differential Bianchi Identity

The differential Bianchi identity is

$$\nabla Rm(X,Y,Z,V,W) + \nabla Rm(Y,Z,X,V,W) + \nabla Rm(Z,X,Y,V,W) = 0. \tag{2.33}$$

This can be easily verified using the definition of the covariant derivative of a $(0,4)$ tensor field which was given in the last lecture, and using normal coordinates to simplify the computation. In coordinates, this is equivalent to

$$\boxed{\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0.} \tag{2.34}$$

Let us raise an index,

$$\nabla_i R_{jkm}{}^l + \nabla_j R_{kim}{}^l + \nabla_k R_{ijm}{}^l = 0. \tag{2.35}$$

Contract on the indices $i$ and $l$,

$$0 = \nabla_l R_{jkm}{}^l + \nabla_j R_{klm}{}^l + \nabla_k R_{ljm}{}^l = \nabla_l R_{jkm}{}^l - \nabla_j R_{km} + \nabla_k R_{jm}. \tag{2.36}$$

This yields the Bianchi identity

$$\boxed{\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}.} \tag{2.37}$$

In invariant notation, this is sometimes written as

$$\delta \mathcal{R} = d^\nabla Ric, \tag{2.38}$$

where $d^\nabla : S^2(T^*M) \to \Lambda^2(T^*M) \otimes T^*M$, is defined by

$$d^\nabla h(X,Y,Z) = \nabla h(X,Y,Z) - \nabla h(Y,Z,X), \tag{2.39}$$

and $\delta$ is the *divergence operator*.

Next, trace (2.37) on the indices $k$ and $m$,

$$g^{km}\nabla_l R_{jkm}{}^l = g^{km}\nabla_j R_{km} - g^{km}\nabla_k R_{jm}. \tag{2.40}$$

Since the metric is parallel, we can move the $g^{km}$ terms inside,

$$\nabla_l g^{km} R_{jkm}{}^l = \nabla_j g^{km} R_{km} - \nabla_k g^{km} R_{jm}. \tag{2.41}$$

The left hand side is

$$\begin{aligned} \nabla_l g^{km} R_{jkm}{}^l &= \nabla_l g^{km} g^{lp} R_{jkpm} \\ &= \nabla_l g^{lp} g^{km} R_{jkpm} \\ &= \nabla_l g^{lp} R_{jp} = \nabla_l R^l_j. \end{aligned} \tag{2.42}$$

So we have the Bianchi identity

$$\boxed{2\nabla_l R^l_j = \nabla_j R.} \tag{2.43}$$

Invariantly, this can be written

$$\delta Rc = \frac{1}{2} dR. \tag{2.44}$$

**Corollary 2.1.** *Let $(M, g)$ be a connected Riemannian manifold. If $n > 2$, and there exists a function $f \in C^\infty(M)$ satisfying $Ric = fg$, then $Ric = (n-1)k_0 g$, where $k_0$ is a constant.*

*Proof.* Taking a trace, we find that $R = nf$. Using (2.43), we have

$$2\nabla_l R^l_j = 2\nabla_l \Big(\frac{R}{n}\delta^l_j\Big) = \frac{2}{n}\nabla_l R = \nabla_l R. \tag{2.45}$$

Since $n > 2$, we must have $dR = 0$, which implies that $R$, and therefore $f$, is constant. □

# 3 Lecture 3: September 13, 2011

## 3.1 Gauss Lemma

We begin with a preliminary lemma.

**Lemma 3.1.** *The radial geodesics from a point $p$ are orthogonal to distance spheres around $p$.*

*Proof.* Let $v(s)$ be a curve in $T_pM$ with $\|v(s)\| = r_0$, and define $f(r, s) = \exp(rv(s))$. We compute

$$\frac{\partial}{\partial r}\Big\{g\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big)\Big\} = g\Big(\frac{D}{\partial r}\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big) + g\Big(\frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial s}\Big) \tag{3.1}$$

$$= 0 + g\Big(\frac{\partial f}{\partial r}, \frac{D}{\partial r}\frac{\partial f}{\partial s}\Big) \text{ (since radial curves are geodesics)} \tag{3.2}$$

$$= g\Big(\frac{\partial f}{\partial r}, \frac{D}{\partial s}\frac{\partial f}{\partial r}\Big) \text{ (symmetry of the connection)} \tag{3.3}$$

$$= \frac{1}{2}\frac{\partial}{\partial s}\Big\{g\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial r}\Big)\Big\} \text{ (compatibility of the connection).} \tag{3.4}$$

Notice that $\partial f/\partial r$ at the point $(r, s)$ is the tangent vector to the geodesic $\gamma(r)$ from $p$, with initial tangent vector $v(s)$. Since the norm of a tangent vector to a geodesic is constant in $r$, we have that

$$g\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial r}\Big) = r_0, \tag{3.5}$$

and is therefore independent of $s$. Consequently, the function

$$g\Big(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}\Big) \tag{3.6}$$

must be constant in $r$. But since $f(0, s) = p$, we have

$$\frac{\partial f}{\partial s}\Big|_{r=0} = 0, \tag{3.7}$$

which finishes the proof. □

## 3.2 Normal Coordinates I

We define Euclidean normal coordinates to be the coordinate system given by the exponential map, together with a Euclidean coordinate system $\{x^i\}$ on $T_pM$. We define radial normal coordinates to be

$$\Phi : \mathbb{R}^+ \times S^{n-1} \to M, \tag{3.8}$$

given by

$$(r, \xi) \mapsto \exp(r\xi). \tag{3.9}$$

**Proposition 3.1.** *In Euclidean normal coordinates,*

$$g = g_{Euc} + O(|x|^2), \ \text{as } x \to 0, \tag{3.10}$$

*where $g_{Euc}$ is the standard Euclidean metric. In radial normal coordinates, we have*

$$\Phi^* g = dr^2 + g_{n-1}, \tag{3.11}$$

*where $g_{n-1}$ is a metric on $S^{n-1}$ depending upon $r$, and satisfying*

$$g_{n-1} = r^2 g_{S^{n-1}} + O(r^2), \ \text{as } r \to 0, \tag{3.12}$$

*where $g_{S^{n-1}}$ is the standard metric on the unit sphere.*

*Proof.* For the first statement, we know that $\exp_*(0) = Id$, so the constant term in the Taylor expansion of $g$ is given by $g_{Euc}$. Next, we recall that the geodesic equation is

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0. \tag{3.13}$$

Since the radial directions are geodesics, we can let $\gamma = rv$, where $v$ is any vector. Evaluating the geodesic equation at the origin, we have

$$\Gamma^i_{jk}(0)v^jv^k = 0, \tag{3.14}$$

for arbitrary $v$, so $\Gamma^i_{jk}(0) = 0$ (using symmetry). It is then easy to see from the definition of the Christoffel symbols that all first derivatives of the metric then vanish at $p$.

In normal coordinates, the lines through the origin are geodesics, and therefore have parallel tangent vector field. This implies that the radial component of the metric is $dr^2$. Then (3.11) follows from the Gauss Lemma. Finally, we see that $g_{Euc} = dr^2 + r^2 g_{S^{n-1}}$, so the second expansion follows from the first.

**Remark 3.1.** Notice that the term $r^2 g_{S^{n-1}}$ is indeed $O(1)$ as $r \to 0$. Write $h = g_{S^{n-1}}$, and then fixing some coordinate system on $S^{n-1}$, we compute

$$|r^2h|^2 = r^4 g^{ip}g^{jq}h_{ij}h_{pq} = h^{ip}h^{jq}h_{ij}h_{pq} = (n-1). \tag{3.15}$$

If that is not convincing, then consider the case of $n = 2$. Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $r^2 = x^2 + y^2$, and $\theta = \arctan y/x$. It is then easy to compute that

$$dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{3.16}$$

Note that, in a computation analogous to the above, that $|d\theta| = r^{-1}$. That is, $d\theta$ is not of unit norm, but rather $rd\theta$ is.

□

## 3.3 Jacobi Fields

Let $\gamma(t)$ be a geodesic, and let $\gamma(s,t)$ be a 1-parameter variation of $\gamma(t)$ through geodesics. Then we have

$$\frac{D^2}{dt^2}\Big(\frac{\partial}{\partial s}\gamma(s,t)\Big) = \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial}{\partial s}\gamma(s,t) \tag{3.17}$$
$$= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial t}\gamma(s,t) \ \ \text{(symmetry of the connection)} \tag{3.18}$$
$$= \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial t}\gamma(s,t) + R\Big(\frac{\partial}{\partial t}\gamma, \frac{\partial}{\partial s}\gamma\Big)\frac{\partial}{\partial t}\gamma(s,t) \tag{3.19}$$
$$= R\Big(\frac{\partial}{\partial t}\gamma, \frac{\partial}{\partial s}\gamma\Big)\frac{\partial}{\partial t}\gamma(s,t), \tag{3.20}$$

where the last line follows since $\gamma(s_0,t)$ is a geodesic for fixed $s_0$. Letting

$$J = \frac{\partial}{\partial s}\gamma, \quad \dot{\gamma} = \frac{\partial}{\partial t}\gamma, \tag{3.21}$$

we have that $J$ satisfies the Jacobi equation:

$$\frac{D^2}{dt^2}J + R(J,\dot{\gamma})\dot{\gamma} = 0. \tag{3.22}$$

This is a second order ODE, and the space of solutions is therefore of dimension $2n$. Obviously, $(at+b)\dot{\gamma}$ is a Jacobi field for any constants $a$ and $b$.

**Proposition 3.2.** *Let $(M,g)$ have constant curvature $k_0$, and $\gamma$ be a unit speed geodesic. Then the Jacobi Fields along $\gamma$ which vanish at $t=0$ and which are orthogonal to $\dot{\gamma}$ are given by $f(t)E$ where $E$ is a parallel normal field, and $f$ is given by*

$$f = \begin{cases} Ct & k_0 = 0 \\ C\sin(\sqrt{k_0}\cdot t) & k_0 > 0 \\ C\sinh(\sqrt{-k_0}\cdot t) & k_0 < 0 \end{cases}. \tag{3.23}$$

*Proof.* Let $E$ be a parallel normal vector field along $\gamma$, and consider $f(t)E$. Since $g$ has constant curvature $k_0$, from (1.60) above, we have

$$R(E,\dot{\gamma})\dot{\gamma} = -k_0(\langle E,\dot{\gamma}\rangle\dot{\gamma} - \langle\dot{\gamma},\dot{\gamma}\rangle E) = k_0 E, \tag{3.24}$$

since by assumption $E$ is orthogonal to $\dot{\gamma}$, and $\gamma$ is a unit speed geodesic. Plugging this into the Jacobi equation,

$$(\ddot{f} + k_0 f)E = 0, \tag{3.25}$$

which has the stated solutions. $\square$

**Corollary 3.1.** *If $g$ has constant curvature $k_0$, then in radial normal coordinates the metric has the form*

$$g = \begin{cases} dr^2 + r^2 g_{S^{n-1}} & k_0 = 0 \\ dr^2 + \frac{1}{k_0}\sin^2(\sqrt{k_0}\cdot r) g_{S^{n-1}} & k_0 > 0 \\ dr^2 + \frac{1}{|k_0|}\sinh^2(\sqrt{|k_0|}\cdot r) g_{S^{n-1}} & k_0 < 0 \end{cases} . \tag{3.26}$$

*Proof.* Pulling the metric back to $T_pM$ using the exponential map, we have a metric on $T_pM$ for which lines through the origin are geodesics. Consider the map $\gamma(s,t) = t\xi(s)$, where $\xi(s)$ is any curve. For $s$ fixed, this is a geodesic, so is a 1-parameter variation of geodesics. Call $\xi(0) = \alpha$ and $\xi'(0) = \beta$. From above, we see that

$$\frac{\partial}{\partial s}\gamma\Big|_{s=0} = t\beta \tag{3.27}$$

is a Jacobi field along the geodesic $t \mapsto t\alpha$. From Proposition 3.1, we already know that the metric in radial normal coordinates has the form (3.11). So assume that $\beta$ is orthogonal to $\alpha$ in the Euclidean metric, and that $|\alpha| = 1$. We claim that the Jacobi Field $t\beta$ is orthogonal to $\alpha$ along this geodesic. To see this we compute

$$\frac{d^2}{dt^2}(g(t\beta,\alpha)) = g\Big(\frac{D^2}{dt^2}(t\beta),\alpha\Big) \text{ (since } \alpha \text{ is parallel)} \tag{3.28}$$

$$= g(-R(t\beta,\alpha)\alpha,\alpha) = 0, \tag{3.29}$$

from the skew-symmetry of the curvature tensor. This obviously implies that $g(\beta,\alpha)$ is constant in $t$, and must vanish identically since it vanishes at the origin. From Proposition 3.2 we conclude that

$$t\beta = \begin{cases} CtE & k_0 = 0 \\ C\sin(\sqrt{k_0}\cdot t)E & k_0 > 0 \\ C\sinh(\sqrt{-k_0}\cdot t)E & k_0 < 0 \end{cases} , \tag{3.30}$$

where $E$ is parallel.

If $k_0 = 0$, this says that $\beta$ is a parallel normal field. In particular, $|\beta|$ is independent of the radius, and $|\beta|(r\alpha) = |\beta|(0)$. So the metric in normal cordinates is the Euclidean metric everywhere, which has the stated form in radial coordinates.

If $k_0 > 0$, then

$$\frac{t}{\sin(\sqrt{k_0}\cdot t)}\beta \tag{3.31}$$

is parallel, which implies that

$$|\beta|(r\alpha) = \frac{\sin(\sqrt{k_0}\cdot r)}{\sqrt{k_0}\cdot r}|\beta|(0). \tag{3.32}$$

In radial coordinates, the metric on the sphere of radius $r$ pulls pack to $r^2 g_{S^{n-1}}$, so the $r$ cancels out and we arrive at (3.26). A similar argument holds in the $k_0 < 0$ case. $\square$

This implies that any two space forms of the same constant curvature are locally isometric, but not necessarily globally! The above coordinate system can fail for two reasons. First, one can hit the cut locus, in which case the coordinate system is not injective. Second, the expression for the metric can become degenerate, this is called a conjugate point. Discuss the cut locus in a few examples, such as tori, spheres, projective spaces, lens spaces.

# 4 Lecture 4: September 15, 2011

## 4.1 Taylor expansion of a metric in normal coordinates

**Theorem 4.1.** *In normal coordinates, a metric $g$ admits the expansion*

$$g_{ij} = \delta_{ij} + \frac{1}{3}R_{kijl}x^k x^l + \frac{1}{6}(\nabla_m R_{kijl})x^m x^k x^l \tag{4.1}$$

$$+ \Big(\frac{1}{20}(\nabla_p \nabla_q R_{kijl}) + \frac{2}{45}R_{kil}{}^{r}R_{pjq}{}^{s}\delta_{rs}\Big)x^k x^l x^p x^q + O(|x|^5), \tag{4.2}$$

*as $x \to 0$, where all coefficients are evaluated at 0.*

*Proof.* To compute this, we argue as in the proof of Corollary 3.1. Choose $\beta$ orthogonal to $\alpha$ in the Euclidean metric, and assume that $|\alpha| = 1$. Then $J = t\beta$ is a Jacobi field along the geodesic $t \mapsto t\alpha$. We want to expand the function $f(t) \equiv g(t\beta, t\beta)(t\alpha)$ as a function of $t$. Obviously, $f(0) = 0$, and

$$\partial_t f = \partial_t(g(J, J)) = 2g(D_t J, J). \tag{4.3}$$

Evaluating at 0, $f'(0) = 0$, since $J(0) = 0$. Next,

$$\partial_t^2 g(J, J) = 2g(D_t^2 J, J) + 2g(D_t J, D_t J). \tag{4.4}$$

Evaluating (4.4) at 0, since $J(0) = 0$, and $D_t J = \beta$, we have

$$f''(0) = 2g_0(\beta, \beta), \tag{4.5}$$

where $g_0$ denotes the Euclidean metric at the origin.

To simplify notation, we will let $R_\alpha$ denote the endomorphism $J \mapsto R(\alpha, J)\alpha$, so we can write

$$\partial_t^2 g(J, J) = 2g(R_\alpha(J), J) + 2g(D_t J, D_t J). \tag{4.6}$$

Note that $R_\alpha$ is self-adjoint, i.e.,

$$g(R_\alpha(X), Y) = g(R(\alpha, X)\alpha, Y) = g(R(\alpha, Y)\alpha, X) = g(X, R_\alpha Y), \tag{4.7}$$

from the symmetry of the curvature tensor (1.39).

Differentiating (4.4),

$$\partial_t^3(g(J, J)) = 2g(D_t^3 J, J) + 6g(D_t^2 J, D_t J). \tag{4.8}$$

Evaluating at 0, since $J(0) = 0$, and $D_t^2 J = R_\alpha(J)$, we have

$$f'''(0) = 0. \tag{4.9}$$

Differentiating (4.8),

$$\partial_t^4(g(J,J)) = 2g(D_t^4 J, J) + 8g(D_t^3 J, D_t J) + 6g(D_t^2 J, D_t^2 J). \tag{4.10}$$

Note that

$$D_t^3 J = D_t(D_t^2 J) = D_t(R_\alpha(J)) = (D_t R_\alpha)(J) + R_\alpha(D_t J). \tag{4.11}$$

Evaluating (4.10) at $t = 0$, we obtain

$$f^{(iv)}(0) = 8g_0(R_\alpha(\beta), \beta). \tag{4.12}$$

Differentiating (4.10), we obtain

$$\partial_t^5(g(J,J)) = 2g(D_t^5 J, J) + 10g(D_t^4 J, D_t J) + 20g(D_t^3 J, D_t^2 J). \tag{4.13}$$

The first and last terms vanish at $t = 0$. We compute

$$\begin{align} D_t^4 J &= D_t^2(D_t^2 J) = D_t D_t(R_\alpha(J)) \tag{4.14}\\ &= D_t((D_t R_\alpha)(J) + R_\alpha(D_t J)) \tag{4.15}\\ &= (D_t^2 R_\alpha)(J) + 2(D_t R_\alpha)(D_t J) + R_\alpha(R_\alpha(J)). \tag{4.16} \end{align}$$

Evaluating (4.13) at $t = 0$, we therefore have

$$f^{(v)}(0) = 20g_0(D_t R_\alpha(\beta), \beta). \tag{4.17}$$

Differentiating (4.10), we obtain

$$\partial_t^6(g(J,J)) = 2g(D_t^6 J, J) + 12g(D_t^5 J, D_t J) + 30g(D_t^4 J, D_t^2 J) + 20g(D_t^3 J, D_t^3 J). \tag{4.18}$$

The first and third term on the right hand side vanish at $t = 0$. Using (4.14), we compute and evaluate at $t = 0$:

$$D_t^5 J(0) = 3(D_t^2 R_\alpha)(\beta) + R_\alpha(R_\alpha(\beta)). \tag{4.19}$$

Also,

$$D_t^3 J(0) = D_t(R_\alpha(J))(0) = R_\alpha(\beta). \tag{4.20}$$

Consequently, evaluating (4.18) at the origin,

$$f^{(vi)}(0) = 36g_0(D_t^2 R_\alpha(\beta), \beta) + 32g_0(R_\alpha(\beta), R_\alpha(\beta)). \tag{4.21}$$

Performing a Taylor expansion around $t = 0$, we have shown that

$$\begin{aligned} g(\beta,\beta)(t\alpha) = g_0(\beta,\beta) &+ \frac{t^2}{3} g_0(R_\alpha(\beta),\beta) + \frac{t^3}{6} g_0(D_t R_\alpha(\beta),\beta) \\ &+ \frac{t^4}{20} g_0(D_t^2 R_\alpha(\beta),\beta) + \frac{2t^4}{45} g_0(R_\alpha(\beta), R_\alpha(\beta)). \end{aligned} \tag{4.22}$$

We let $\alpha = (x^i/t)\partial_i$, and $\beta = \beta^j \partial_j$. The first term on the right hand side of (4.22) is simply

$$g_0(\beta,\beta) = \delta_{ij}\beta^i\beta^j. \tag{4.23}$$

The second term on the right hand side of (4.22) is

$$\frac{t^2}{3} g_0(R_\alpha(\beta),\beta) = \frac{t^2}{3} g_0(R(\alpha,\beta)\alpha,\beta) \tag{4.24}$$

$$= \frac{1}{3} g_0(R(x^k\partial_k, \beta^i\partial_i)x^l\partial_l, \beta^j\partial_j) \tag{4.25}$$

$$= \frac{1}{3} x^k x^l R_{kil}{}^m \delta_{mj}\beta^i\beta^j \tag{4.26}$$

$$= \frac{1}{3} R_{kijl} x^k x^l \beta^i\beta^j \tag{4.27}$$

Also, since the Christoffel symbols vanish at $p$, covariant derivatives are just ordinary partial derivatives. We also have that $\partial_t = (x^i/t)\partial_i$. The third term on the right hand side of (4.22) is

$$\frac{t^3}{6} g_0(D_t R_\alpha(\beta),\beta) = \frac{t^3}{6} g_0(\partial_t R(\alpha,\beta)\alpha,\beta) \tag{4.28}$$

$$= \frac{1}{6} g_0(x^m\partial_m R(x^k\partial_k, \beta^i\partial_i)x^l\partial_l, \beta^j\partial_j) \tag{4.29}$$

$$= \frac{1}{6} \nabla_m R_{kil}{}^p \delta_{pj} x^m k^k x^l \beta^i\beta^j \tag{4.30}$$

$$= \frac{1}{6} \nabla_m R_{kijl} x^m k^k x^l \beta^i\beta^j \tag{4.31}$$

The fourth term on the right hand side of (4.22) is

$$\frac{t^4}{20} g_0(D_t^2 R_\alpha(\beta),\beta) = \frac{1}{20} g_0(x^p x^q \partial_p\partial_q R(x^k\partial_k, \beta^i\partial_i)x^l\partial_l, \beta^j\partial_j) \tag{4.32}$$

$$= \frac{1}{20} \nabla_p\nabla_q R_{kil}{}^m \delta_{mj} x^p x^q x^k x^l \beta^i\beta^j \tag{4.33}$$

$$= \frac{1}{20} \nabla_p\nabla_q R_{kijl} x^p x^q x^k x^l \beta^i\beta^j \tag{4.34}$$

The fifth term on the right hand side of (4.22) is

$$\frac{2t^4}{45} g_0(R_\alpha(\beta), R_\alpha(\beta)) = \frac{2}{45} g_0(R(\alpha,\beta)\alpha, R(\alpha,\beta)\alpha)) \tag{4.35}$$

$$= \frac{2}{45} g_0(R(x^k\partial_k, \beta^i\partial_i)x^l\partial_l, R(x^p\partial_p, \beta^j\partial_j)x^q\partial_q) \tag{4.36}$$

$$= \frac{2}{45} R_{kil}{}^r R_{pjq}{}^s \delta_{rs} x^k x^l x^p x^q \beta^i\beta^j. \tag{4.37}$$

Adding together these 5 terms, we obtain the expansion (4.1). □

### 4.2 Original definition of the curvature tensor

In fact, Riemann used the expansion of a metric (4.1) in normal coordinates to originally *define* the curvature tensor. We have

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{kijl} x^k x^l + O(|x|^3), \tag{4.38}$$

as $|x| \to 0$. Then

$$\partial_p g_{ij} = \frac{1}{3} R_{pijl} x^l + \frac{1}{3} R_{kijp} x^k + O(|x|^3) \tag{4.39}$$

and at the origin,

$$\partial_q \partial_p g_{ij}(0) = \frac{1}{3}(R_{pijq} + R_{qijp}), \tag{4.40}$$

so we have the expansion

$$g_{ij} = \delta_{ij} + \frac{1}{3}(R_{kijl} + R_{lijk}) x^k x^l + O(|x|^3), \tag{4.41}$$

as $|x| \to 0$.

But the Taylor series expansion is also written

$$g_{ij} = \delta_{ij} + \frac{1}{2} \partial_k \partial_l g_{ij}(0) x^k x^l + O(|x|^3) \tag{4.42}$$

as $|x| \to 0$, and therefore we must have

$$\frac{1}{2} \partial_k \partial_l g_{ij}(0) = \frac{1}{3}(R_{kijl} + R_{lijk}). \tag{4.43}$$

Riemann used this equation to show there is a unique such $R_{ijkl}$ with the algebraic symmetries of the curvature tensor, defined by

$$R_{ijkl} = -\frac{1}{2}\Big( \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il} + \partial_j \partial_l g_{ik} \Big), \tag{4.44}$$

and proved directly that this defines a tensor. This is opposite to how we define things today, see [Spi79, Chapter 4].

**Exercise 1.** *Directly prove* (4.43) *using* (4.44)*.*

# 5 Lecture 5: September 20, 2011

In this lecture, will give an asymptotic expansion for the volume of geodesic balls. We begin with the volume element.

## 5.1 Expansion of volume element

We will next give an asymptotic expansion for the volume of geodesic balls. We begin with the volume element.

**Claim 5.1.** *We may write* $g = \exp(C)$*, where*

$$C_{ij} = \frac{1}{3}R_{kijl}x^k x^l + \frac{1}{6}(\nabla_m R_{kijl})x^m x^k x^l \tag{5.1}$$

$$+ \Big(\frac{1}{20}(\nabla_p \nabla_q R_{kijl}) - \frac{1}{90}R_{kil}{}^r R_{pjq}{}^s \delta_{rs}\Big)x^k x^l x^p x^q + O(|x|^5), \tag{5.2}$$

*as* $|x| \to 0$*.*

*Proof.* We write

$$g = I + G_2 + G_3 + G_4 + O(r^5), \tag{5.3}$$

and

$$C = C_2 + C_3 + C_4 + O(r^5), \tag{5.4}$$

where the index corresponds to the degree of the term. Then

$$\exp(C) = I + C + \frac{1}{2}C^2 + O(r^5) = 1 + C_2 + C_3 + (C_4 + \frac{1}{2}C_2^2) + O(r^5). \tag{5.5}$$

So we take $C_2 = G_2$, $C_3 = G_3$. The last equation is $C_4 + (1/2)C_2^2 = G_4$, or $C_4 = G_4 - (1/2)G_2^2$. The coefficient of the quadratic curvature term in the expansion is then $(2/45) - (1/18) = -1/90$. □

**Claim 5.2.** *We have*

$$\det(g) = 1 - \frac{1}{3}R_{kl}x^k x^l - \frac{1}{6}(\nabla_m R_{kl})x^m k^k x^l \tag{5.6}$$

$$- \Big(\frac{1}{20}(\nabla_p \nabla_q R_{kl}) + \frac{1}{90}R_{kil}{}^r R_{pjq}{}^s \delta_{rs}\delta^{ij} - \frac{1}{18}R_{kl}R_{pq}\Big)x^k x^l x^p x^q + O(r^5), \tag{5.7}$$

*as* $r \to 0$*.*

*Proof.* We use the formula $det(g) = \det(\exp(C)) = \exp(tr(C))$, and compute (keeping in mind that $R_{ij} = \delta^{pq}R_{piqj}$),

$$tr(C) = -\frac{1}{3}R_{kl}x^k x^l - \frac{1}{6}(\nabla_m R_{kl})x^m x^k x^l \tag{5.8}$$

$$- \Big(\frac{1}{20}(\nabla_p \nabla_q R_{kl}) + \frac{1}{90}R_{kil}{}^r R_{pjq}{}^s \delta_{rs}\delta^{ij}\Big)x^k x^l x^p x^q + O(|x|^5), \tag{5.9}$$

then taking exp as above, we obtain the stated expansion. □

**Theorem 5.1.** *The volume element in normal coordinates has the expansion*

$$\sqrt{\det(g)} = 1 - \frac{1}{6}R_{kl}x^k x^l - \frac{1}{12}(\nabla_m R_{kl})x^m x^k x^l \tag{5.10}$$

$$- \Big(\frac{1}{40}(\nabla_p \nabla_q R_{kl}) + \frac{1}{180}R_{kil}{}^{r}R_{pjq}{}^{s}\delta_{rs}\delta^{ij} - \frac{1}{72}R_{kl}R_{pq}\Big)x^k x^l x^p x^q + O(r^5), \tag{5.11}$$

*Proof.* The Taylor expansion of $\sqrt{y}$ around $y = 1$ is

$$\sqrt{y} = 1 + \frac{1}{2}(y-1) - \frac{1}{8}(y-1)^2 + O(|y-1|^3), \tag{5.12}$$

as $|y| \to 1$. □

**Corollary 5.1.** *If the Ricci tensor is strictly positive definite at $p$ then*

$$\sqrt{\det(g)} < \sqrt{\det(g_{Euc})} \tag{5.13}$$

*in a neighborhood of $p$. If the Ricci tensor is strictly negative definite at $p$ then*

$$\sqrt{\det(g)} > \sqrt{\det(g_{Euc})}, \tag{5.14}$$

*in a neighborhood of $p$.*

The case (5.33) is related to (but not equivalent to) Bishop's volume comparison theorem, which we will discuss later.

## 5.2 Expansion of volume

**Theorem 5.2.** *Denote the volume of a ball of radius $r$ in Euclidean space by $\omega_n r^n$. The expansion for the volume of a metric ball is given by*

$$Vol(r) = \omega_n r^n \Big\{ 1 - \frac{R(p)}{6(n+2)}r^2 + \Big(\frac{-3|Rm|^2 + 8|Ric|^2 + 5R^2 - 18\Delta R}{360(n+2)(n+4)}\Big)r^4 + O(r^6)\Big\}, \tag{5.15}$$

*as $r \to 0$.*

*Proof.* Obviously, no odd terms will appear in this expansion. Moreover, any polynomial of the form

$$(x^1)^{i_1} \dots (x^n)^{i_n} \tag{5.16}$$

restricted to a sphere $S(r)$ will have integral zero if any of the $i_k$ are odd. This follows from the change of variables formula applied to the transformation

$$(x^1, \dots, x^k, \dots, x^n) \mapsto (x^1, \dots, -x^k, \dots, x^n). \tag{5.17}$$

Also, from invariance under permutations of the coordinates, we have

$$\int_{S(1)} (x^i)^2 d\sigma = \int_{S(1)} (x^j)^2 d\sigma, \tag{5.18}$$

which implies that

$$\int_{S(1)} (x^i)^2 d\sigma = \frac{1}{n} \int_{S(1)} d\sigma = \omega_n. \tag{5.19}$$

Next,

$$\begin{aligned}
\int_{B(p,r)} x^k x^l &= \int_0^r \int_{S(r)} x^k x^l r^{n-1} d\sigma dt && (5.20)\\
&= \int_0^r \int_{S(1)} x^k x^l r^{n+1} d\sigma dt && (5.21)\\
&= \frac{1}{n+2} r^{n+2} \int_{S(1)} x^k x^l d\sigma && (5.22)\\
&= \frac{1}{n+2} r^{n+2} \delta_{kl} \int_{S(1)} (x^l)^2 d\sigma && (5.23)\\
&= \frac{\omega_n}{n+2} r^{n+2} \delta_{kl}. && (5.24)
\end{aligned}$$

Consequently,

$$\int_{B(p,r)} R_{kl} x^k x^l dx = \frac{\omega_n R(p)}{n+2} r^{n+2} \tag{5.25}$$

For the next term, we need to consider quartic integrals.

**Claim 5.3.** *For any $i$ and $j$ such that $i \neq j$,*

$$n\omega_n = \int_{S(1)} (x^i)^4 d\sigma + n(n-1) \int_{S(1)} (x^i)^2 (x^j)^2 d\sigma. \tag{5.26}$$

*Proof.* From invariance under permutations of the coordinates, we have

$$\begin{aligned}
n\omega_n = \int_{S(1)} (\sum_i (x^i)^2)^2 d\sigma &= \sum_i \int_{S(1)} (x^i)^4 d\sigma + 2 \sum_{i,j} \int_{S(1)} (x^i)^2 (x^j)^2 d\sigma && (5.27)\\
&= n \int_{S(1)} (x^i)^4 d\sigma + n(n-1) \int_{S(1)} (x^i)^2 (x^j)^2 d\sigma. && (5.28)
\end{aligned}$$

□

**Claim 5.4.** *For any $i$ and $j$ such that $i \neq j$,*

$$\int_{S(1)} (x^i)^4 d\sigma = 3 \int_{S(1)} (x^i)^2 (x^j)^2 d\sigma \tag{5.29}$$

*Proof.* Define a change of coordinates $\tilde{x}^1 = (1/\sqrt{2})(x^1 + x^2)$, $\tilde{x}^2 = (1/\sqrt{2})(x^1 + x^2)$, and $\tilde{x}^j = x^j$ for $j \geq 2$. This is an orthonormal change of basis, so from rotational invariance we have

$$\int_{S(1)} (x^1)^4 d\sigma = \int_{S(1)} (\tilde{x}^1)^4 d\sigma = \frac{1}{4} \int_{S(1)} (x^1 + x^2)^4 d\sigma = \frac{1}{2} \int_{S(1)} ((x^1)^4 + (x^1)^2 (x^2)^2) d\sigma, \tag{5.30}$$

Invariance under permutations of the coordinates then implies the claim. □

Putting together the two claims implies that

$$\int_{S(1)} (x^i)^4 d\sigma = \frac{3}{(n+2)} \omega_n, \tag{5.31}$$

and for $i \neq j$,

$$\int_{S(1)} (x^i)^2 (x^j)^2 d\sigma = \frac{1}{(n+2)} \omega_n, \tag{5.32}$$

The next term in the expansion follows from these formulas. □

**Exercise 2.** *Fill in the details of the last step.*

So the sign of the scalar curvature has a very important geometric meaning: locally it measures the deviation of the volume of balls from the volume of Euclidean balls, to highest order:

**Corollary 5.2.** *If $R(p) > 0$ then*

$$Vol(B(p, r)) < \omega_n r^n, \tag{5.33}$$

*for $r$ sufficiently small. If $R(p) < 0$, then*

$$Vol(B(p, r)) > \omega_n r^n, \tag{5.34}$$

*for $r$ sufficiently small.*

A fantastic reference for the material in this section is [Gra04].

# 6 Lecture 6: September 22, 2011

## 6.1 Geometry of level sets

Fix a point $p$, and let $r(x) = d(p, x)$ denote the distance function to $p$. We know that $\partial_r$ is the unit normal to level sets $S(r)$.We next consider the Hessian of $r$, which is given by the following $\nabla^2 r = \nabla(\partial_r)$. We will view this as an endomorphism of the tangent space of $S(r)$ at any point, and denote this by $S$. Note that of course,

$$S(X) = \nabla_X \partial_r. \tag{6.1}$$

**Proposition 6.1.** *We have*

$$\nabla_{\partial_r} S + S^2 = R_{\partial_r}, \tag{6.2}$$
$$(L_{\partial_r} g) = 2\nabla^2 r, \tag{6.3}$$
$$L_{\partial_r} S = \nabla_{\partial_r} S, \tag{6.4}$$

*where* $R_{\partial_r}$ *is the endomorphism* $X \mapsto R(\partial_r, X)\partial_r$.

*Proof.* For (22.7), we have

$$\begin{aligned}
\nabla_{\partial_r} S(X) + S^2(X) &= \nabla_{\partial_r}(S(X)) - S(\nabla_{\partial_r} X) + S(S(X)) && (6.5)\\
&= \nabla_{\partial_r}(\nabla_X(\partial_r)) - \nabla_{\nabla_{\partial_r} X}(\partial_r) + \nabla_{\nabla_X \partial_r}(\partial_r) && (6.6)\\
&= \nabla_{\partial_r}(\nabla_X(\partial_r)) - \nabla_{[\partial_r, X]}(\partial_r) && (6.7)\\
&= R(\partial_r, X)\partial_r - \nabla_X(\nabla_{\partial_r}(\partial_r)). && (6.8)
\end{aligned}$$

However, the second term is zero. To see this, for any $X$, we have

$$\begin{aligned}
g(\nabla_{\partial_r}\partial_r, X) &= g(S(\partial_r), X) && (6.9)\\
&= g(\partial_r, S(X)) \ \text{ from symmetry of the Hessian} && (6.10)\\
&= g(\partial_r, \nabla_X \partial_r) = \frac{1}{2} X|\partial_r|^2 = 0, && (6.11)
\end{aligned}$$

since $\partial_r$ is a unit vector field.

For (22.12), we compute

$$\begin{aligned}
&(L_{\partial_r} g(X, Y) = \partial_r(g(X, Y)) - g([\partial_r, X], Y) - g(X, [\partial_r, Y]) && (6.12)\\
&= g(\nabla_{\partial_r} X, Y) + g(X, \nabla_{\partial_r} Y) - g(\nabla_{\partial_r} X - \nabla_X \partial_r, Y) - g(X, \nabla_{\partial_r} Y - \nabla_Y \partial_r) && (6.13)\\
&= g(\nabla_X \partial_r, Y) + g(X, \nabla_Y \partial_r) = 2g(S(X), Y) = 2\nabla^2 r(X, Y). && (6.14)
\end{aligned}$$

For (6.4), we have

$$\begin{aligned}
(L_{\partial_r} S)(X) &= L_{\partial_r}(S(X)) - S(L_{\partial_r} X) && (6.15)\\
&= \nabla_{\partial_r}(S(X)) - \nabla_{S(X)}\partial_r - S(\nabla_{\partial_r} X - \nabla_X \partial_r) && (6.16)\\
&= (\nabla_{\partial_r} S)(X) + S(\nabla_{\partial_r}(X)) - S(S(X)) - S(\nabla_{\partial_r} X) + S(S(X) && (6.17)\\
&= (\nabla_{\partial_r} S)(X). && (6.18)
\end{aligned}$$

□

In the case $n = 2$, this is very simple In radial normal coordinates, the metric is

$$g = dr^2 + f^2 d\theta^2, \tag{6.19}$$

With respect to the orthonormal basis $\{\partial_r, f^{-1}\partial_\theta\}$, the hessian of $r$ is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \Delta r \end{pmatrix}. \tag{6.20}$$

We compute

$$
\begin{aligned}
2f\partial_r f &= \partial_r f^2 = \partial_r|\partial_\theta|^2 && (6.21)\\
&= 2g(\nabla_{\partial_r}\partial_\theta, \partial_\theta) = 2g(S(\partial_\theta, \partial_\theta)) && (6.22)\\
&= 2f^2 S(f^{-1}\partial_\theta, f^{-1}\partial_\theta)) = 2f^2(\Delta r). && (6.23)
\end{aligned}
$$

That is,

$$
\Delta r = \frac{\partial_r f}{f}. \tag{6.24}
$$

Plugging in $\partial_\theta$ to (22.7), and then taking an inner product with $\partial_\theta$, we obtain the scalar equation

$$
\partial_r(\Delta r) + (\Delta r)^2 = g(R(\partial_r, \partial_\theta)\partial_r, \partial_\theta) = -K, \tag{6.25}
$$

where $K$ is the Gaussian curvature. Note that we used (6.4) to replace the covariant derivative with an ordinary coordinate derivative. This simplifies to

$$
\partial_r^2 f = -Kf. \tag{6.26}
$$

So the curvature completely determines the metric in dimension 2, since we have the initial conditions $f(0) = 0$, and $f'(0) = 1$ (recalling that $f = r + O(r^2)$ as $r \to 0$). Notice that (6.26) is exactly the Jacobi equation corresponding to $\partial_\theta$, so this viewpoint is equivalent to the previous Jacobi field method.

## 6.2 Bishop's Theorem and Myers' Theorem

We will denote $dV_g = \sqrt{\det(g)}dx = \lambda dx$.

**Proposition 6.2.** *We have*

$$
\partial_r\lambda = (\Delta r)\cdot\lambda \tag{6.27}
$$

$$
\partial_r(\Delta r) + \frac{1}{n-1}(\Delta r)^2 \leq -Ric(\partial_r, \partial_r) \tag{6.28}
$$

$$
\partial_r^2 \sqrt[n-1]{\lambda} \leq -\frac{Ric(\partial_r, \partial_r)}{n-1}\cdot \sqrt[n-1]{\lambda}. \tag{6.29}
$$

*Proof.* For (6.27), we begin with (22.12), which says that $\partial_r g_{ij} = 2\nabla_i\nabla_j r$. We note the formula

$$
\partial_r(\det(g)) = T^{ij}_{n-1}\partial_r g_{ij}, \tag{6.30}
$$

where $T_{n-1}$ is the cofactor matrix. However $g^{-1} = (1/\det(g))T_{n-1}$, so

$$
\partial_r(\det(g)) = \det(g)\cdot(g^{-1})^{ij}\partial_r g_{ij} = 2\det(g)\cdot\Delta r. \tag{6.31}
$$

So we have

$$\partial_r \lambda = \frac{\partial_r \det(g)}{2\sqrt{\det(g)}} = \frac{2\Delta r \cdot \det(g)}{2\sqrt{\det(g)}} = (\Delta r) \cdot \lambda. \tag{6.32}$$

For (6.28), we trace (22.7), which yields

$$\partial_r(\Delta r) + tr(S^2) = tr(X \mapsto R(\partial_r, X)\partial_r) = -Ric(\partial_r, \partial_r). \tag{6.33}$$

However, for any $(n-1) \times (n-1)$ matrix $S$, we have

$$0 \leq |S - (1/n-1)tr(S)g|^2 = |S|^2 - (1/n-1)(tr(S))^2, \tag{6.34}$$

so (6.28) follows upon employing this matrix inequality.

Finally, a simple calculation shows that (6.27) and (6.28) imply (6.29). □

**Proposition 6.3.** *If $Ric \geq (n-1)k$, then*

$$\sqrt[n-1]{\lambda} \leq \begin{cases} r & k = 0 \\ \frac{1}{\sqrt{k}} \sin(\sqrt{k} \cdot r) & k > 0 \\ \frac{1}{\sqrt{|k|}} \sinh(\sqrt{|k|} \cdot r) & k < 0 \end{cases}. \tag{6.35}$$

*Furthermore,*

$$\Delta r \leq \begin{cases} \frac{n-1}{r} & k = 0 \\ (n-1)\cot(\sqrt{k} \cdot r) & k > 0 \\ (n-1)\coth(\sqrt{|k|} \cdot r)E & k < 0 \end{cases}. \tag{6.36}$$

*Proof.* The inequality (6.35) follows upon integration of (6.29). For (6.36), we note that the inequality

$$\partial_r(\Delta r) + \frac{1}{n-1}(\Delta r)^2 \leq -(n-1)k \tag{6.37}$$

can be integrated explicitly using separation of variables. □

**Remark 6.1.** A useful way to write $\Delta r \leq (n-1)/r$ is $\Delta(r^2) \leq 2n$.

**Corollary 6.1.** *If $(M, g)$ is complete, and satisfies $Ric \geq (n-1)k$ then*

$$Vol(B(p, r)) \leq Vol_{g_k}(B_{g_k}(p, r)) \tag{6.38}$$

*where $g_k$ is a metric of constant sectional curvature $k$. Furthermore, if $k > 0$, then*

$$diam(M, g) \leq \frac{\pi}{\sqrt{k}}. \tag{6.39}$$

*Proof.* The estimate (6.38) follows easily from the estimate on the volume element (6.35). For (6.39), (6.36) shows that any geodesic longer that $\pi/\sqrt{k}$ must contain a conjugate point, and therefore cannot be minimizing past this length. □

A nice reference for this section is [Pet06].

# 7 Lecture 7: September 27, 2011

## 7.1 Algebraic study of the curvature tensor

We say that a tensor $A \subset \otimes^4 T^*M$ is in $S^2(\Lambda^2(T^*M))$ if

$$\begin{aligned} A(x,y,z,t) &= -A(y,x,z,t) = -A(x,y,t,z) \\ A(x,y,z,t) &= A(z,t,x,y). \end{aligned} \tag{7.1}$$

Recall that the curvature tensor $Rm$ as a $(0,4)$-tensor satisfies

$$Rm \in S^2(\Lambda^2 T^*M) \subset \otimes^4 T^*M. \tag{7.2}$$

Define a map $b : S^2\Lambda^2 \to \otimes^4 T^*M$ by

$$bA(x,y,z,t) = \frac{1}{3}\Big(A(x,y,z,t) + A(y,z,x,t) + A(z,x,y,t)\Big), \tag{7.3}$$

this is called the *Bianchi symmetrization map.*

**Claim 7.1.** *The Image of $b$ is contained in $S^2\Lambda^2$.*

*Proof.* First,

$$bA(y,x,z,t) = \frac{1}{3}\Big(A(y,x,z,t) + A(x,z,y,t) + A(z,y,x,t)\Big) \tag{7.4}$$

$$= \frac{1}{3}\Big(-A(x,y,z,t) - A(z,x,y,t) - A(y,z,x,t)\Big) = -bA(x,y,z,t), \tag{7.5}$$

and skew-symmetry in the last two indices is proved similarly. Next,

$$\begin{aligned} bA(z,t,x,y) &= \frac{1}{3}\Big(A(z,t,x,y) + A(t,x,z,y) + A(x,z,t,y)\Big) &(7.6)\\ &= \frac{1}{3}\Big(A(x,y,z,t) + A(z,y,t,x) - A(z,x,t,y)\Big) &(7.7)\\ &= \frac{1}{3}\Big(A(x,y,z,t) + A(y,z,x,t) + A(z,x,y,t)\Big) = bA(x,y,z,t). &(7.8) \end{aligned}$$

$\square$

Next, we have

**Claim 7.2.** *The space $S^2(\Lambda^2)$ decomposes as*

$$S^2(\Lambda^2) = Ker(b) \oplus Im(b), \tag{7.9}$$

*which is an orthogonal direct sum (using the tensor inner product defined above in Subsection 1.3).*

*Proof.* It is very easy to that $b$ is an idempotent, that is $b^2 = b$. Next, given $A \in S^2(\Lambda^2)$, consider $B = A - f(A)$. Then $f(B) = f(A) - f(f(A)) = 0$. We next claim that $b$ is self-adjoint. To see, we compute in an orthonormal basis

$$g(A, bB) = \frac{1}{3} A_{ijkl}(B_{ijkl} + B_{jkil} + B_{kijl}) \tag{7.10}$$

$$= \frac{1}{3}(A_{ijkl}B_{ijkl} + A_{ijkl}B_{jkil} + A_{ijkl}B_{kijl}) \tag{7.11}$$

$$= \frac{1}{3}(A_{ijkl}B_{ijkl} + A_{kijl}B_{ijkl} + A_{jkil}B_{ijkl}) \tag{7.12}$$

$$= g(bA, B). \tag{7.13}$$

Clearly then, if $A \in Ker(b)$, and $B = b(C) \in Im(b)$, we have

$$g(A, B) = g(A, b(C)) = g(bA, C) = 0, \tag{7.14}$$

that is, $b$ is an orthogonal projection. □

Next, we identify the image of $b$.

**Claim 7.3.** *We have*

$$Im(b) = \Lambda^4 T^* M. \tag{7.15}$$

*Proof.* To see this, we claim that

$$b(\alpha \odot \beta) = \frac{1}{3} \alpha \wedge \beta, \tag{7.16}$$

where $\alpha, \beta \in \Lambda^2(T^*M)$, and $\odot$ denotes the symmetric product. Here, we are thinking of $\Lambda^2(T^*M)$ as skew-symmetric $(0,2)$ tensors. We have that

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij} \tag{7.17}$$

(note that our symmetric product does not have a factor of 1/2, just like our wedge product). So the left hand side of (7.16) is

$$(b(\alpha \odot \beta))_{ijkl} = \frac{1}{3}(\alpha_{ij}\beta_{kl} + \beta_{ij}\alpha_{kl} + \alpha_{jk}\beta_{il} + \beta_{jk}\alpha_{il} + \alpha_{ki}\beta_{jl} + \beta_{ki}\alpha_{jl}). \tag{7.18}$$

Under our identification of 2-forms with $(0,2)$ tensors, the wedge product is given by

$$\alpha \wedge \beta(e_i, e_j, e_k, e_l) = \frac{1}{2!\ 2!} \sum_{\sigma} \alpha(e_{\sigma(1)}, e_{\sigma(2)}) \cdot \beta(e_{\sigma(3)}, e_{\sigma(4)}), \tag{7.19}$$

and the sum is over all permutations of length 4. This can be rewritten as

$$\alpha \wedge \beta(e_i, e_j, e_k, e_l) = \sum_{\sigma(1)<\sigma(2), \sigma(3)<\sigma(4)} \alpha(e_{\sigma(1)}, e_{\sigma(2)}) \cdot \beta(e_{\sigma(3)}, e_{\sigma(4)}) \tag{7.20}$$

$$= \alpha(e_i, e_j) \cdot \beta(e_k, e_l) - \alpha(e_i, e_k) \cdot \beta(e_j, e_l) + \alpha(e_i, e_l) \cdot \beta(e_j, e_k) \tag{7.21}$$

$$+ \alpha(e_j, e_k) \cdot \beta(e_i, e_l) - \alpha(e_j, e_l) \cdot \beta(e_i, e_k) + \alpha(e_k, e_l) \cdot \beta(e_i, e_j) \tag{7.22}$$

$$= \alpha_{ij}\beta_{kl} - \alpha_{ik}\beta_{jl} + \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{jl}\beta_{ik} + \alpha_{kl}\beta_{ij}. \tag{7.23}$$

Comparing (7.18), we see that the terms agree, up to the factor of 1/3. This clearly implies the claim. □

**Remark 7.1.** Note that this implies $b \equiv 0$ if $n = 2, 3$, and $dim(Im(b)) = 1$ if $n = 4$.

**Definition 1.** The space of *curvature-like tensors* is

$$\mathcal{C} = Ker(b) \subset S^2(\Lambda^2). \tag{7.24}$$

Consider the decomposition

$$S^2(\Lambda^2) = \mathcal{C} \oplus \Lambda^4. \tag{7.25}$$

If $V$ is a vector space of dimension $p$, then

$$dim(S^2(V)) = \frac{p(p+1)}{2}. \tag{7.26}$$

Since

$$dim(\Lambda^2) = \frac{n(n-1)}{2}, \tag{7.27}$$

we find that

$$dim S^2(\Lambda^2) = \frac{1}{8}n(n-1)(n^2 - n + 2). \tag{7.28}$$

Also,

$$dim(\Lambda^4) = \binom{n}{4}, \tag{7.29}$$

which yields

$$\begin{aligned} dim(\mathcal{C}) &= \frac{1}{8}n(n-1)(n^2 - n + 2) - \frac{1}{24}n(n-1)(n-2)(n-3) \\ &= \frac{1}{12}n^2(n^2 - 1). \end{aligned} \tag{7.30}$$

Recall the *Ricci contraction*, $c : \mathcal{C} \to S^2(T^*M)$, defined by

$$(c(Rm))(X, Y) = tr(U \to \sharp Rm(U, X, \cdot, Y)). \tag{7.31}$$

In components, we have

$$c(Rm) = R_{lij}{}^{l} dx^i \otimes dx^j = g^{pq} R_{ipjq} dx^i \otimes dx^j. \tag{7.32}$$

Recall the Kulkarni-Nomizu product $\owedge : S^2(T^*M) \times S^2(T^*M) \to \mathcal{C}$ defined by

$$\begin{aligned} h \owedge k(X, Y, Z, W) &= h(X, Z)k(Y, W) - h(Y, Z)k(X, W) \\ &\quad - h(X, W)k(Y, Z) + h(Y, W)k(X, Z). \end{aligned} \tag{7.33}$$

Note that $h \owedge k = k \owedge h$.

**Proposition 7.1.** *The map $\psi : S^2(T^*M) \to \mathcal{C}$ defined by*

$$\psi(h) = h \owedge g, \tag{7.34}$$

*is injective for $n > 2$.*

*Proof.* First note that

$$\langle f, h \owedge g\rangle = 4\langle cf, h\rangle. \tag{7.35}$$

To see this, we compute (in an orthonormal basis)

$$\begin{aligned} &f_{ijkl}(h_{ik}g_{jl} - h_{jk}g_{il} - h_{il}g_{jk} + h_{jl}g_{ik}) \\ &= f_{ijkj}h_{ik} - f_{ijki}h_{jk} - f_{ijjl}h_{il} + f_{ijil}h_{jl} \\ &= 4f_{ijkj}h_{ik}. \end{aligned} \tag{7.36}$$

Also note that

$$c(h \owedge g) = (n-2)h + (tr(h))g. \tag{7.37}$$

To see this

$$\begin{aligned} c(h \owedge g) &= \sum_{j,l}(h \owedge g)_{ijkl} \\ &= \sum_{j,l}(h_{ik}g_{jl} - h_{jk}g_{il} - h_{il}g_{jk} + h_{jl}g_{ik} \\ &= nh_{ik} - h_{jk}g_{ij} - h_{ij}g_{jk} + (tr(h))g_{ik} \\ &= (n-2)h + (tr(h))g. \end{aligned} \tag{7.38}$$

To prove the proposition, assume that $h \owedge g = 0$. Then

$$\begin{aligned} 0 &= \langle h \owedge g, h \owedge g\rangle \\ &= 4\langle h, c(h \owedge g)\rangle \\ &= 4\langle h, (n-2)h + (tr(h))g\rangle \\ &= 4\Big((tr(h))^2 + (n-2)|h|^2\Big), \end{aligned} \tag{7.39}$$

which clearly implies that $h = 0$ if $n > 2$. ☐

**Corollary 7.1.** *For $n = 2$, the scalar curvature determines the full curvature tensor. For $n = 3$, the Ricci curvature determines the full curvature tensor.*

*Proof.* The $n = 2$ case is trivial, since the only non-zero component of $R$ can be $R_{1212}$. For any $n$, define the *Schouten tensor*

$$A = \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)}g\right). \tag{7.40}$$

We claim that

$$c(Rm - A \owedge g) = 0. \tag{7.41}$$

To see this, we first compute

$$tr(A) = \frac{1}{n-2}\left(R - \frac{nR}{2(n-1)}\right) = \frac{R}{2(n-1)}. \tag{7.42}$$

Then

$$\begin{aligned} c(Rm - A \owedge g) = c(Rm) - c(A \owedge g) &= Ric - \Big((n-2)A + (tr(A))g\Big) \\ &= Ric - \Big(Ric - \frac{R}{2(n-1)}g + \frac{R}{2(n-1)}g\Big) \\ &= 0. \end{aligned} \tag{7.43}$$

For $n = 3$, we have $dim(\mathcal{C}) = 6$. From the proposition, we also have

$$\psi : S^2(T^*M) \hookrightarrow \mathcal{C}. \tag{7.44}$$

But $dim(S^2(T^*)) = 6$, so $\psi$ is an isomorphism. This implies that

$$Rm = A \owedge g. \tag{7.45}$$

□

**Remark 7.2.** The above argument of course implies that, in any dimension, the curvature tensor can always be written as

$$Rm = W + A \owedge g, \tag{7.46}$$

where $W \in Ker(c)$. The tensor $W$ is called the *Weyl tensor*, which we will study in depth a bit later.

# 8 Lecture 8: September 29

## 8.1 Orthogonal decomposition of the curvature tensor

Last time we showed that the curvature tensor may be decomposed as

$$Rm = W + A \owedge g, \tag{8.1}$$

where $W \in Ker(c)$ is the *Weyl tensor*, and $A$ is the *Schouten tensor*. We can rewrite this as

$$Rm = W + \frac{1}{n-2}E \owedge g + \frac{R}{2n(n-1)}g \owedge g, \tag{8.2}$$

where

$$E = Ric - \frac{R}{n} g \tag{8.3}$$

is the *traceless Ricci tensor*. In general, we will have

$$\begin{aligned} S^2(\Lambda^2(T^*M)) &= \Lambda^4(T^*M) \oplus \mathcal{C} \\ &= \Lambda^4 \oplus \mathcal{W} \oplus \psi(S_0^2(T^*M)) \oplus \psi(\mathbb{R}g), \end{aligned} \tag{8.4}$$

where $\mathcal{W} = Ker(c) \cap Ker(b)$. This turns out to be an irreducible decomposition as an $SO(n)$-module, except in dimension 4. In this case, the $\mathcal{W}$ splits into two irreducible pieces $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$. We will discuss this in detail later.

**Proposition 8.1.** *The decomposition (8.2) is orthogonal.*

*Proof.* From above,

$$\langle W, h \owedge g \rangle = 4 \langle cW, h \rangle = 0, \tag{8.5}$$

so the Weyl tensor is clearly orthogonal to the other 2 terms. Next,

$$\langle E \owedge g, g \owedge g \rangle = \langle E, c(g \owedge g) \rangle = \langle E, 2(n-1)g \rangle = 0. \tag{8.6}$$

□

To compute these norms, note that for any tensor $B$,

$$\begin{aligned} |B \owedge g|^2 &= \langle B \owedge g, B \owedge g \rangle \\ &= 4 \langle B, c(B \owedge g) \rangle \\ &= 4 \langle B, (n-2)B + tr(B) g \rangle \\ &= 4(n-2)|B|^2 + 4(tr(B))^2. \end{aligned} \tag{8.7}$$

The decomposition (7.46) yields

$$|Rm|^2 = |W|^2 + 4(n-2)|A|^2 + 4(tr(A))^2, \tag{8.8}$$

while the decomposition (8.2) yields

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|E|^2 + \frac{2}{n(n-1)} R^2. \tag{8.9}$$

Note that

$$\begin{aligned} |E|^2 = E_{ij}E_{ij} &= (R_{ij} - \frac{R}{n} g_{ij})(R_{ij} - \frac{R}{n} g_{ij}) \\ &= |Ric|^2 - \frac{2}{n} R^2 + \frac{1}{n} R^2 \\ &= |Ric|^2 - \frac{1}{n} R^2, \end{aligned} \tag{8.10}$$

so we obtain

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|Ric|^2 - \frac{2}{(n-1)(n-2)} R^2. \tag{8.11}$$

## 8.2 The curvature operator

Above, we said that $P \in S^2(\Lambda^2)$ if $P$ is a $(0,4)$ tensor satisfying (7.1). But now we would like to view such an element as a symmetric mapping $\mathcal{P} : \Lambda^2 \to \Lambda^2$. In the remainder of this section, we will perform computations in a local oriented ONB. For a 2-form $\omega$, the components of $\omega$ are defined by

$$\omega_{ij} = \omega(e_i, e_j), \tag{8.12}$$

so that the 2-form can be written

$$\omega = \frac{1}{2} \sum_{i,j} \omega_{ij} e^i \wedge e^j. \tag{8.13}$$

We define the operator $\mathcal{P}$ as follows: in an local ONB, we write

$$(\mathcal{P}\omega)_{ij} \equiv \frac{1}{2} \sum_{k,l} P_{ijkl} \omega_{kl}. \tag{8.14}$$

Another way to write this is

$$\mathcal{P}\omega = \frac{1}{2} \sum_{i,j} (\mathcal{P}\omega)_{ij} e^i \wedge e^j = \frac{1}{4} \sum_{i,j,k,l} P_{ijkl} \omega_{kl} e^i \wedge e^j. \tag{8.15}$$

That is,

$$\mathcal{P}\omega = \frac{1}{4} \sum_{i,j,k,l} P_{ijkl} \omega_{kl} e^i \wedge e^j. \tag{8.16}$$

It is easy to see that $\mathcal{P}$ is well-defined and independent of the particular basis chosen.

**Claim 8.1.** *The operator $\mathcal{P}$ is symmetric. That is,*

$$\langle \mathcal{P}\omega_1, \omega_2 \rangle_{\Lambda^2} = \langle \omega_1, \mathcal{P}\omega_2 \rangle_{\Lambda^2}. \tag{8.17}$$

*Proof.* Using the identity

$$P(X,Y,Z,W) = P(Z,W,X,Y), \tag{8.18}$$

we compute in an ONB

$$\langle \mathcal{P}\alpha, \beta \rangle_{\Lambda^2} = \frac{1}{2} \sum_{k,l} (\mathcal{P}\alpha)_{kl} \beta_{kl} = \frac{1}{4} \sum_{i,j,k,l} P_{ijkl} \alpha_{ij} \beta_{kl} \tag{8.19}$$

$$= \frac{1}{4} \sum_{i,j,k,l} \alpha_{ij} (P_{ijkl} \beta_{kl}) = \langle \alpha, \mathcal{P}\beta \rangle_{\Lambda^2}. \tag{8.20}$$

□

Conversely, any symmetric operator $\mathcal{P} : \Lambda^2 \to \Lambda^2$ is equivalent to a $(0,4)$ tensor, by

$$P_{pqrs} = \langle \mathcal{P}(e^p \wedge e^q), e^r \wedge e^s \rangle_{\Lambda^2}. \tag{8.21}$$

**Claim 8.2.** *These maps are inverses of each other.*

*Proof.* We first write down the components of $e^p \wedge e^q$:

$$e^p \wedge e^q = \frac{1}{2} \sum_{i,j} (e^p \wedge e^q)_{ij} e^i \wedge e^j = \frac{1}{2} \sum_{i,j} \delta^{pq}_{ij} e^i \wedge e^j, \tag{8.22}$$

so the components of $e^p \wedge e^q$ are given by $(e^p \wedge e^q)_{ij} = \delta^{pq}_{ij}$, the generalized Kronecker delta symbol, which is defined to be $+1$ if $(p,q) = (i,j)$, $-1$ if $(p,q) = (j,i)$, and 0 otherwise. Next, for the operator $\mathcal{P}$ associated to $P_{ijkl}$, its components (as defined in (8.21)) are given by

$$\begin{aligned}
P_{pqrs} &= \langle \mathcal{P}(e^p \wedge e^q), e^r \wedge e^s \rangle_{\Lambda^2} \\
&= \langle \frac{1}{4} \sum_{i,j,k,l} P_{ijkl} (e^p \wedge e^q)_{kl} e^i \wedge e^j, e^r \wedge e^s \rangle_{\Lambda^2} \\
&= \frac{1}{4} \langle \sum_{i,j,k,l} P_{ijkl} \delta^{pq}_{kl} e^i \wedge e^j, e^r \wedge e^s \rangle_{\Lambda^2} \\
&= \frac{1}{4} \langle \sum_{i,j} (P_{ijpq} - P_{ijqp}) e^i \wedge e^j, e^r \wedge e^s \rangle_{\Lambda^2} \\
&= \frac{1}{2} \langle \sum_{i,j} P_{ijpq} e^i \wedge e^j, e^r \wedge e^s \rangle_{\Lambda^2} \\
&= \frac{1}{2} (P_{rspq} - P_{srpq}) = P_{pqrs}.
\end{aligned} \tag{8.23}$$

□

The above construction applied to $Rm$ yields

$$\mathcal{R} \in \Gamma\Big( End\big(\Lambda^2(T^*M)\big)\Big), \tag{8.24}$$

which is called the *curvature operator*. Note that since any symmetric matrix can be diagonalized, $\mathcal{R}$ has $n(n-1)/2$ real eigenvalues, counted with multiplicity.

## 8.3 Curvature in dimension three

For $n = 3$, the Weyl tensor vanishes, so the curvature decomposes as

$$Rm = A \owedge g = (Ric - \frac{R}{4} g) \owedge g = Ric \owedge g - \frac{R}{4} g \owedge g, \tag{8.25}$$

in coordinates,

$$R_{ijkl} = R_{ik} g_{jl} - R_{jk} g_{il} - R_{il} g_{jk} + R_{jl} g_{ik} - \frac{R}{2}(g_{ik} g_{jl} - g_{jk} g_{il}). \tag{8.26}$$

The sectional curvature in the plane spanned by $\{e_i, e_j\}$ is

$$\begin{aligned} R_{ijij} &= R_{ii}g_{jj} - R_{ji}g_{ij} - R_{ij}g_{ji} + R_{jj}g_{ii} - \frac{R}{2}(g_{ii}g_{jj} - g_{ji}g_{ij}) \\ &= R_{ii}g_{jj} - 2R_{ij}g_{ij} + R_{jj}g_{ii} - \frac{R}{2}(g_{ii}g_{jj} - g_{ij}g_{ij}). \end{aligned} \tag{8.27}$$

Note we do not sum repeated indices in the above equation! Choose an ONB so that the $Rc$ is diagonalized at a point $p$,

$$Rc = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{8.28}$$

In this ONB, $R_{ij} = \lambda_i \delta_{ij}$ (again we do not sum!). Then the sectional curvature is

$$\begin{aligned} R_{ijij} &= \lambda_i g_{jj} - 2\lambda_i g_{ij} g_{ij} + \lambda_j g_{ii} - \frac{\lambda_1 + \lambda_2 + \lambda_3}{2}(g_{ii}g_{jj} - g_{ij}g_{ij}) \\ &= \lambda_i - 2\lambda_i \delta_{ij} + \lambda_j - \frac{\lambda_1 + \lambda_2 + \lambda_3}{2}(1 - \delta_{ij}). \end{aligned} \tag{8.29}$$

We obtain

$$\begin{aligned} R_{1212} &= \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3) \\ R_{1313} &= \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3) \\ R_{2323} &= \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3). \end{aligned} \tag{8.30}$$

We can also express the Ricci eigenvalues in terms of the sectional curvatures

$$Rc = \begin{pmatrix} R_{1212} + R_{1313} & 0 & 0 \\ 0 & R_{1212} + R_{2323} & 0 \\ 0 & 0 & R_{1313} + R_{2323} \end{pmatrix}. \tag{8.31}$$

We note the following, define

$$T_1(A) = -A + tr(A)g = -Ric + \frac{R}{2}g. \tag{8.32}$$

Since $Rc$ is diagonal, $T_1(A)$ takes the form

$$T_1(A) = \begin{pmatrix} R_{2323} & 0 & 0 \\ 0 & R_{1313} & 0 \\ 0 & 0 & R_{1212} \end{pmatrix}. \tag{8.33}$$

That is, the eigenvalue of $T_1(A)$ with eigenvector $e_i$ is equal to the sectional curvature of the 2-plane orthogonal to $e_i$.

# 9 Lecture 9: October 4

## 9.1 Dimension 3 continued

We recall the Hodge star operator in dimension 3, $* : \Lambda^k \to \Lambda^{3-k}$. In coordinates, this operator is given as follows. If $\alpha$ is a 1-form, then

$$(*\alpha)_{ij} = \epsilon_{ij}{}^{p}\alpha_p = g^{kp}\epsilon_{ijk}\alpha_p, \tag{9.1}$$

where $\epsilon_{ijk}$ are the components of the volume form. If $\omega$ is a 2-form, then

$$(*\omega)_i = \epsilon_i{}^{pq}\omega_{pq} = g^{ip}g^{jq}\epsilon_{ijk}\omega_{pq}. \tag{9.2}$$

**Proposition 9.1.** *For 1-forms $\alpha$ and $\beta$,*

$$\langle \mathcal{R}(*\alpha), *\beta \rangle_{\Lambda^2} = G(\alpha, \beta), \tag{9.3}$$

*where $G$ is the Einstein tensor $G = T_1(A) = -Ric + (R/2)g$.*

*Proof.* We compute in an orthonormal basis

$$\begin{aligned}\langle \mathcal{R}(*\alpha), *\beta \rangle_{\Lambda^2} &= \frac{1}{2}\sum_{i,j}(\mathcal{R}(*\alpha))_{ij}(*\beta)_{ij} &(9.4)\\ &= \frac{1}{4}\sum_{i,j,k,l} R_{ijkl}(*\alpha)_{kl}(*\beta)_{ij} = \frac{1}{4}\sum_{i,j,k,l,p,q} R_{ijkl}\epsilon_{klp}\epsilon_{ijq}\alpha_p\beta_q. &(9.5)\end{aligned}$$

So we must show that

$$\frac{1}{4}\sum_{i,j,k,l} R_{ijkl}\epsilon_{klp}\epsilon_{ijq} = G_{pq}. \tag{9.6}$$

As above, we choose an orthonormal basis which diagonalizes $Ric$, then of course $G$ is also diagonalized, so we only need to examine the entries for which $p = q$. For example, when $p = q = 1$,

$$\frac{1}{4}\sum_{i,j,k,l} R_{ijkl}\epsilon_{kl1}\epsilon_{ij1} = \frac{1}{4}\sum_{kl}(R_{23kl} - R_{32kl})\epsilon_{kl1} = \frac{1}{2}\sum_{kl} R_{23kl}\epsilon_{kl1} = R_{2323}. \tag{9.7}$$

But from (8.33) above, this is equal to $T_1(A)_{11}$. A similar computation handles the cases $p = q = 2$ and $p = q = 3$. $\square$

**Remark 9.1.** At a point $p$, the sectional curvature is really a function on the Grassmannian of 2-planes $G(2, T_pM)$. But in dimension 3, $G(2, T_pM) = G(1, T_pM) = \mathbb{P}(T_pM)$, so sectional curvature can be viewed as a function on $\mathbb{RP}^2$, and the above says this function is simply $\{G(\alpha, \alpha), |\alpha| = 1\}$.

Recall that a linear operator is said to be 2-*positive* if the sum of the two smallest eigenvalues is positive, and 2-*negative* if the sum of the two largest eigenvalues is negative. The above implies the following:

**Proposition 9.2.** *In dimension* 3*, we have the following implications:*

$$
\begin{aligned}
K(\sigma) > 0 \textit{ for any 2-plane } \sigma &\iff Ric < (R/2)g, && (9.8)\\
K(\sigma) < 0 \textit{ for any 2-plane } \sigma &\iff Ric > (R/2)g, && (9.9)\\
\mathcal{R} \textit{ is positive (negative) definite} &\iff K \textit{ is positive (negative)}, && (9.10)\\
Ric \textit{ is positive (negative) definite} &\iff \mathcal{R} \textit{ is 2-positive(negative)}. && (9.11)
\end{aligned}
$$

*Proof.* Implications (9.8) and (9.9) follow easily from (9.3). For (9.10), we see that definiteness of the curvature operator is equivalent to definiteness of $G$, which we have just seen is equivalent to a sign on the sectional curvature. Next, in a basis which diagonalizes Ricci, (9.3) and (8.31) show that $\mathcal{R}$ is also diagonalized with eigenvalues $R_{1212}, R_{1313}$, and $R_{2323}$, and (9.11) follows. □

In general, positive Ricci curvature is a much weaker assumption that positive sectional curvature.

## 9.2 Symmetric Powers

In this subsection, we do some basic counting which will be useful later. Let $V$ be a vector space of dimension $n$, and consider $S^k(V)$ the space of symmetric tensors on $V$. Let $S^k_0(V) \subset S^k(V)$ be the symmetric tensors which are totally traceless, that is, traceless on any pair of indices.

**Proposition 9.3.** *We have*

$$\dim(S^k(V)) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}, \tag{9.12}$$

$$\dim(S^k_0(V)) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}. \tag{9.13}$$

*Proof.* The space $S^k(V)$ can be identified with the space of homogeneous polynomials of degree $k$ on $V$, by sending $a_{i_1\dots i_k}$ to

$$\sum_{i_1,\dots,i_k} a_{i_1\dots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{9.14}$$

We can identify a basis for $S^k(V)$ with the number of integer solutions to

$$i_1 + i_2 + \cdots + i_n = k,\ i_p \geq 0,\ 1 \leq p \leq n, \tag{9.15}$$

by sending $(i_1, \dots, i_k)$ to the polynomial

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}. \tag{9.16}$$

First, let us count the number of integer solutions to

$$j_1 + j_2 + \cdots + j_n = k,\ i_p \geq 1,\ 1 \leq p \leq n. \tag{9.17}$$

This problem can be thought of as putting $k$ balls into $n$ urns, with at least 1 ball in each urn. We can specify this by listing $k$ identical objects, and choosing $n-1$ of the spaces in between. Since there are $k-1$ spaces in between, this has dimension

$$\binom{k-1}{n-1}. \tag{9.18}$$

By letting $i_r = j_r + 1,\ i = 1 \dots n$, the first problem is transformed into the second problem with $n+k$ on the right hand side, and we obtain (9.12).

We will let $\mathcal{P}^k(\mathbb{R}^n)$ denote the space of homogeneous polynomials of degree $k$ on $\mathbb{R}^n$ (which we have identified with $S^k(\mathbb{R}^n)$). Then $S^k_0(\mathbb{R}^n)$ corresponds to a subspace, which we denote by $\mathcal{H}^k(\mathbb{R}^n)$. It is easy to see that this is the space of homogeneous *harmonic* polynomials of degree $k$ on $\mathbb{R}^n$. Obviously, $\mathcal{P}^0 = \mathcal{H}^0$, and $\mathcal{P}^1 = \mathcal{H}^1$. We claim that for $k \geq 2$, we have

$$\mathcal{P}^k(\mathbb{R}^n) = \mathcal{H}^k(\mathbb{R}^n) \oplus |x|^2 \mathcal{P}^{k-2}(\mathbb{R}^n). \tag{9.19}$$

This follows since these spaces are clearly orthogonal complements of each other under the tensor inner product (the latter space consisting of tensors which are pure trace in a pair of indices). The formula (9.13) follows from this and (9.12). □

Iterating (9.19) yields the following decomposition. For $k$ even,

$$S^k(\mathbb{R}^n) = \mathcal{H}^k(\mathbb{R}^n) \oplus |x|^2 \mathcal{H}^{k-2}(\mathbb{R}^n) \oplus \dots \oplus |x|^k \mathcal{H}^0(\mathbb{R}^n), \tag{9.20}$$

and if $k$ is odd,

$$S^k(\mathbb{R}^n) = \mathcal{H}^k(\mathbb{R}^n) \oplus |x|^2 \mathcal{H}^{k-2}(\mathbb{R}^n) \oplus \dots \oplus |x|^{k-1} \mathcal{H}^1(\mathbb{R}^n). \tag{9.21}$$

This is in fact an irreducible decomposition of $S^k(\mathbb{R}^n)$ under the orthogonal group $\mathrm{O}(n)$.

## 9.3 Representations of SO(3)

In this subsection, we present an alternate way of obtaining the decomposition of the curvature tensor, using representation theory.

**Proposition 9.4.** *We have the following isomorphisms of Lie groups*

$$Spin(3) = Sp(1) = SU(2). \tag{9.22}$$

*Proof.* Recall that $Sp(1)$ is the group of unit quaternions,

$$Sp(1) = \{q \in \mathbb{H} : q\overline{q} = |q|^2 = 1\}, \tag{9.23}$$

where for $q = x_0 + x_1 i + x_2 j + x_3 k$, the conjugate is $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. We note the identities

$$q\overline{q} = |q|^2 \cdot 1 \tag{9.24}$$

$$\overline{q_1 \cdot q_2} = \overline{q_2} \cdot \overline{q_1}. \tag{9.25}$$

The first isomorphism is, for $q_1 \in Sp(1)$, and $q \in Im(\mathbb{H}) = \{x_1 i + x_2 j + x_3 k\}$,

$$q \mapsto q_1 q \overline{q}_1 \in GL(Im(\mathbb{H})), \tag{9.26}$$

From (9.24) and (9.25), this is moreover in $\mathrm{O}(Im(\mathbb{H}))$. This is a continuous map, and $Sp(1) = S^3$ is connected, with 1 mapping onto the identity, so the image is in $\mathrm{SO}(Im(\mathbb{H}))$. The kernel is clearly $\{\pm 1\}$, so this map is a double covering of $SO(3) = \mathbb{RP}^3$. For the isomorphism $Sp(1) = SU(2)$, we send

$$q = \alpha + j\beta \mapsto \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \tag{9.27}$$

where $\alpha, \beta \in \mathbb{C}$. □

We let $V$ denote the standard 2-dimensional complex representation of $SU(2)$, which is just matrix multiplication of (9.27) on column vectors. Let $S^r(V)$ denote the space of complex totally symmetric $r$-tensors. From (9.12) above, $\dim_{\mathbb{C}}(S^r(V)) = r + 1$.

**Proposition 9.5.** *If $W$ is an irreducible complex representation of $Spin(3) = SU(2)$ then $W$ is equivalent to $S^r(V)$ for some $r \geq 0$. Such a representation descends to $SO(3)$ if and only if $r$ is even, in which case $W$ is a complexification of a real representation of $SO(3)$. Furthermore,*

$$S^p(V) \otimes S^q(V) = \bigoplus_{r=0}^{\min(p,q)} S^{p+q-2r} V. \tag{9.28}$$

*Proof.* On the Lie algebra level, it is proved in [FH91, Chapter 11] that the weights of any representation are given by

$$\{r, r-2, \ldots, \ldots, -r+2, -r\}, \tag{9.29}$$

and there is a unique such representation for any integer $r \geq 0$ of dimension $r + 1$. Going back to the Lie group level, since SU(2) is a double cover of SO(3), only the representations for which $-I$ acts trivially will descend, which is only true for $r$ even. The decomposition (9.28) follows since the weights of a tensor product are sums of the weights of each factor, counted with multiplicity. □

Using this, it is quite easy to decompose the curvature tensor directly in dimension 3. The standard representation of SO(3), call it $T$, is irreducible, So we must have $T \otimes \mathbb{C} = S^2 V$. Also, $\Lambda^2(T)$ is irreducible and of dimension 3, so it is also equal to $S^2 V$, so $\Lambda^2(T) = T$. Of course, the Hodge star gives the isomorphism between these spaces. Next, since $\Lambda^4(T) = 0$, we know that

$$\mathcal{C} = S^2(\Lambda^2) = S^2(T) = S^2_0(T) \oplus \mathbb{R}, \tag{9.30}$$

which is the exactly the statement that

$$Rm = E \owedge g + \frac{R}{12} g \owedge g, \tag{9.31}$$

which we derived above.

We next identify $S^2_0(T) \otimes \mathbb{C}$ in terms of $V$. From above, we have that

$$S^2_0(T) \otimes \mathbb{C} = S^2_0(S^2(V)). \tag{9.32}$$

From (9.28), we have the irreducible decomposition

$$S^2(V) \otimes S^2(V) = S^4(V) \oplus S^2(V) \oplus \mathbb{C}. \tag{9.33}$$

But we also have

$$S^2(V) \otimes S^2(V) = End(S^2(V)) = S^2_0(S^2(V)) \oplus \Lambda^2(S^2(V)) \oplus \mathbb{C}. \tag{9.34}$$

Comparing these, by counting dimensions we must have

$$S^2_0(S^2(V)) = S^4(V), \tag{9.35}$$

so we have the irreducible decomposition.

$$\mathcal{C} \otimes \mathbb{C} = S^4(V) \oplus \mathbb{C}. \tag{9.36}$$

# 10 Lecture 10: October 6

For illustration, we will study the curvature tensor in dimension 4 in two ways. First, we will give a proof which involves direct computation. Then we will give a proof using representation theory.

## 10.1 Curvature in dimension 4

Recall the Hodge star operator on $\Lambda^p$ in dimension $n$ satisfies

$$*^2 = (-1)^{p(n-p)} I. \tag{10.1}$$

In the case of $\Lambda^2$ in dimension 4, $*^2 = I$. The space of 2-forms decomposes into

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{10.2}$$

the $+1$ and $-1$ eigenspaces of the Hodge star operator, respectively. Note that $dim_{\mathbb{R}}(\Lambda^2) = 6$, and $dim_{\mathbb{R}}(\Lambda^2_\pm) = 3$. Elements of $\Lambda^2_+$ are called *self-dual* 2-forms, and elements of $\Lambda^2_-$ are called *anti-self-dual* 2-forms

We fix an oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with dual basis $\{e^1, e^2, e^3, e^4\}$, and let

$$\begin{aligned}
\omega_1^\pm &= e^1 \wedge e^2 \pm e^3 \wedge e^4,\\
\omega_2^\pm &= e^1 \wedge e^3 \pm e^4 \wedge e^2,\\
\omega_3^\pm &= e^1 \wedge e^4 \pm e^2 \wedge e^3,
\end{aligned}$$

note that $*\omega_i^\pm = \pm\omega_i^\pm$, and $\frac{1}{\sqrt{2}}\omega_i^\pm$ is an orthonormal basis of $\Lambda^2_\pm$.

**Remark 10.1.** In dimension 6, on $\Lambda^3$, we have $*^2 = -1$, so $\Lambda^3 \otimes \mathbb{C} = \Lambda^3_+ \oplus \Lambda^3_-$, the $+i$ and $-i$ eigenspaces of the Hodge star. That is, $*$ gives a complex structure on $\Lambda^3$ in dimension 6. In general, in dimensions $n = 4m$, $\Lambda^{2m} = \Lambda^{2m}_+ \oplus \Lambda^{2m}_-$, the $\pm 1$ eigenspaces of Hodge star, while in dimensions $n = 4m + 2$, the Hodge star gives a complex structure on $\Lambda^{2m+1}$.

In dimension 4 there is the special coincidence that the curvature operator acts on 2-forms, and the space of 2-forms decomposes as above. Recall from Section 8.1, the full curvature tensor decomposes as

$$Rm = W + \frac{1}{2}E \owedge g + \frac{R}{24} g \owedge g, \tag{10.3}$$

where

$$E = Ric - \frac{R}{4} g \tag{10.4}$$

is the *traceless Ricci tensor.*

Corresponding to this decomposition, we define the *Weyl curvature operator*, $\mathcal{W} : \Lambda^2 \to \Lambda^2$ as

$$(\mathcal{W}\omega)_{ij} = \frac{1}{2} \sum_{k,l} W_{ijkl}\omega_{kl}. \tag{10.5}$$

We also define $\mathcal{W}^{\pm} : \Lambda^2 \to \Lambda^2$ as

$$\mathcal{W}^{\pm}\omega = \pi_{\pm}\mathcal{W}\pi_{\pm}\omega, \tag{10.6}$$

where $\pi_{\pm} : \Lambda^2 \to \Lambda^2_{\pm}$ is the projection $\frac{1}{2}(I \pm *)$. Note that

$$\begin{aligned}
\langle \mathcal{W}^+\omega_1, \omega_2\rangle &= \langle \pi_+\mathcal{W}\pi_+\omega_1, \omega_2\rangle && (10.7)\\
&= \langle \mathcal{W}\pi_+\omega_1, \pi_+\omega_2\rangle && (10.8)\\
&= \langle \pi_+\omega_1, \mathcal{W}\pi_+\omega_2\rangle \text{ (since } \mathcal{W} \text{ is symmetric)} && (10.9)\\
&= \langle \omega_1, \mathcal{W}^+\omega_2\rangle. && (10.10)
\end{aligned}$$

This says that $\mathcal{W}^+$ is a symmetric operator, so by the above procedure, it corresponds to a curvature-like tensor $W^+$, the components of which are defined by

$$\begin{aligned}
W^+_{pqrs} &= \langle \mathcal{W}^+(e^p \wedge e^q), e^r \wedge e^s\rangle\\
&= \langle \pi_+\mathcal{W}\pi_+(e^p \wedge e^q), e^r \wedge e^s\rangle\\
&= \frac{1}{4}\langle \mathcal{W}(e^p \wedge e^q + *(e^p \wedge e^q)), e^r \wedge e^s + *(e^r \wedge e^s)\rangle.
\end{aligned} \tag{10.11}$$

For example,

$$\begin{aligned}
W^+_{1234} &= \frac{1}{4}\langle \mathcal{W}(e^1 \wedge e^2 + e^3 \wedge e^4), e^1 \wedge e^2 + e^3 \wedge e^4\rangle\\
&= \frac{1}{2}(W_{1212} + 2W_{1234} + W_{3434}).
\end{aligned} \tag{10.12}$$

We will prove that

$$\pi_+ \mathcal{W} = \mathcal{W} \pi_+, \tag{10.13}$$

which is equivalent to saying that $\mathcal{W}$ commutes with the Hodge star operator. This in turn is equivalent to proving certain curvature identities for $W$. For example, we can use this to alternatively compute

$$\begin{aligned} W^+_{1234} &= \langle \pi_+ \mathcal{W}(e^1 \wedge e^2), e^3 \wedge e^4 \rangle \\ &= \langle \mathcal{W} \pi_+(e^1 \wedge e^2), e^3 \wedge e^4 \rangle \\ &= \frac{1}{2} \langle \mathcal{W}(e^1 \wedge e^2 + e^3 \wedge e^4), e^3 \wedge e^4 \rangle \\ &= \frac{1}{2}(W_{1234} + W_{3434}). \end{aligned} \tag{10.14}$$

Comparing (10.12), this yields the identity

$$W_{1212} = W_{3434}. \tag{10.15}$$

So (10.13) is equivalent to various Weyl curvature identites like (10.15).

We can decompose the Weyl curvature tensor as

$$W = W^+ + W^-, \tag{10.16}$$

the *self-dual* and *anti-self-dual* components of the Weyl curvature, respectively. Therefore in dimension 4 we have the further orthogonal decomposition of the curvature tensor

$$Rm = W^+ + W^- + \frac{1}{2} E \owedge g + \frac{R}{24} g \owedge g. \tag{10.17}$$

The *traceless Ricci curvature operator* $\mathcal{E}$ is the operator associated to the curvature-like tensor $E \owedge g$, and the *scalar curvature operator* $\mathcal{S}$ is the operator associated to $Rg \owedge g$.

**Proposition 10.1.** *The Weyl curvature operator commutes with the Hodge star operator, $*\mathcal{W} = \mathcal{W}*$, and therefore preserves the type of forms, $\mathcal{W}(\Lambda^2_\pm) \subset \Lambda^2_\pm$. Furthermore,*

$$*\mathcal{W}^+ = \mathcal{W}^+ * = \mathcal{W}^+ \tag{10.18}$$
$$*\mathcal{W}^- = \mathcal{W}^- * = -\mathcal{W}^-. \tag{10.19}$$

*The scalar curvature operator acts as a multiple of the identity*

$$\mathcal{S}\omega = 2R\omega. \tag{10.20}$$

*The traceless Ricci operator anti-commutes with the Hodge star operator,*

$$*\mathcal{E} = -\mathcal{E}*, \tag{10.21}$$

*and therefore reverses types, $\mathcal{E}(\Lambda^2_{\pm}) \subset \Lambda^2_{\mp}$. In block form corresponding to the decomposition (10.2), the full curvature operator is*

$$\mathcal{R} = \left( \begin{array}{c|c} \mathcal{W}^+ + \frac{R}{12} I & \frac{1}{2}\mathcal{E}\pi_- \\ \hline \frac{1}{2}\mathcal{E}\pi_+ & \mathcal{W}^- + \frac{R}{12} I \end{array} \right). \tag{10.22}$$

*Proof.* We first consider the traceless Ricci operator. We compute

$$\begin{aligned} ((E \owedge g)\omega)_{ij} &= \frac{1}{2}(E \owedge g)_{ijkl}\omega_{kl} \\ &= \frac{1}{2}(E_{ik}g_{jl}\omega_{kl} - E_{jk}g_{il}\omega_{kl} - E_{il}g_{jk}\omega_{kl} + E_{jl}g_{ik}\omega_{kl}) \\ &= \frac{1}{2}(E_{ik}\omega_{kj} - E_{jk}\omega_{ki} - E_{il}\omega_{jl} + E_{jl}\omega_{il}) \\ &= E_{ik}\omega_{kj} - E_{jk}\omega_{ki}, \end{aligned} \tag{10.23}$$

since $\omega$ is skew-symmetric. Next assume that $E_{ij}$ is diagonal, so that $E_{ij} = \lambda_i g_{ij}$, and we have

$$\begin{aligned} \frac{1}{2}(E \owedge g)_{ijkl}\omega_{kl} &= \lambda_i \delta_{ik}\omega_{kj} - \lambda_j \delta_{jk}\omega_{ki} \\ &= \lambda_i \omega_{ij} - \lambda_j \omega_{ji} \\ &= (\lambda_i + \lambda_j)\omega_{ij}. \end{aligned} \tag{10.24}$$

Next compute

$$\begin{aligned} \frac{1}{2}(E \owedge g)_{ijkl}(\omega_1^+)_{kl} &= \frac{1}{2}(E \owedge g)_{ijkl}(\delta^{kl}_{12} + \delta^{kl}_{34}) \\ &= (\lambda_i + \lambda_j)(\delta^{ij}_{12} + \delta^{ij}_{34}) \\ &= (\lambda_1 + \lambda_2)\delta^{ij}_{12} + (\lambda_3 + \lambda_4)\delta^{ij}_{34}. \end{aligned} \tag{10.25}$$

Since $E$ is traceless, $\lambda_1 + \lambda_2 = -\lambda_3 - \lambda_4$, so we have

$$\frac{1}{2}(E \owedge g)_{ijkl}(\omega_1^+)_{kl} = (\lambda_1 + \lambda_2)(\delta^{ij}_{12} - \delta^{ij}_{34}), \tag{10.26}$$

which equivalently is

$$(E \owedge g)(\omega_1^+) = (\lambda_1 + \lambda_2)\omega_1^-. \tag{10.27}$$

Similar computations for the other components (which we leave to the reader) shows that

$$(E \owedge g)(\Lambda^2_{\pm}) \subset \Lambda^2_{\mp}. \tag{10.28}$$

This is equivalent to $\pi_{\pm}\mathcal{E}\pi_{\pm} = 0$, which is easily seen to be equivalent to (10.21).

Next, the dimension of the space $\{M : \Lambda^2 \to \Lambda^2, M \text{ symmetric}, M* = - * M\}$ is 9. The dimension of the maps of the form $E \owedge g$, where is a traceless symmetric tensor is also 9, since the map $E \to E \owedge g$ is injective for $n > 2$. We conclude that the remaining part of the curvature tensor

$$\Big(\mathcal{W}^{\pm} + \frac{1}{24}\mathcal{S}\Big)(\Lambda^2_{\pm}) \subset (\Lambda^2_{\pm}), \tag{10.29}$$

which is equivalent to (10.18) and (10.19).

Finally, the proposition follows, noting that $g \owedge g = 2I$, twice the identity. To see this, we have

$$\begin{aligned}\big((g \owedge g)\omega\big)_{ij} &= \frac{1}{2}(g \owedge g)_{ijkl}\omega_{kl} \\ &= \frac{1}{2}(g_{ik}g_{jl} - g_{jk}g_{il} - g_{il}g_{jk} + g_{jl}g_{ik})\omega_{kl} \\ &= (g_{ik}g_{jl} - g_{jk}g_{il})\omega_{kl} \\ &= (\omega_{ij} - \omega_{ji}) = 2\omega_{ij}.\end{aligned} \tag{10.30}$$

□

Of course, instead of appealing to the dimension argument, one can show directly that (10.29) is true, using the fact that the Weyl is in the kernel of Ricci contraction, that is, the Weyl tensor satisfies $W_{iljl} = 0$. For example,

$$\begin{aligned}(\mathcal{W}\omega_1^+)_{ij} &= \frac{1}{2}W_{ijkl}(\delta^{12}_{kl} + \delta^{34}_{kl}) \\ &= W_{ij12} + W_{ij34},\end{aligned} \tag{10.31}$$

taking an inner product,

$$\begin{aligned}\langle \mathcal{W}\omega_1^+, \omega_1^- \rangle &= \frac{1}{2}(W_{ij12} + W_{ij34})(\delta^{12}_{ij} - \delta^{34}_{ij}) \\ &= W_{1212} - W_{3412} + W_{1234} - W_{3434} \\ &= W_{1212} - W_{3434}.\end{aligned} \tag{10.32}$$

But we have

$$\begin{aligned}W_{1212} + W_{1313} + W_{1414} = 0 \\ W_{1212} + W_{3232} + W_{4242} = 0,\end{aligned} \tag{10.33}$$

adding these,

$$2W_{1212} = -W_{1313} - W_{1414} - W_{3232} - W_{4242}. \tag{10.34}$$

We also have

$$\begin{aligned}W_{1414} + W_{2424} + W_{3434} = 0 \\ W_{3131} + W_{3232} + W_{3434} = 0,\end{aligned} \tag{10.35}$$

adding,

$$2W_{3434} = -W_{1414} - W_{2424} - W_{3131} - W_{3232} = 2W_{1212}, \tag{10.36}$$

which shows that

$$\langle \mathcal{W}\omega_1^+, \omega_1^- \rangle = 0. \tag{10.37}$$

Next

$$\begin{aligned}\langle \mathcal{W}\omega_1^+, \omega_2^- \rangle &= \frac{1}{2}(W_{ij12} + W_{ij34})(\delta_{ij}^{13} - \delta_{ij}^{42}) \\ &= W_{1312} - W_{4212} + W_{1334} - W_{4234} \\ &= -W_{1231} - W_{4212} - W_{4313} - W_{4234}.\end{aligned} \tag{10.38}$$

But from vanishing Ricci contraction, we have

$$\begin{aligned}W_{4212} + W_{4313} =& 0, \\ W_{1231} + W_{4234} =& 0,\end{aligned} \tag{10.39}$$

which shows that

$$\langle \mathcal{W}\omega_1^+, \omega_2^- \rangle = 0.$$

A similar computation can be done for the other components.

# 11 Lecture 11: October 11

Since $Weyl \in Ker(c)$, the operator $\mathcal{W} : \Lambda^2 \to \Lambda^2$ is clearly traceless. But we have the stronger statement:

**Proposition 11.1.** *Both $\mathcal{W}^+$ and $\mathcal{W}^-$ are traceless.*

*Proof.* We compute

$$tr(\mathcal{W}^+) = \sum_{i=1}^{3} \langle \mathcal{W}^+(\omega_i^+), \omega_i^+ \rangle_{\Lambda^2} \tag{11.1}$$

$$= W_{1212} + 2W_{1234} + W_{3434} \tag{11.2}$$

$$+ W_{1313} + 2W_{1342} + W_{4242} \tag{11.3}$$

$$+ W_{1414} + 2W_{1423} + W_{2323}. \tag{11.4}$$

The first column sums to zero by Ricci contraction. Since $*\mathcal{W} = \mathcal{W}*$, we know that $W_{1212} = W_{3434}$, $W_{4242} = W_{1313}$, and $W_{1414} = W_{2323}$, so the last column is the same as the first column. The algebraic Bianchi identity says that

$$R_{1234} + R_{1342} + R_{1423} = 0. \tag{11.5}$$

Substituting $Rm = Weyl + A \owedge g$ into this, all terms arising from $A \owedge g$ are zero since all of the indices are different. Consequently,

$$W_{1234} + W_{1342} + W_{1423} = 0. \tag{11.6}$$

so the middle column sums to zero. A similar computation deals with $\mathcal{W}^-$. $\square$

We next have a corollary of our computations from last time.

**Corollary 11.1.** *Let $(M, g)$ be a Riemannian 4-manifold. The following are equivalent*

- $*\mathcal{R} = \mathcal{R}*$.
- *For any 2-plane $\sigma$, $K(\sigma) = K(\sigma^{\perp})$.*
- *$g$ is Einstein.*

*Furthermore, the following are equivalent*

- $*\mathcal{R} = -\mathcal{R}*$.
- *For any 2-plane $\sigma$, $K(\sigma) = -K(\sigma^{\perp})$.*
- *$Weyl_g \equiv 0$ and $R_g \equiv 0$.*

*Proof.* Obviously, $\mathcal{R}$ commutes with $*$ if and only if the part of the curvature tensor which anti-commutes with $*$ must vanish if and only if $E = 0$. Next, take any 2-plane $\sigma \subset T_pM$. Choose an oriented ONB $\{e_1, e_2, e_3, e_4\}$ such that $\mathrm{span}\{e_1, e_2\} = \sigma$, and let $\{e^1, e^2, e^2, e^4\}$ be the dual basis of $T_p^*M$. Then

$$K(\sigma) = Rm(e_1, e_2, e_1, e_2) = \langle \mathcal{R}(e^1 \wedge e^2), (e^1 \wedge e^2)\rangle_{\Lambda^2}. \tag{11.7}$$

If $\mathcal{R}$ commutes with $*$,

$$K(\sigma^{\perp}) = Rm(e_3, e_4, e_3, e_4) = \langle \mathcal{R} * (e^1 \wedge e^2), *(e^1 \wedge e^2)\rangle_{\Lambda^2} \tag{11.8}$$
$$= \langle *\mathcal{R}(e^1 \wedge e^2), *(e^1 \wedge e^2)\rangle_{\Lambda^2} = \langle \mathcal{R}(e^1 \wedge e^2), (e^1 \wedge e^2)\rangle_{\Lambda^2} = K(\sigma). \tag{11.9}$$

Conversely, if $K(\sigma) = K(\sigma^{\perp})$ for any 2-plane $\sigma$, then for any oriented ONB as above

$$\langle \mathcal{R}(e^1 \wedge e^2), (e^1 \wedge e^2)\rangle_{\Lambda^2} = \langle \mathcal{R}(e^3 \wedge e^4), (e^3 \wedge e^4)\rangle_{\Lambda^2}, \tag{11.10}$$

which shows that $\pi_+\mathcal{R}\pi_- = 0$ and $\pi_-\mathcal{R}\pi_+ = 0$, which are equivalent to $\mathcal{R}* = *\mathcal{R}$. A similar argument works to prove the second set of equivalences. $\square$

An example for the first case is $S^2 \times S^2$ with the product metric

$$g = \pi_1^* g_S + \pi_2^* g_S, \tag{11.11}$$

where $g_S$ the round metric on $S^2$ with constant curvature $K = 1$, and $\pi_i$ is the projection to the $i$th factor, $i = 1, 2$. Note in general for product metrics we have

$$Rm(g) = \pi_1^* Rm(g_S) + \pi_2^* Rm(g_S), \tag{11.12}$$

so we have

$$Ric(g) = \pi_1^* Ric(g_S) + \pi_2^* Ric(g_S) = \pi_1^* g_S + \pi_2^* g_S = g, \tag{11.13}$$

so the product metric is Einstein. However, it does not have constant curvature, since the Weyl tensor is given by

$$\begin{aligned}
Weyl(g) &= Rm(g) - A \owedge g = \frac{1}{2}\pi_1^* g_S \owedge \pi_1^* g_S + \frac{1}{2}\pi_2^* g_S \owedge \pi_2^* g_S - \frac{1}{6} g \owedge g \\
&= \frac{1}{2}\Big(\pi_1^* g_S \owedge \pi_1^* g_S + \pi_2^* g_S \owedge \pi_2^* g_S\Big) - \frac{1}{6}(\pi_1^* g_S + \pi_2^* g_S) \owedge (\pi_1^* g_S + \pi_2^* g_S) \\
&= \frac{1}{3}\Big(\pi_1^* g_S \owedge \pi_1^* g_S + \pi_2^* g_S \owedge \pi_2^* g_S - \pi_1^* g_S \owedge \pi_2^* g_S\Big),
\end{aligned} \tag{11.14}$$

which is not zero. An important fact is that this metric has non-negative sectional curvature. To see this, for $e_1$ and $e_2$ orthonormal we compute

$$\begin{aligned}
Rm(e_1, e_2, e_1, e_2) &= (\pi_1^* g_S)(e_1, e_1)(\pi_1^* g_S)(e_2, e_2) - ((\pi_1^* g_S)(e_1, e_2))^2 \\
&\quad + (\pi_2^* g_S)(e_1, e_1)(\pi_2^* g_S)(e_2, e_2) - ((\pi_2^* g_S)(e_1, e_2))^2 \\
&= g_S(f_1, f_1) g_S(f_2, f_2) - (g_S(f_1, f_2))^2 \\
&\quad + g_S(h_1, h_1) g_S(h_2, h_2) - (g_S(h_1, h_2))^2,
\end{aligned} \tag{11.15}$$

where $f_i = (\pi_1)_* e_i$, and $h_i = (\pi_2)_* e_i$, for $i = 1, 2$. This is clearly non-negative by the Cauchy-Schwartz inequality. Note that this is zero if $(\pi_2)_*(e_1) = 0$, and $(\pi_1)_*(e_2) = 0$, that is, if $e_1$ is tangent to the first factor, and $e_2$ is tangent to the second factor. Thus this metric has many zero sectional curvature planes at every point. The following is a very famous conjecture:

**Conjecture 11.1** (Hopf Conjecture)**.** *The manifold $S^2 \times S^2$ does not admit a metric of positive sectional curvature.*

An example for the second case in Corollary 11.1 is $S^2 \times \mathcal{H}^2$ with the product metric $g = \pi_1^*(g_S) + \pi_2^*(g_H)$, where $g_S$ is the round metric with constant curvature $K = 1$, and $g_H$ is a hyperbolic metric with constant curvature $K = -1$. The Ricci tensor is given by

$$Ric(g) = \pi_1^* Ric(g_S) + \pi_2^* Ric(g_H) = \pi_1^* g_S - \pi_2^* g_H, \tag{11.16}$$

so $g$ is scalar-flat. To see that $Weyl = 0$,

$$\begin{aligned}
Rm(g) = \pi_1^* Rm(g_S) + \pi_2^* Rm(g_H) &= \frac{1}{2}\pi_1^* g_S \owedge \pi_1^* g_S - \frac{1}{2}\pi_1^* g_H \owedge \pi_1^* g_H \\
&= \frac{1}{2}(\pi_1^* g_S - \pi_2^* g_H) \owedge (\pi_1^* g_S + \pi_2^* g_H) = \frac{1}{2}(\pi_1^* g_S - \pi_2^* g_H) \owedge g.
\end{aligned} \tag{11.17}$$

This says the curvature tensor is in the Image of $\Psi$, which implies that the Weyl tensor vanishes.

## 11.1 The Grassmannian

A mentioned above, we can view the sectional curvature as a function

$$K : G_o(2, T_p M) \to \mathbb{R}, \tag{11.18}$$

where $G(2, T_p M)$ is the Grassmannian of oriented 2-planes in the tangent space. There is a nice description of the Grassmannian in dimension 4:

**Proposition 11.2.**

$$G_o(2,\mathbb{R}^4) = S^2 \times S^2 = \Big\{(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_- \ \big| \ |\alpha|_{\Lambda^2} = |\beta|_{\Lambda^2} = \frac{1}{\sqrt{2}}\Big\}. \tag{11.19}$$

*Proof.* A 2-plane is determined by a orthonormal basis $\{e_1, e_2\}$, which determines a unit-norm 2-form $e_1 \wedge e_2$. Conversely, any non-zero 2-form of the form $e_1 \wedge e_2$ with $e_1$ and $e_2$ linearly independent determines a 2-plane. Such a 2-form is called *decomposable*. We claim that a 2-form $\omega$ is decomposable if and only if $\omega \wedge \omega = 0$. Obviously $e_1 \wedge e_2 \wedge e_1 \wedge e_2 = 0$. For the converse, if $\omega \wedge \omega = 0$, then the linear map $L_\omega : \Lambda^1 \to \Lambda^3$ defined by $L_\omega(\theta) = \omega \wedge \theta$ has rank 2, and therefore $\omega = \pm e_1 \wedge e_2$ where $\{e_1, e_2\}$ is any orthonormal basis for $Ker(L)$. Writing $\omega = \alpha + \beta$, where $\alpha \in \Lambda^2_+$ and $\beta \in \Lambda^2_-$,

$$0 = \omega \wedge \omega = \langle \omega, *\omega\rangle_{\Lambda^2} dV = \langle \alpha + \beta, \alpha - \beta\rangle_{\Lambda^2} dV = (|\alpha|^2 - |\beta|^2) dV. \tag{11.20}$$

Consequently,

$$\begin{aligned} G_o(2,\mathbb{R}^4) &= \{\omega \in \Lambda^2 \ \big| \ \omega \wedge \omega = 0, |\omega|^2_{\Lambda^2} = 1\} &(11.21)\\ &= \Big\{(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_- \ \big| \ |\alpha|_{\Lambda^2} = |\beta|_{\Lambda^2} = \frac{1}{\sqrt{2}}\Big\}. &(11.22)\end{aligned}$$

□

**Remark 11.1.** The un-oriented Grassmannian is a 2-fold quotient of $G_0(2,\mathbb{R}^4)$ given by $\xi \sim -\xi$, which in the above description is $(\alpha,\beta) \sim (-\alpha,-\beta)$.

Using this description, we can think of the sectional curvature as $K(\alpha,\beta) \equiv K(\sigma)$ where $\sigma$ is the 2-plane determined by the unit-norm decomposable 2-form $\alpha + \beta$. We can add another equivalent condition to Corollary 11.1:

**Corollary 11.2.** *Let $(M,g)$ be a Riemannian 4-manifold. The following are equivalent*

- $*\mathcal{R} = \mathcal{R}*$.
- *For a 2-plane $\sigma$ corresponding to $(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_-$, we have*

$$K(\alpha,\beta) = \langle \mathcal{R}(\alpha), \alpha\rangle_{\Lambda^2} + \langle \mathcal{R}(\beta), \beta\rangle_{\Lambda^2}. \tag{11.23}$$

*Furthermore, the following are equivalent*

- $*\mathcal{R} = -\mathcal{R}*$.
- *For a 2-plane $\sigma$ corresponding to $(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_-$, we have*

$$K(\alpha,\beta) = 2\langle \mathcal{R}(\alpha), \beta\rangle_{\Lambda^2}. \tag{11.24}$$

*Proof.* Since $\alpha + \beta$ is a unit norm 2-form, we have

$$K(\alpha,\beta) = \langle \mathcal{R}(\alpha+\beta), \alpha+\beta\rangle_{\Lambda^2} \tag{11.25}$$
$$= \langle \mathcal{R}(\alpha), \alpha\rangle_{\Lambda^2} + 2\langle \mathcal{R}(\alpha), \beta\rangle_{\Lambda^2} + \langle \mathcal{R}(\beta), \beta\rangle_{\Lambda^2}. \tag{11.26}$$

In the first case, $\mathcal{R}$ preserves types of forms, so the middle term vanishes. In the second case, $\mathcal{R}$ reverses types of forms, so the first and third terms vanish. □

Next, we revisit the product examples from above. In the following, we let $\mathcal{A}$ denote the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11.27}$$

In the case of the product metric on $S^2 \times S^2$, we have the following.

**Proposition 11.3.** *Let $M = S^2 \times S^2$ with the product metric $g = \pi_1^* g_S + \pi_2^* g_S$. Define*

$$\omega_1^{\pm} = \pi_1^* dV_{g_S} \pm \pi_2^* dV_{g_S}. \tag{11.28}$$

*Then for a 2-plane $\sigma$ corresponding to $(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_-$, we have*

$$K(\alpha,\beta) = \langle \alpha, \omega_1^+\rangle^2_{\Lambda^2} + \langle \beta, \omega_1^-\rangle^2_{\Lambda^2}. \tag{11.29}$$

*Equivalently, extending $(1/\sqrt{2})\omega_1^{\pm}$ to an ONB of $\Lambda^2$, the curvature operator has the form*

$$\mathcal{R} = \left(\begin{array}{c|c} \mathcal{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{A} \end{array}\right). \tag{11.30}$$

*Proof.* We already know that the curvature operator preserves types of forms. It is easy to see that $\mathcal{R}\omega_1^{\pm} = \omega_1^{\pm}$, and that the curvature operator annihilates anything orthogonal to $span\{\omega_1^+, \omega_1^-\}$. □

**Corollary 11.3.** *For the product metric $(S^2 \times S^2, g)$, at any point, we have*

$$0 \le K(\sigma) \le 1, \tag{11.31}$$

*and the zero sectional curvature set is a torus $S^1 \times S^1 \subset G_o(2, T_pM)$ defined by*

$$\Big\{(\alpha,\beta) \in \Lambda^2_+ \times \Lambda^2_- \;\Big|\; |\alpha| = |\beta| = \frac{1}{\sqrt{2}}, \langle \alpha, \omega^1_+\rangle = \langle \beta, \omega^1_-\rangle = 0\Big\}. \tag{11.32}$$

Revisiting the second example,

**Proposition 11.4.** *Let $M = S^2 \times \mathcal{H}^2$ with the product metric $g = \pi_1^* g_S + \pi_2^* g_H$. Then for a 2-plane $\sigma$ corresponding to $(\alpha, \beta) \in \Lambda^2_+ \times \Lambda^2_-$, we have*

$$K(\alpha, \beta) = \langle \alpha, \omega_1^+ \rangle_{\Lambda^2} \cdot \langle \beta, \omega_1^- \rangle_{\Lambda^2}. \tag{11.33}$$

*Equivalently, extending $(1/\sqrt{2})\omega_1^\pm$ to an ONB of $\Lambda^2$, the curvature operator has the form*

$$\mathcal{R} = \left( \begin{array}{c|c} \mathbf{0} & \mathcal{A} \\ \hline \mathcal{A} & \mathbf{0} \end{array} \right). \tag{11.34}$$

*Proof.* We already know that the curvature operator reverses types of forms. It is easy to see that $\mathcal{R}\omega_1^\pm = \omega_1^\mp$, and that the curvature operator annihilates anything orthogonal to $span\{\omega_1^+, \omega_1^-\}$. $\square$

**Corollary 11.4.** *For the product metric $(S^2 \times \mathcal{H}^2, g)$, at any point, we have*

$$-1 \leq K(\sigma) \leq 1, \tag{11.35}$$

*and the zero sectional curvature set is $S^1 \times S^2 \cup S^2 \times S^1 \subset G_o(2, T_pM)$ defined by*

$$\Big\{(\alpha, \beta) \in \Lambda^2_+ \times \Lambda^2_- \;\Big|\; |\alpha| = |\beta| = \frac{1}{\sqrt{2}}, \langle \alpha, \omega^1_+ \rangle = 0 \text{ or } \langle \beta, \omega^1_- \rangle = 0\Big\}. \tag{11.36}$$

# 12 Lecture 12

## 12.1 Representations of $Spin(4)$

Next, we give a representation-theoretic description of the curvature decomposition in dimension 4. As $SO(4)$ modules, we have the decomposition

$$\begin{aligned} S^2(\Lambda^2) &= S^2(\Lambda^2_+ \oplus \Lambda^2_-) \\ &= S^2(\Lambda^2_+) \oplus (\Lambda^2_+ \otimes \Lambda^2_-) \oplus S^2(\Lambda^2_-), \end{aligned} \tag{12.1}$$

which is just the "block form" decomposition in (24.32).

**Proposition 12.1.** *We have the following isomorphisms of Lie groups*

$$Spin(4) = Sp(1) \times Sp(1) = SU(2) \times SU(2). \tag{12.2}$$

*Proof.* To see that $Sp(1) \times Sp(1) = Spin(4)$, take $(q_1, q_2) \in Sp(1) \times Sp(1)$, and define $\phi : \mathbb{H} \to \mathbb{H}$ by

$$\phi(q) = q_1 q \overline{q}_2. \tag{12.3}$$

We claim that this defines a homomorpishm $f : Sp(1) \times Sp(1) \to SO(4)$, with

$$ker(f) = \{(1, 1), (-1, -1)\}, \tag{12.4}$$

so $f$ is a non-trivial double covering.

To see that $f$ is a homomorphism:

$$\begin{aligned} f((q_1, q_2) \cdot (q_1', q_2'))(q) &= f(q_1 q_1', q_2 q_2')(q) = q_1 q_1' q \overline{q_2 q_2'} \\ &= q_1 q_1' q \overline{q_2'} \overline{q_2} = f(q_1, q_2)(f(q_1', q_2')(q)). \end{aligned} \tag{12.5}$$

The inverse map to $f(q_1, q_2)$ is $f(q_1^{-1}, q_2^{-1})$, so the image of $f$ lies in $GL(4, \mathbb{R})$. We compute

$$|f(q_1, q_2)(q)|^2 = |q_1 q \overline{q}_2|^2 = q_1 q \overline{q}_2 \overline{q_1 q \overline{q}_2} = q_1 q \overline{q}_2 q_2 \overline{q q_1} = |q|^2, \tag{12.6}$$

since $q_1$ and $q_2$ are unit quaternions, so the image of $f$ lies in $\mathrm{O}(4, \mathbb{R})$. Since $Sp(1) \times Sp(1) = S^3 \times S^3$ is connected, the image must lie in the identity component of $\mathrm{O}(4)$, which is $\mathrm{SO}(4)$. Finally, assume

$$q = q_1 q \overline{q_2}, \tag{12.7}$$

for every $q \in \mathbb{H}$. Letting $q = 1$, we see that $q_1 = q_2$, which implies that

$$q q_1 = q_1 q, \tag{12.8}$$

for every $q \in \mathbb{H}$, thus $q_1$ is in the center of $Sp(1)$, which says that $q_1 \in \mathbb{R}$, so $q_1 = \pm 1$. $\square$

For $G_1$ and $G_2$ compact Lie groups, it is well-known that the irreducible representations of $G_1 \times G_2$ are exactly those of the form $V_1 \otimes V_2$ for irreducible representations $V_1$ and $V_2$ of $G_1$ and $G_2$, respectively [FH91]. For $Spin(4) = SU(2) \times SU(2)$, let $V_+$ and $V_-$ denote the standard irreducible complex 2-dimensional representations of the first and second factors, respectively. The representation theory of $Spin(4)$ is given by the following.

**Proposition 12.2.** *If $W$ is an irreducible complex representation of $Spin(4) = SU(2) \times SU(2)$ then $W$ is equivalent to*

$$S^{p,q} = S^p(V_+) \otimes S^q(V_-), \tag{12.9}$$

*for some non-negative integers $p, q$. Such a representation $W$ descends to $SO(4)$ if and only if $p+q$ is even, in which case $W$ is a complexification of a real representation of $SO(4)$.*

The following Lemma will also be very useful:

**Lemma 12.1** (Schur's Lemma)**.** *Let $V$ and $W$ be irreducible $\mathrm{SO}(4)$ modules, and $f : V \to W$ an equivariant map. Then $f$ is either an isomorphism or identically zero. Moreover, any equivariant map $f : V \to V$ has the form $f(v) = \lambda \cdot v$ for some $\lambda \in \mathbb{C}$.*

*Proof.* The spaces $Ker(f)$ and $Im(f)$ are invariant subspaces, so are either trival or the entire space. For the second part, any non-trivial eigenspace is an invariant subspace, so must be the entire space. $\square$

We begin by noting that

$$dim_{\mathbb{C}}(S^{p,q}) = (p+1)(q+1), \tag{12.10}$$

which yields that $dim_{\mathbb{C}}(S^{1,1}) = 4$. Since $p+q = 2$ is even, this corresponds to an irreducible real representation of SO(4). Clearly, the standard real 4-dimensional representation of SO(4), call it $T$, is irreducible. If $(p+1)(q+1) = 4$, then $(p,q) \in \{(1,1),(3,0),(0,3)\}$. The latter two cases have $p+q$ odd, and do not descend to representations of SO(4). Therefore, we must have

$$T \otimes \mathbb{C} = V_+ \otimes_{\mathbb{C}} V_-. \tag{12.11}$$

We know from above that $\Lambda^2(T) = \Lambda^2_+ \oplus \Lambda^2_-$, using the Hodge star. Assuming these are irreducible, then if $(p+1)(q+1) = 3$, then $(p,q) \in \{(2,0),(0,2)\}$, so from Proposition 12.2, the only 3-dimensional irreducible representations of $SO(4)$ are $S^2(V_+)$ and $S^2(V_-)$. But we can directly prove these are irreducible as follows:

**Claim 12.1.** *We have the irreducible decomposition*

$$\Lambda^2(T) \otimes \mathbb{C} = S^2(V_+) \oplus S^2(V_-). \tag{12.12}$$

*Consequently, the spaces $\Lambda^2_\pm$ are irreducible, and up to a choice of orientation,*

$$\Lambda^2_\pm \otimes \mathbb{C} = S^2(V_\pm). \tag{12.13}$$

*Proof.* We begin by noting that for any vector spaces $V$ and $W$,

$$\Lambda^2(V \otimes W) = \Lambda^2(V) \otimes S^2(W) \oplus S^2(V) \otimes \Lambda^2(W). \tag{12.14}$$

This is a decomposition with respect to $GL(n,\mathbb{R})$, and an explicit isomorphism is seen by sending

$$(a \otimes b) \wedge (c \otimes d) \tag{12.15}$$

to

$$(a \otimes c - c \otimes a) \otimes (b \otimes d + d \otimes b) + (a \otimes c + c \otimes a) \otimes (b \otimes d - d \otimes b). \tag{12.16}$$

Applying this to (12.11), we obtain (12.12) since $\Lambda^2(V_\pm) = \mathbb{C}$, these spaces being 2-dimensional. $\square$

Note also that

$$End(T) = T \otimes T = S^2_0 T \oplus \mathbb{R} \cdot I \oplus \Lambda^2 T, \tag{12.17}$$

where $S^2_0(T)$ are the traceless symmetric endomorphisms. Note that $dim_{\mathbb{R}}(S^2_0 T) = 9$, assuming this is irreducible, if $(p+1)(q+1) = 9$, then $(p,q) \in \{(2,2),(8,0),(0,8)\}$. All 3 of these descend to SO(4), so we can not see which space this is using only Proposition 12.2. We identify this space in the following:

**Claim 12.2.** *We have the isomorphism of real representations of* $\mathrm{SO}(4)$*:*

$$S_0^2(T) = \Lambda^2_+ \otimes \Lambda^2_-. \tag{12.18}$$

*In terms of spin representations,*

$$S_0^2(T) \otimes \mathbb{C} = S^{2,2} = S^2 V_+ \otimes S^2 V_-. \tag{12.19}$$

*Proof.* We begin by noting that for any vector spaces $V$ and $W$,

$$S^2(V \otimes W) = S^2(V) \otimes S^2(W) \oplus \Lambda^2(V) \otimes \Lambda^2(W). \tag{12.20}$$

This is a decomposition with respect to $GL(n, \mathbb{R})$, and an explicit isomorphism is seen by writing

$$(a \otimes b) \odot (c \otimes d) \tag{12.21}$$

to

$$(a \otimes c + c \otimes a) \otimes (b \otimes d + d \otimes b) + (a \otimes c - c \otimes a) \otimes (b \otimes d - d \otimes b). \tag{12.22}$$

So we have

$$S^2(V_+ \otimes V_-) = S^2 V_+ \otimes S^2 V_- \oplus \mathbb{C}, \tag{12.23}$$

since $\Lambda^2(V_\pm) = \mathbb{C}$. But under $\mathrm{O}(n)$, taking the traceless part of a matrix yields the decomposition

$$S^2(T) = S_0^2(T) \oplus \mathbb{R}. \tag{12.24}$$

Comparing (12.23) and (12.24) yields the claim. $\square$

We can see the isomorphism, and the inverse map in (12.18) explicitly as follows. In the previous lecture, sending $E \in S_0^2(T)$ to the operator $\mathcal{E}$ corresponding to $E \owedge g$ is an isomorphism to $Hom(\Lambda^2_+, \Lambda^2_-) = \Lambda^2_+ \otimes \Lambda^2_-$. For the inverse map, take $\omega^1 \in \Lambda^2_+$, and $\omega^2 \in \Lambda^2_-$. Consider

$$E_{ij} = \sum_k \omega^1_{ik} \omega^2_{kj}. \tag{12.25}$$

It is easy to see that this is traceless because

$$trE = \sum_{p,k} \omega^1_{pk} \omega^2_{kp} = -2 \langle \omega^1, \omega^2 \rangle_{\Lambda^2} = 0, \tag{12.26}$$

since $\Lambda^2_+$ and $\Lambda^2_-$ are orthogonal subspaces. However, symmetry is a little more difficult to prove directly, so we argue as follows. The mapping in (12.25) induces a mapping

$$\Lambda^2_+ \otimes \Lambda^2_- \to T \otimes T = S_0^2 \oplus \Lambda^2_+ \oplus \Lambda^2_- \oplus \mathbb{R}. \tag{12.27}$$

From Schur's Lemma, our map must be a multiple of projection onto the first factor. The main point is that symmetrization in $(i, j)$ in (12.25) is not necessary!

The above shows that

$$End(T) \otimes \mathbb{C} = S^2V_+ \otimes S^2V_- \oplus \mathbb{C} \oplus S^2V_+ \oplus S^2V_-. \tag{12.28}$$

Returning to the curvature tensor, recall from Section 7.1 that

$$S^2(\Lambda^2 T) = \mathcal{C} \oplus \Lambda^4(T), \tag{12.29}$$

where $\mathcal{C}$ is the space of curvature-like tensors. So in dimension 4,

$$S^2(\Lambda^2 T) = \mathcal{C} \oplus \mathbb{R}. \tag{12.30}$$

The left hand side decomposes as

$$\begin{aligned} S^2(\Lambda^2 T) &= S^2(\Lambda^2_+ T \oplus \Lambda^2_- T) & (12.31)\\ &= S^2(\Lambda^2_+ T) \oplus (\Lambda^2_+ T \otimes \Lambda^2_- T) \oplus S^2(\Lambda^2_- T), & (12.32)\end{aligned}$$

so we have

$$S^2(\Lambda^2 T) \otimes \mathbb{C} = S^2(S^2V_+) \oplus (S^2V_+ \otimes S^2V_-) \oplus S^2(S^2V_-). \tag{12.33}$$

From Proposition 9.5,

$$S^2V_+ \otimes S^2V_+ = S^4V_+ \oplus S^2V_+ \oplus \mathbb{C}. \tag{12.34}$$

Also

$$End(S^2V_+) = \Lambda^2(S^2V_+) \oplus S^2_0(S^2V_+) \oplus \mathbb{C}. \tag{12.35}$$

By counting dimensions, this implies that

$$S^4V_+ = S^2_0(S^2V_+). \tag{12.36}$$

Putting all of these facts together, we have

$$\begin{aligned} \mathcal{C} \otimes \mathbb{C} &= \mathbb{C} \oplus (S^2V_+ \otimes S^2V_-) \oplus S^4V_+ \oplus S^4V_-.\\ &= S^{0,0} \oplus S^{2,2} \oplus S^{4,0} \oplus S^{0,4}. \end{aligned} \tag{12.37}$$

We relate this to the decomposition of the curvature tensor from the previous section. The trivial summand determines the scalar curvature. By (12.18) the second piece is traceless endomorphisms of $T$, so this piece gives the traceless Ricci tensor. Finally, by (12.36), $S^4V_\pm$ are traceless endomorphisms of $S^2V_\pm = \Lambda^2_\pm T$, so these pieces determines the self-dual and anti-self-dual Weyl curvatures.

## 12.2 Some identities in dimension 4

**Proposition 12.3.** *We have the following identities in dimension* 4*.*

$$\sum_{r,s,t} W^+_{irst} W^+_{jrst} = \frac{1}{4}|W^+|^2 g_{ij} \tag{12.38}$$

$$\sum_{r,s,t} W^+_{irst} W^-_{jrst} = 0 \tag{12.39}$$

$$\sum_{r,s,t} W^-_{irst} W^-_{jrst} = \frac{1}{4}|W^-|^2 g_{ij} \tag{12.40}$$

$$\sum_{r,s,t} W_{irst} W_{jrst} = \frac{1}{4}|W|^2 g_{ij} \tag{12.41}$$

*Proof.* For $Y, Z \in S^2_0(\Lambda^2_+)$, consider the mapping

$$Y \odot Z \mapsto \sum_{r,s,t} Y_{irst} Z_{jrst}. \tag{12.42}$$

This extends to an equivariant map

$$\phi : S^2(S^2_0(\Lambda^2_+)) \to S^2(T^*M). \tag{12.43}$$

From the previous lecture, we know that $S^2_0(\Lambda^2_+) \otimes \mathbb{C} = S^4(V_+)$, and $S^2_0(T^*M) \otimes \mathbb{C} = S^{2,2}$, so upon complexifying $\phi$,

$$\phi : S^2(S^4(V_+)) \to S^{2,2} \oplus \mathbb{C}. \tag{12.44}$$

To find the irreducible decomposition of the left hand side, we argue as follows. From (9.28),

$$S^4(V_+) \otimes S^4(V_+) = S^8(V_+) \oplus S^6(V_+) \oplus S^4(V_+) \oplus S^2(V_+) \oplus \mathbb{C}. \tag{12.45}$$

On the other hand,

$$S^4(V_+) \otimes S^4(V_+) = S^2(S^4(V_+)) \oplus \Lambda^2(S^4(V_+)). \tag{12.46}$$

Counting dimensions, we see that

$$S^2(S^4(V_+)) = S^8(V_+) \oplus S^4(V_+) \oplus \mathbb{C}. \tag{12.47}$$

So $\phi$ is an equivariant mapping,

$$\phi : S^{8,0} \oplus S^{4,0} \oplus \mathbb{C} \to S^{2,2} \oplus \mathbb{C}. \tag{12.48}$$

Since $\mathbb{C}$ is the only module in common between the domain and range, we must have that $\phi = \pi_0 \lambda \pi_0$, where $\lambda \in \mathbb{C}$, and $\pi_0$ denotes the projections onto the $\mathbb{C}$ modules.

That is, $\phi$ is identically zero on $S^{8,0}$ and $S^{4,0}$, the image of $\phi$ lies in $\mathbb{C}$, and $\phi$ is just multiplication by a constant. Therefore,

$$\phi(W^+ \odot W^+) = \sum_{r,s,t} W^+_{irst} W^+_{jrst} = \lambda g_{ij}, \tag{12.49}$$

and taking a trace shows that $\lambda = (1/4)|W^+|^2$. And identical argument with $V_-$ replacing $V_+$ proves (12.39).

If we consider $\phi$ as a mapping

$$\phi : S^2_0(\Lambda^2_+) \otimes S^2_0(\Lambda^2_-) \to S^2(T^*M) \tag{12.50}$$

then tensoring with $\mathbb{C}$ yields a mapping

$$\phi : S^{4,4} \to S^{2,2} \oplus \mathbb{C}, \tag{12.51}$$

and since there are no modules in common in the domain and range, this mapping is identically zero, which proves, (12.40). Then (12.41) follows since $W = W^+ + W^-$ is an orthogonal decomposition. □

# 13 Lecture 13

## 13.1 Another identity in dimension four

**Proposition 13.1.** *In dimension* 4*, we have*

$$\sum_{r,s,t} R_{irst} R_{jrst} = \frac{1}{4}|Rm|^2 g_{ij} + \frac{R}{3} E_{ij} + 2 \sum_{p,q} W_{ipjq} E_{pq}. \tag{13.1}$$

*Proof.* We use the formula

$$Rm = W + \frac{1}{2} E \owedge g + \frac{R}{24} g \owedge g, \tag{13.2}$$

to compute

$$\begin{aligned}
\sum_{r,s,t} R_{irst} R_{jrst} &= \sum_{r,s,t} \Big( W_{irst} + \frac{1}{2}(E \owedge g)_{irst} + \frac{R}{24}(g \owedge g)_{irst} \Big) \\
&\qquad \cdot \Big( W_{jrst} + \frac{1}{2}(E \owedge g)_{jrst} + \frac{R}{24}(g \owedge g)_{jrst} \Big) \\
&= \sum_{r,s,t} \Big\{ W_{irst} W_{jrst} + \frac{1}{4}(E \owedge g)_{irst}(E \owedge g)_{jrst} + \frac{R^2}{24^2}(g \owedge g)_{irst}(g \owedge g)_{jrst} \\
&+ \frac{1}{2}(W_{irst}(E \owedge g)_{jrst} + W_{jrst}(E \owedge g)_{irst}) + \frac{R}{24}(W_{irst}(g \owedge g)_{jrst} + W_{jrst}(g \owedge g)_{irst}) \\
&+ \frac{R}{2 \cdot 24}((E \owedge g)_{irst}(g \owedge g)_{jrst} + (E \owedge g)_{jrst}(g \owedge g)_{irst}) \Big\} \\
&= I + \frac{1}{4} II + \frac{R^2}{24^2} III + \frac{1}{2} IV + \frac{R}{24} V + \frac{R}{2 \cdot 24} VI.
\end{aligned}$$

From (12.41), we have

$$I = \sum_{r,s,t} W_{irst} W_{jrst} = \frac{1}{4}|W|^2 g_{ij}. \tag{13.3}$$

Next, direct computation shows the following:

$$\begin{aligned}
II &= \sum_{r,s,t} (E \owedge g)_{irst} (E \owedge g)_{jrst} = 2|E|^2 g_{ij} \\
III &= \sum_{r,s,t} (g \owedge g)_{irst} (g \owedge g)_{jrst} = 24 g_{ij} \\
IV &= \sum_{r,s,t} (W_{irst}(E \owedge g)_{jrst} + W_{jrst}(E \owedge g)_{irst}) = 4 \sum_{pq} W_{ipjq} E_{pq} \\
V &= \sum_{r,s,t} (W_{irst}(g \owedge g)_{jrst} + W_{jrst}(g \owedge g)_{irst}) = 0 \\
VI &= \sum_{r,s,t} (E \owedge g)_{irst}(g \owedge g)_{jrst} + (E \owedge g)_{jrst}(g \owedge g)_{irst}) = 16 E_{ij}.
\end{aligned}$$

Using these, we obtain

$$\begin{aligned}
\sum_{r,s,t} R_{irst} R_{jrst} &= I + \frac{1}{4} II + \frac{R^2}{24^2} III + \frac{1}{2} IV + \frac{R}{24} V + \frac{R}{2 \cdot 24} VI \\
&= \frac{1}{4}|W|^2 g_{ij} + \frac{1}{2}|E|^2 g_{ij} + \frac{R^2}{24} g_{ij} + 2 \sum_{pq} W_{ipjq} E_{pq} + \frac{R}{3} E_{ij}.
\end{aligned}$$

The formula (8.9) in dimension 4 is

$$|Rm|^2 = |W|^2 + 2|E|^2 + \frac{1}{6} R^2, \tag{13.4}$$

and the proof is completed. $\square$

## 13.2 Curvature operator on symmetric tensors

The curvature tensor can also be viewed as an operator on symmetric tensors

$$\mathring{\mathcal{R}} : S^2(T^*M) \to S^2(T^*M), \tag{13.5}$$

by defining

$$(\mathring{\mathcal{R}} h)_{ij} \equiv \sum_{p,q} R_{ipjq} h_{pq}. \tag{13.6}$$

It is easily seen that this operator is a symmetric operator. Of course, the space of symmetric tensors has the orthogonal decomposition

$$S^2(T^*M) = S^2_0(T^*M) \oplus \mathbb{R}. \tag{13.7}$$

**Proposition 13.2.** *With respect to the above decomposition, we have*

$$\overset{\circ}{Weyl} : \mathbb{R} \to 0,\ \overset{\circ}{Weyl} : S^2_0(T^*M) \to S^2_0(T^*M), \tag{13.8}$$

$$(E \overset{\circ}{\owedge} g) : \mathbb{R} \to 0,\ (E \overset{\circ}{\owedge} g) : S^2_0(T^*M) \to \mathbb{R}, \tag{13.9}$$

*and $R(g \owedge g)$ acts diagonally. Consequently, $g$ is Einstein if and only if*

$$\overset{\circ}{\mathcal{R}} : S^2_0(T^*M) \to S^2_0(T^*M). \tag{13.10}$$

*Proof.* This follows from a simple computation. □

## 13.3 Differential Bianchi Identity in dimension 3

In any dimension, we recall that there are 3 Bianchi identities: the full Bianchi identity

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0, \tag{13.11}$$

the once-contracted Bianchi identity

$$\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}, \tag{13.12}$$

and the twice-contracted Bianchi identity

$$2\nabla_l R^l_j = \nabla_j R. \tag{13.13}$$

**Proposition 13.3.** *In dimension* 3*,* (13.11)*,* (13.12)*, and* (13.13) *are equivalent.*

*Proof.* We know that the curvature tensor is determined by the Ricci tensor in dimension 3, so each Bianchi identity is equivalent to some linear relation in first covariant derivatives of the Ricci tensor. In terms of representations, we have

$$\nabla Ric \in T \otimes S^2(T) = T \otimes (S^2_0(T) \oplus \mathbb{R}) = (T \otimes S^2_0(T)) \oplus T. \tag{13.14}$$

Upon complexification, we have

$$(T \otimes S^2_0(T)) \oplus T \otimes \mathbb{C} = (S^2(V) \otimes S^4(V)) \oplus S^2(V) \tag{13.15}$$

$$= S^6(V) \oplus S^4(V) \oplus S^2(V) \oplus S^2(V). \tag{13.16}$$

Let us write the corresponding projections as $\Pi_1, \Pi_2, \Pi_3$, and $\Pi_4$, respectively. Clearly, we can assume that $\Pi_3$ is the divergence operator $\delta$, and $\Pi_4$ is $d(trace)$. The Bianchi identity (13.13) can then be written $(2\Pi_3 - \Pi_4)Ric = 0$.

The quantity on the left hand side of the full Bianchi identity (13.11) is easily seen to be skew-symmetric in the first three indices, so it lives in the space $\Lambda^3 \otimes \Lambda^2 = T$. Consequently, by Schur's lemma, (13.11) must be a linear combination of $\Pi_3$ and $\Pi_4$. But obviously, this must be the same linear combination as in (13.13), so (13.11) and (13.13) are equivalent. A similar argument applies to see that (13.12) is also equivalent to (13.13). □

**Remark 13.1.** In terms of real representations, from Section 9.2 above, the decomposition (13.16) can be written

$$(T \otimes S^2_0(T)) \oplus T = \mathcal{H}^3(\mathbb{R}^3) \oplus S^2_0(T) \oplus T \oplus T. \tag{13.17}$$

# 14 Lecture 14

## 14.1 Example of $\mathbb{R}^4 = \mathbb{C}^2$

We consider $\mathbb{R}^4$ and take coordinates $x_1, y_1, x_2, y_2$. Letting $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$, define complex one-forms

$$\begin{aligned} dz_j &= dx_j + idy_j, \\ d\bar{z}_j &= dx_j - idz_j, \end{aligned}$$

and tangent vectors

$$\begin{aligned} \partial/\partial z_j &= (1/2)\left(\partial/\partial x_j - i\partial/\partial y_j\right), \\ \partial/\partial \bar{z}_j &= (1/2)\left(\partial/\partial x_j + i\partial/\partial y_j\right). \end{aligned}$$

Note that

$$\begin{aligned} dz_j(\partial/\partial z_k) &= d\bar{z}_j(\partial/\partial \bar{z}_k) = \delta_{jk}, \\ dz_j(\partial/\partial \bar{z}_k) &= d\bar{z}_j(\partial/\partial z_k) = 0. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ denote the complexified Euclidean inner product, so that

$$\begin{aligned} \langle \partial/\partial z_j, \partial/\partial z_k \rangle &= \langle \partial/\partial \bar{z}_j, \partial/\partial \bar{z}_k \rangle = 0, \\ \langle \partial/\partial z_j, \partial/\partial \bar{z}_k \rangle &= \frac{1}{2}\delta_{jk}. \end{aligned}$$

Similarly, on 1-forms we have

$$\begin{aligned} \langle dz_j, dz_k \rangle &= \langle d\bar{z}_j, d\bar{z}_k \rangle = 0, \\ \langle dz_j, d\bar{z}_k \rangle &= 2\delta_{jk}. \end{aligned}$$

The standard complex structure $J_0 : T\mathbb{R}^4 \to T\mathbb{R}^4$ on $\mathbb{R}^4$ is given by

$$J_0(\partial/\partial x_j) = \partial/\partial y_j, \quad J_0(\partial/\partial y_j) = -\partial/\partial x_j,$$

which in matrix form is written

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{14.1}$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \mathrm{span}\{\partial/\partial z_1, \partial/\partial z_2\} = \{X - iJ_0 X, X \in T_p\mathbb{R}^4\} \tag{14.2}$$

be the $i$-eigenspace and

$$T^{(0,1)}(J_0) = \mathrm{span}\{\partial/\partial \bar{z}_1, \partial/\partial \bar{z}_2\} = \{X + iJ_0 X, X \in T_p\mathbb{R}^4\} \tag{14.3}$$

be the $-i$-eigenspace of $J_0$, so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \tag{14.4}$$

The map $J_0$ also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0^{\top} v_1) = -\omega(J_0 v_1),$$

which satisfies

$$J_0(dx_j) = dy_j, \quad J_0(dy_j) = -dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \mathrm{span}\{dz_1, dz_2\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^4\} \tag{14.5}$$

is the $-i$-eigenspace, and

$$\Lambda^{0,1}(J_0) = \mathrm{span}\{d\bar{z}_1, d\bar{z}_2\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^4\} \tag{14.6}$$

is the $+i$-eigenspace of $J_0$, and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \tag{14.7}$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{14.8}$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{14.9}$$

## 14.2 Complex structure in $\mathbb{R}^{2n}$

The above works in a more general setting, in any even dimension. We only need assume that $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is linear and satisfies $J^2 = -I$. In this more general setting, we have

$$T \otimes \mathbb{C} = T^{(1,0)}(J) \oplus T^{(0,1)}(J), \tag{14.10}$$

where

$$T^{(1,0)}(J) = \{X - iJX, X \in T_p\mathbb{R}^{2n}\} \tag{14.11}$$

is the $i$-eigenspace of $J$ and

$$T^{(0,1)}(J) = \{X + iJX, X \in T_p\mathbb{R}^{2n}\} \tag{14.12}$$

is the $-i$-eigenspace of $J$.

As above, The map $J$ also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(J^{\top} v_1) = -\omega(J v_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J) \oplus \Lambda^{0,1}(J), \tag{14.13}$$

where

$$\Lambda^{1,0}(J) = \{\alpha + iJ\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{14.14}$$

is the $-i$-eigenspace of $J$, and

$$\Lambda^{0,1}(J) = \{\alpha - iJ\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{14.15}$$

is the $+i$-eigenspace of $J$.

Again, we have the characterizations

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{14.16}$$

and

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{14.17}$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly $p$ elements in $\Lambda^{1,0}$ and exactly $q$ elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{14.18}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{14.19}$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \dots, v_{p+q}) = 0, \tag{14.20}$$

if more than $p$ if the $v_j$-s are in $T^{(1,0)}$ or if more than $q$ of the $v_j$-s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{q-p}\alpha, \tag{14.21}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

## 14.3 Hermitian metrics

We next consider $(\mathbb{R}^{2n}, J, g)$ where $g$ is a Riemannian metric, and we assume that $g$ and $J$ are compatible. That is,

$$g(X, Y) = g(JX, JY), \tag{14.22}$$

the metric $g$ is then called a Hermitian metric. We extend $g$ by complex linearity to a symmetric inner product on $T \otimes \mathbb{C}$. The following will be useful later.

**Proposition 14.1.** *There exist elements $\{X_1, \dots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \dots, X_n, JX_n\} \tag{14.23}$$

*is an ONB for $\mathbb{R}^{2n}$ with respect to $g$.*

*Proof.* We use induction on the dimension. First we note that if $X$ is any unit vector, then $JX$ is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{14.24}$$

so $X$ and $JX$ are orthonormal. This handles $n = 1$. In general, start with any $X_1$, and let $W$ be the orthogonal complement of *span*$\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using $J$-invariance of $g$, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since $W$ is of dimension $2n - 2$. $\square$

To a Hermitian metric $(\mathbb{R}^{2n}, J, g)$ we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{14.25}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{14.26}$$

This form is in fact of type $(1, 1)$, and is called the *Kähler form.*

# 15 Lecture 15

## 15.1 Hermitian symmetric tensors

More generally, we say that any symmetric 2-tensor is *hermitian* if

$$b(JX, JY) = b(X, Y). \tag{15.1}$$

We have following property of hermitian symmetric 2-tensors:

**Proposition 15.1.** *If $b$ is any symmetric 2-tensor which is hermitian, then*

$$\beta(X,Y) = b(JX,Y) \tag{15.2}$$

*is skew-symmetric and $\beta \in \Lambda^{1,1}$. Furthermore, define an endomorphism $I$ by*

$$g(I(X),Y) = \beta(X,Y), \tag{15.3}$$

*then*

$$IJ = JI, \tag{15.4}$$

*that is, $I$ commutes with $J$.*

*Proof.* To check this, we need to show that

$$\beta(X,Y) = 0 \tag{15.5}$$

if either both $X$ and $Y$ are in $T^{(1,0)}$ or both are in $T^{(0,1)}$. For the first case,

$$\begin{aligned}\beta(X,Y) &= \beta(X' - iJX', Y' - iJY') = b(J(X' - iJX'), Y' - iJY') \\ &= b(JX' + iX', Y' - iJY') \\ &= b(JX',Y') + b(X',JY') + i(b(X',Y') - b(JX',JY')) = 0,\end{aligned}$$

since $b$ is $J$-invariant. The second case is similar. Next,

$$g(IJ(X),Y) = \beta(JX,Y) = b(J^2X,Y) = -b(X,Y). \tag{15.6}$$

On the other hand, since $g$ is $J$-invariant,

$$\begin{aligned}g(JI(X),Y) &= g(J^2I(X),JY) = -g(I(X),JY) \qquad (15.7)\\ &= -\beta(X,JY) = -b(JX,JY) = -b(X,Y), \qquad (15.8)\end{aligned}$$

and therefore $IJ = JI$. $\square$

We can view the above proposition in matrix form. Choose a basis so that

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{15.9}$$

Since $\beta$ is skew-symmetric, the endomorphism $I$ is also. Therefore we can write $I$ in block form

$$I = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}, \tag{15.10}$$

where $A$ and $D$ are skew-symmetric, and $B$ is an $n \times n$ matrix. Then $IJ = JI$ is

$$\begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} \tag{15.11}$$

which is

$$\begin{pmatrix} B & -A \\ D & B^T \end{pmatrix} = \begin{pmatrix} B^T & -D \\ A & B \end{pmatrix}, \tag{15.12}$$

so we conclude that $A = D$, and $B$ is symmetric, so

$$I = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{15.13}$$

where $A$ is skew-symmetric, and $B$ is symmetric. The total number of parameters is

$$\frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2, \tag{15.14}$$

which of course agrees with $\dim(\Lambda^{1,1}) = n^2$.

## 15.2 The Unitary Group

We embed $GL(n, \mathbb{C})$ in $GL(2n, \mathbb{R})$ by

$$A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{15.15}$$

These are exactly the matrices which commute with $J$. The condition for a matrix to be unitary is that $M\overline{M}^T = I_n$. The Lie algebra consists of skew-hermitian matrices, that is, matrices with $M + \overline{M}^T = 0$. Using the above embedding to $GL(2n, \mathbb{R})$, this says that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} + \begin{pmatrix} A & -B \\ B & A \end{pmatrix}^T = \begin{pmatrix} A + A^T & B - B^T \\ -B + B^T & A + A^T \end{pmatrix} = \mathbf{0}, \tag{15.16}$$

which says that $A$ is skew-symmetric, and $B$ is symmetric. This is exactly what we found above, thus $\Lambda^{1,1} \cong \mathfrak{u}(n)$, is identified with the Lie algebra of the unitary group. Note that hermitian symmetric 2-tensors yield skew-hermitian matrices.

## 15.3 Skew-hermitian tensors

We say that a symmetric 2-tensor $b$ is *skew-hermitian* if

$$b(JX, JY) = -b(X, Y), \tag{15.17}$$

These have the following property:

**Proposition 15.2.** *If $b$ is a symmetric 2-tensor which is skew-hermitian, then*

$$\beta(X, Y) = b(JX, Y) \tag{15.18}$$

*is also a symmetric 2-tensor. Define an endomorphism $I$ by*

$$g(I(X), Y) = \beta(X, Y), \tag{15.19}$$

*then*

$$IJ + JI = 0, \tag{15.20}$$

*that is $I$ anti-commutes with $J$. Furthermore $I(T^{0,1}) \subset T^{1,0}$, or equivalently, $I \in \Lambda^{0,1} \otimes T^{1,0}$.*

*Proof.* For the first statement

$$\beta(Y, X) = b(JY, X) = -b(J^2Y, JX) = b(Y, JX) = \beta(X, Y). \tag{15.21}$$

Next,

$$g(IJ(X), Y) = \beta(JX, Y) = b(J^2X, Y) = -b(X, Y). \tag{15.22}$$

On the other hand, since $g$ is $J$-invariant,

$$\begin{aligned} g(JI(X), Y) &= g(J^2I(X), JY) = -g(I(X), JY) && (15.23)\\ &= -\beta(X, JY) = -b(JX, JY) = b(X, Y), && (15.24) \end{aligned}$$

and therefore $IJ = -JI$.

Finally, if $X \in T^{0,1}$, any $Y \in T^{1,0}$, then

$$\begin{aligned} g(IX, Y) &= \beta(X, Y) = \beta(X' + iJX', Y' - iJY')\\ &= b(J(X' + iJX'), Y' - iJY')\\ &= b(JX' - iX', Y' - iJY')\\ &= b(JX', Y') - b(X', JY') - i(b(X', Y') + b(JX', JY')) = 0 \end{aligned}$$

□

We can do a similar matrix analysis as above. Since $\beta$ is symmetric, the endomorphism $I$ is also. Therefore we can write $I$ in block form

$$I = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \tag{15.25}$$

where $A$ and $D$ are symmetric, and $B$ is an $n \times n$ matrix. Then $IJ = -JI$ is

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = - \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \tag{15.26}$$

which is

$$\begin{pmatrix} B & -A \\ D & -B^T \end{pmatrix} = \begin{pmatrix} B^T & D \\ -A & -B \end{pmatrix}, \tag{15.27}$$

so we conclude that $A = -D$, and $B$ is symmetric, so

$$I = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \tag{15.28}$$

where both $A$ and $B$ are symmetric. The total number of parameters is now $n(n+1)$.

## 15.4 Representations

In terms of representations, what we have seen is that the space of symmetric 2-tensors, which has the irreducible decomposition

$$S^2(\mathbb{R}^{2n}) = S^2_0(\mathbb{R}^{2n}) \oplus \mathbb{R} \tag{15.29}$$

over $SO(2n)$ is *not* irreducible when the group is reduced to $U(n)$. It decomposes into 3 pieces:

$$S^2(\mathbb{R}^{2n}) = \Lambda^{1,1}_0 \oplus \mathbb{R} \oplus V, \tag{15.30}$$

where $\dim(V) = n^2 + n$, and $\Lambda^{1,1}_0 \subset \Lambda^{1,1}$ is the orthogonal complement of the Kähler form. We can understand this on the matrix level as follows. In the above we started with a symmetric 2 tensor $b$, which, after converting to an endomorphism is

$$b = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \tag{15.31}$$

where $A$ and $D$ are symmetric, and $B$ is an arbitrary $n \times n$. We then applied $J$, which yields the matrix.

$$\beta = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} B & -A \\ D & -B^T \end{pmatrix}. \tag{15.32}$$

Decomposing $B = B^s + B^{ss}$ into its symmetric and skew-symmetric parts, we then write this as

$$\beta = \begin{pmatrix} B & -A \\ D & -B^T \end{pmatrix} = \begin{pmatrix} B^s + B^{ss} & M_1 + M_2 \\ M_1 - M_2 & -B^s + B^{ss} \end{pmatrix} = \begin{pmatrix} B^s & M_1 \\ M_1 & -B^s \end{pmatrix} + \begin{pmatrix} B^{ss} & M_2 \\ -M_2 & B^{ss} \end{pmatrix}, \tag{15.33}$$

where $M_1 = -\frac{A-D}{2}$ and $M_2 = -\frac{A+D}{2}$. Converting back to $b$, we have the explicit decomposition corresponding to the pieces in (15.30):

$$b = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} = \begin{pmatrix} (M_2)_0 & B^{ss} \\ -B^{ss} & (M_2)_0 \end{pmatrix} + c_1 \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} + \begin{pmatrix} M_1 & B^s \\ B^s & -M_1 \end{pmatrix}, \tag{15.34}$$

where $B^{ss}$ is skew-symmetric, $(M_2)_0$ is traceless and symmetric, and $B^s$ and $M_1$ are symmetric.

# 16 Lecture 16

## 16.1 Two-forms

In the last lecture, we decomposed symmetric 2-tensors into hermitian and skew-hermitian parts. We can do the same thing for 2-forms. First, we say that a 2-form $\beta$ is *hermitian* if

$$\beta(JX, JY) = \beta(X, Y). \tag{16.1}$$

From Proposition 15.1, we already know that this is a 2-form of type $(1,1)$ and the associated endomorphism satisfies $IJ = JI$. So we only need consider a *skew-hermitian* 2-form, that is,

$$\beta(JX, JY) = -\beta(X, Y). \tag{16.2}$$

These have the following property:

**Proposition 16.1.** *If $\beta$ is a skew-hermitian 2-form, then $\beta \in \Lambda^{2,0} \oplus \Lambda^{0,2}$. Furthermore, define an endomorphism $I$ by*

$$g(I(X), Y) = \beta(X, Y), \tag{16.3}$$

*then*

$$IJ + JI = 0, \tag{16.4}$$

*that is, $I$ anti-commutes with $J$.*

*Proof.* To check this, we need to show that

$$\beta(X, Y) = 0 \tag{16.5}$$

if $X \in T^{(1,0)}$ and $Y \in T^{(0,1)}$. We compute

$$\begin{aligned}
\beta(X, Y) &= \beta(X' - iJX', Y' + iJY') \\
&= \beta(X', Y') + \beta(JX', JY') + i(\beta(X', JY') - \beta(JX', Y') \\
&= 0 + i(-\beta(JX', J^2Y') - \beta(JX', Y')) = i(\beta(JX', Y') - \beta(JX', Y')) = 0.
\end{aligned}$$

Next,

$$g(IJ(X), Y) = \beta(JX, Y) \tag{16.6}$$

On the other hand, since $g$ is $J$-invariant,

$$\begin{aligned}
g(JI(X), Y) &= g(J^2I(X), JY) = -g(I(X), JY) \\
&= -\beta(X, JY) = \beta(JX, J^2Y) = -\beta(JX, Y),
\end{aligned}$$

and therefore $IJ = -JI$. $\square$

In terms of representations, combining results from the previous section, we have shown the following

$$T^* \otimes T^* = S^2(T^*) \oplus \Lambda^2(T^*) = \Lambda_0^{1,1} \oplus \mathbb{R} \oplus V \oplus \Lambda_0^{1,1} \oplus \mathbb{R} \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}), \tag{16.7}$$

where the final summand means the real elements in this complex vector space.

Note that the 2-tensors which anti-commute with $J$, that is, satisfy $IJ + JI = 0$, are given by

$$V \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}). \tag{16.8}$$

But as shown in Proposition 15.2, these type of tensors can be viewed as sections of $\Lambda^{0,1} \otimes T^{1,0}$. Counting dimensions

$$dim_{\mathbb{R}}(V \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2})) = n^2 + n + 2\binom{n}{2} = 2n^2, \tag{16.9}$$

Also,

$$dim_{\mathbb{R}}(\Lambda^{0,1} \otimes T^{1,0}) = 2n^2, \tag{16.10}$$

so these spaces are the same. We actually have the more general statment, without reference to any metric, only the complex structure:

**Proposition 16.2.** *The space of endomorphisms $I$ satisfying $IJ + JI = 0$ can be identified with $\Lambda^{0,1} \otimes T^{1,0}$.*

*Proof.* Assume $J$ is in standard form (15.9) and write the endomorphism $I$ in matrix form as

$$I = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{16.11}$$

where $A, B, C, D$ are arbitrary $n \times n$ matrices. Then $IJ = -JI$ is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{16.12}$$

which is

$$\begin{pmatrix} B & -A \\ D & -C \end{pmatrix} = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}, \tag{16.13}$$

which says that $I$ is of the form

$$I = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \tag{16.14}$$

where $A$ and $B$ are arbitrary $n \times n$ real matrics. The dimension of this space is thus $2n^2$.

Finally, as shown in Proposition 15.2, these type of tensors can be viewed as sections of $\Lambda^{0,1} \otimes T^{1,0}$, which also has real dimension $2n^2$, so these spaces are therefore equivalent. □

Note if we take a path of complex structures $J(t)$ with $J(0) = J$ and $J'(0) = I$, then differentiating $J^2 = -I_n$ an evaluating at $t = 0$ yields $IJ + JI = 0$. So elements of $\Lambda^{0,1} \otimes T^{1,0}$ are infinitesimal deformations of the complex structure.

## 16.2 Complex manifolds and the Nijenhuis tensor

We begin with a defintion

**Definition 2.** A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is *holomorphic* if $f_* \circ J_0 = J_0 \circ f_*$, where we view $\mathbb{C}^m = (\mathbb{R}^{2m}, J_0)$ and $\mathbb{C}^n = (\mathbb{R}^{2n}, J_0)$.

We have the following characterization of holomorphic maps

**Proposition 16.3.** *A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots u_n + iv_n), \tag{16.15}$$

*and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have*

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \qquad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \tag{16.16}$$

*for each $k = 1 \dots m$, and these equations are equivalent to*

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \tag{16.17}$$

*for each $j = 1 \dots n$ and each $k = 1 \dots m$*

*Proof.* First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{16.18}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{16.19}$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \dots, x^m, y^1, \dots, y^m)$ are the real coordinates on $\mathbb{R}^{2m}$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$, such that the complex structure $J_0$ is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{16.20}$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\Big)(u_j + iv_j) \tag{16.21}$$

$$= \frac{1}{2}\Big\{\frac{\partial}{\partial x^k}u_j - \frac{\partial}{\partial y^k}v_j + i\Big(\frac{\partial}{\partial x^k}v_j + \frac{\partial}{\partial y^k}u_j\Big)\Big\}, \tag{16.22}$$

the vanishing of which again yields the Cauchy-Riemann equations. □

Now we can define a complex manifold

**Definition 3.** A *complex manifold* of dimension $n$ is a smooth manifold of real dimension $2n$ with a collection of coordinate charts $(U_\alpha, \phi_\alpha)$ covering $M$, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \to \phi_\alpha(U_\alpha)$ satisfying the Cauchy-Riemann equations.

Such spaces have a complex structure on the tangent bundle.

**Proposition 16.4.** *In any coordinate chart, define $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{16.23}$$

*Then $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$ and therefore gives a global complex structure $J : TM \to TM$ satisfying $J^2 = -Id$.*

*Proof.* On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{16.24}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{16.25}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{16.26}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned}
J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\
&= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\
&= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id.
\end{aligned}$$

□

Consequently, we can apply all of the linear algebra from the previous sections to complex manifolds.

**Definition 4.** An *almost complex structure* is an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$. An almost complex structure $J$ is said to be *integrable* if $J$ is induced from a collection of holomorphic coordinates on $M$.

Let $(M^2, g)$ be any oriented Riemannian surface. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. In the case of surfaces, this always comes from a collection of holomorphic coordinate charts, but this is not true in higher dimensions. To understand this we proceed as follows:

**Proposition 16.5.** *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX, JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \tag{16.27}$$

*is a tensor of type* $(1,2)$ *and satisfies* $N(Y,X) = -N(X,Y)$.

*Proof.* Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned}
N(fX,Y) &= 2\{[J(fX), JY] - [fX,Y] - J[fX,JY] - J[J(fX),Y]\} \\
&= 2\{[fJX, JY] - [fX,Y] - J[fX,JY] - J[fJX,Y]\} \\
&= 2\{f[JX,JY] - (JY(f))JX - f[X,Y] + (Yf)X \\
&- J(f[X,JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\} \\
&= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X.
\end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X,fY) = fN(X,Y)$. Consequently, $N$ is a tensor. The skew-symmetry in $X$ and $Y$ is obvious. □

# 17 Lecture 17

We have the following local formula for the Nijenhuis tensor.

**Proposition 17.1.** *In local coordinates, the Nijenhuis tensor is given by*

$$N_{jk}^i = 2\sum_{h=1}^{2n}(J_j^h \partial_h J_k^i - J_k^h \partial_h J_j^i - J_h^i \partial_j J_k^h + J_h^i \partial_k J_j^h) \tag{17.1}$$

*Proof.* We compute

$$\begin{aligned}
&\frac{1}{2}N(\partial_j,\partial_k) = [J\partial_j, J\partial_k] - [\partial_j,\partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j,\partial_k] \\
&= [J_j^l\partial_l, J_k^m\partial_m] - [\partial_j,\partial_k] - J[\partial_j, J_k^l\partial_l] - J[J_j^l\partial_l,\partial_k] \\
&= I + II + III + IV.
\end{aligned}$$

The first term is

$$\begin{aligned}
I &= J_j^l\partial_l(J_k^m\partial_m) - J_k^m\partial_m(J_j^l\partial_l) \\
&= J_j^l(\partial_l J_k^m)\partial_m + J_j^l J_k^m \partial_l\partial_m - J_k^m(\partial_m J_j^l)\partial_l - J_k^m J_j^l \partial_m\partial_l \\
&= J_j^l(\partial_l J_k^m)\partial_m - J_k^m(\partial_m J_j^l)\partial_l.
\end{aligned}$$

The second term is obviously zero. The third term is

$$III = -J(\partial_j(J_k^l)\partial_l) = -\partial_j(J_k^l)J_l^m\partial_m. \tag{17.2}$$

Finally, the fourth term is

$$III = \partial_k(J_j^l)J_l^m\partial_m. \tag{17.3}$$

Combining these, we are done. □

Next, we have

**Theorem 17.1.** *An almost complex structure $J$ is integrable if and only if the Nijenhuis tensor vanishes.*

*Proof.* If $J$ is integrable, then we can always find local coordinates so that $J = J_0$, and Proposition 17.1 shows that the Nijenhuis tensor vanishes. For the converse, the vanishing of the Nijenhuis tensor is the integrability condition for $T^{1,0}$ as a complex sub-distribution of $T \otimes \mathbb{C}$. To see this, if $X$ and $Y$ are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields $X'$ and $Y'$. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \tag{17.4}$$

But this is also a $(1,0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \tag{17.5}$$

applying $J$, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \tag{17.6}$$

which is exactly the vanishing of the Nijenhuis tensor. In the analytic case, the converse then follows using a complex version of the Frobenius Theorem. The $C^\infty$-case is more difficult, and is the content of the Newlander-Nirenberg Theorem, which we will not prove here. □

**Proposition 17.2.** *For an almost complex structure $J$*

$$d(\Lambda^{p,q}) \subset \Lambda^{p+2,q-1} + \Lambda^{p+1,q} + \Lambda^{p,q+1} + \Lambda^{p-1,q+2}, \tag{17.7}$$

*and $J$ is integrable if and only if*

$$d(\Lambda^{p,q}) \subset \Lambda^{p+1,q} + \Lambda^{p,q+1}. \tag{17.8}$$

*Proof.* Let $\alpha \in \Lambda^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) &= \sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{17.9}$$

This is easily seen to vanish if more than $p+2$ of the $X_j$ are of type $(1,0)$ or if more than $q+2$ are of type $(0,1)$.

If $J$ is integrable, then in a local complex coordinate system, (17.8) is easily seen to hold. For the converse we have the inclusions,

$$d(\Lambda^{1,0}) \subset \Lambda^{2,0} + \Lambda^{1,1} \text{ and } d(\Lambda^{0,1}) \subset \Lambda^{1,1} + \Lambda^{0,2}. \tag{17.10}$$

The formula

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{17.11}$$

then implies that if both $X$ and $Y$ are in $T^{1,0}$ then so is their bracket $[X,Y]$. So write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields $X'$ and $Y'$. Define $Z = [X,Y]$, then $Z$ is also of type $(1,0)$, so

$$Z + iJZ = 0. \tag{17.12}$$

Writing this in terms of $X'$ and $Y'$ we see that

$$0 = 2(Z + iJZ) = -N(X',Y') - iJN(X',Y'), \tag{17.13}$$

which implies that $N \equiv 0$. □

**Corollary 17.1.** *On a complex manifold, $d = \partial + \overline{\partial}$ where $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $\overline{\partial} : \Lambda^{p,q} \to \Lambda^{p,q+1}$, and these operators satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{17.14}$$

*Proof.* These relations follow simply from $d^2 = 0$. □

## 17.1 Automorphisms

**Definition 5.** An *infinitesimal automorphism* of a complex manifold is a real vector field $X$ such that $\mathcal{L}_X J = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

It is straightforward to see that $X$ is an infinitesimal automorphism if and only if its 1-parameter group of diffeomorphisms are holomorphic automorphisms, that is, $(\phi_s)_* \circ J = J \circ (\phi_s)_*$.

**Proposition 17.3.** *A vector field $X$ is an infinitesimal automorphism if and only if*

$$J([X,Y]) = [X, JY], \tag{17.15}$$

*for a vector fields $X$ and $Y$.*

*Proof.* We compute

$$[X, JY] = \mathcal{L}_X(JY) = \mathcal{L}_X(J)Y + J(\mathcal{L}_X Y) = \mathcal{L}_X(J)Y + J([X,Y]), \tag{17.16}$$

and the result follows. □

**Definition 6.** A *holomorphic vector field* on a complex manifold $(M,J)$ is vector field $Z \in \Gamma(T^{1,0})$ which satisfies $Zf$ is holomorphic for every locally defined holomorphic function $f$.

In complex coordinates, a holomorphic vector field can locally be written as

$$Z = \sum f_i \frac{\partial}{\partial z^i}, \tag{17.17}$$

where the $f_i$ are locally defined holomorphic functions.

**Proposition 17.4.** *For $X \in \Gamma(TM)$, associate a vector field of type $(1,0)$ by mapping $X \mapsto Z = X - iJX$. Then $X$ is an infinitesimal automorphism if and only if $Z$ is a holomorphic vector field.*

*Proof.* Choose a local holomorphic coordinate system $\{z^i\}$, and for real vector fields $X'$ and $Y'$, write

$$X = X' - iJX' = \sum X^j \frac{\partial}{\partial z^j}, \tag{17.18}$$

$$Y = Y' - iJY' = \sum Y^j \frac{\partial}{\partial z^j}. \tag{17.19}$$

We know that $X'$ is an infinitesimal automorphism if and only if

$$J([X', Y']) = [X', JY'], \tag{17.20}$$

for all real vector fields $Y'$. This condition is equivalent to

$$\sum_j \overline{Y}^j \frac{\partial X^k}{\partial \overline{z}^j}, \tag{17.21}$$

for each $k = 1 \dots n$, which is equivalent to $X$ being a holomorphic vector field.

To see this, we rewrite (17.20) in terms of complex vector fields. We have

$$X' = \frac{1}{2}(X + \overline{X}) \quad JX' = \frac{i}{2}(X - \overline{X})$$
$$Y' = \frac{1}{2}(Y + \overline{Y}) \quad JY' = \frac{i}{2}(Y - \overline{Y})$$

The left hand side of (17.20) is

$$\begin{aligned} J([X', Y']) &= J([\frac{1}{2}(X + \overline{X}), \frac{1}{2}(Y + \overline{Y})]) \\ &= \frac{1}{4} J([X, Y] + [X, \overline{Y}] + [\overline{X}, Y] + [\overline{X}, \overline{Y}]). \end{aligned}$$

But from integrability, $[X, Y]$ is also of type $(1,0)$, and $[\overline{X}, \overline{Y}]$ is of type $(0,1)$. So we can write this as

$$J([X', Y']) = \frac{1}{4}(i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]). \tag{17.22}$$

Next, the right hand side of (17.20) is

$$[\frac{1}{2}(X + \overline{X}), \frac{i}{2}(Y - \overline{Y})] = \frac{i}{4}([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{17.23}$$

Then (17.22) equals (17.23) if and only if

$$J[X,\overline{Y}] + J[\overline{X},Y] = -i[X,\overline{Y}] + i[\overline{X},Y]. \tag{17.24}$$

This is equivalent to

$$J(Re([X,\overline{Y}])) = Im([X,\overline{Y}]). \tag{17.25}$$

This says that $[X,\overline{Y}]$ is a vector field of type $(0,1)$. We can write the Lie bracket as

$$\begin{aligned}[X,\overline{Y}] &= \Big[\sum_j X^j \frac{\partial}{\partial z^j}, \sum_k \overline{Y}^k \frac{\partial}{\partial \overline{z}^k}\Big]\\ &= \sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j)\frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j}\overline{Y}^k)\frac{\partial}{\partial \overline{z}^k},\end{aligned}$$

and the vanishing of the $(1,0)$ component is exactly (17.21). □

# 18 Lecture 18

We next give an alternate proof of Proposition 16.2.

**Proposition 18.1.** *The space of endomorphisms $I$ satisfying $IJ + JI = 0$ can be identified with $\Lambda^{0,1} \otimes T^{1,0}$. Furthermore, the space of endomorphisms $I$ satisfying $IJ - JI = 0$ can be identified with $\Lambda^{1,0} \otimes T^{1,0}$.*

*Proof.* An element $\mathcal{I} \in \Lambda^{0,1} \otimes T^{1,0}$ is a complex linear mapping from $T^{0,1}$ to $T^{1,0}$, that is $\mathcal{I} \in Hom_{\mathbb{C}}(T^{0,1}, T^{1,0})$. Writing $X \in T^{0,1}$ as $X = X' + iJX'$ for real $X' \in T$ and since $\mathcal{I}$ maps to $T^{1,0}$, $\mathcal{I}$ can be written as

$$\mathcal{I} : X' + iJX' \mapsto I(X') - iJI(X'), \tag{18.1}$$

for some real endomorphism of the tangent space $I : T \to T$, by defined by

$$I(X') = Re(\mathcal{I}(X' + iJX')). \tag{18.2}$$

To show that $IJ = -JI$, we first compute

$$IJ(X') = Re\{\mathcal{I}(JX' + iJJX')\} = Re\{\mathcal{I}(J(X' + iJX')\},$$

but since $X' + iJX' \in T^{0,1}$, we have $J(X' + iJX') = -i(X' + iJX')$, so

$$IJ(X') = Re\{\mathcal{I}(-i(X' + iJX')\},$$

using complex linearity of $\mathcal{I}$,

$$IJ(X') = Re\{-i\mathcal{I}(X' + iJX')\} = Im(\mathcal{I}(X' + iJX'))$$

Next, we have

$$JI(X') = JRe(\mathcal{I}(X' + iJX')) = -Im(\mathcal{I}(X' + iJX')), \tag{18.3}$$

since $\mathcal{I}(X' + iJX')$ is a $(1,0)$ vector field, and we have shown that $IJ = -JI$.

For the converse, given a real mapping satisfying $IJ + JI = 0$, writing $X \in T^{0,1}$ as $X = X' + iJX'$ define $\mathcal{I} : T^{0,1} \to T^{1,0}$ by

$$\mathcal{I} : X' + iJX' \mapsto I(X') - iJI(X'). \tag{18.4}$$

This map is clearly real linear, and we claim that this map is moreover complex linear. To see this,

$$\begin{aligned}\mathcal{I}(i(X' + iJX')) &= \mathcal{I}(-J(X' + iJX')) \\ &= -\mathcal{I}(JX' + iJ(JX')) = -I(JX') + iJI(JX').\end{aligned}$$

Using $IJ = -JI$, this is

$$\mathcal{I}(i(X' + iJX')) = JI(X') - iIJ(JX') = JI(X') + iI(X').$$

Next,

$$i\mathcal{I}(X' + iJX') = i(I(X') - iJI(X')) = JI(X') + iI(X'),$$

so $\mathcal{I}$ is indeed complex linear.

A similar argument proves the second case, and we are done. □

## 18.1 The $\overline{\partial}$ operator on holomorphic vector bundles

We first illustrate this operator for the holomorphic tangent bundle $T^{1,0}$.

**Proposition 18.2.** *There is an first order differential operator*

$$\overline{\partial} : \Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}), \tag{18.5}$$

*such that a vector field $Z$ is holomorphic if and only if $\overline{\partial}(Z) = 0$.*

*Proof.* Choose local holomorphic coordinates $\{z^j\}$, and write any section of $Z$ of $T^{1,0}$, locally as

$$Z = \sum Z^j \frac{\partial}{\partial z^j}. \tag{18.6}$$

Then define

$$\overline{\partial}(Z) = \sum_j (\overline{\partial} Z^j) \otimes \frac{\partial}{\partial z^j}. \tag{18.7}$$

This is in fact a well-defined global section of $\Lambda^{0,1} \otimes T^{1,0}$ since the transition functions of the bundle $T^{1,0}$ corresponding to a change of holomorphic coordinates are holomorphic.

To see this, if we have an overlapping coordinate system $\{w^j\}$ and

$$Z = \sum W^j \frac{\partial}{\partial w^j}. \tag{18.8}$$

Note that

$$\frac{\partial}{\partial z^j} = \frac{\partial w^k}{\partial z^j} \frac{\partial}{\partial w^j}, \tag{18.9}$$

which implies that

$$W^j = Z^p \frac{\partial w^j}{\partial z^p}. \tag{18.10}$$

We compute

$$\begin{aligned}
\overline{\partial}(Z) &= \sum \overline{\partial}(W^j) \otimes \frac{\partial}{\partial w^j} = \sum \overline{\partial}(Z^p \frac{\partial w^j}{\partial z^p}) \otimes \frac{\partial z^q}{\partial w^j} \frac{\partial}{\partial z^q} \\
&= \sum \frac{\partial w^j}{\partial z^p} \frac{\partial z^q}{\partial w^j} \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \delta^q_p \overline{\partial}(Z^p) \otimes \frac{\partial}{\partial z^q} = \sum \overline{\partial}(Z^j) \otimes \frac{\partial}{\partial z^j}.
\end{aligned}$$

□

Recall that the transition functions of a complex vector bundle are locally defined functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$, satisfying

$$\phi_{\alpha\beta} = \phi_{\alpha\gamma}\phi_{\gamma\beta}. \tag{18.11}$$

Notice the main property we used in the proof of Proposition 18.2 is that the transition functions of the bundle are holomorphic. Thus we make the following definition.

**Definition 7.** A vector bundle $\pi : E \to M$ is a *holomorphic vector bundle* if in complex coordinates the transitition functions $\phi_{\alpha\beta}$ are holomorphic.

Recall that a section of a vector bundle is a mapping $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$. In local coordinates, a section satisfies

$$\sigma_\alpha = \phi_{\alpha\beta}\sigma_\beta, \tag{18.12}$$

and conversely any locally defined collection of functions $\sigma_\alpha : U_\alpha \to \mathbb{C}^m$ satisfying (18.12) defines a global section. A section is *holomorphic* if in complex coordinates, the $\sigma_\alpha$ are holomorphic.

We next have the generalization of Proposition 18.2.

**Proposition 18.3.** *If $\pi : E \to M$ is a holomorphic vector bundle, then there is an first order differential operator*

$$\overline{\partial} : \Gamma(E) \to \Gamma(\Lambda^{0,1} \otimes E), \tag{18.13}$$

*such that a section $\sigma$ is holomorphic if and only if $\overline{\partial}(\sigma) = 0$.*

*Proof.* Let $\sigma_j$ be a local basis of holomorphic sections in $U_\alpha$, and write any section $\sigma$ as

$$\sigma = \sum s_j \sigma_j. \tag{18.14}$$

Then define

$$\overline{\partial}\sigma = \sum (\overline{\partial} s_j) \otimes \sigma_j. \tag{18.15}$$

We claim this is a global section of $\Gamma(\Lambda^{0,1} \otimes E)$. If we choose a local basis $\sigma_j'$ of holomorphic sections in $U_\beta$, and write $\sigma$ as

$$\sigma = \sum s_j' \sigma_j'. \tag{18.16}$$

We can write

$$s_j' = A_{jl} s_l, \tag{18.17}$$

where $A : U_\alpha \cap U_\beta \to GL(m, \mathbb{C})$ is holomorphic. We also have

$$\sigma_j' = A^{-1}_{jl} \sigma_l. \tag{18.18}$$

Consequently

$$\begin{aligned}
\overline{\partial}\sigma &= \sum (\overline{\partial} s_j') \otimes \sigma_j' = \sum \overline{\partial}(A_{jk} s_k) \otimes A^{-1}_{jl}\sigma_l \\
&= \sum A_{jk} \overline{\partial}(s_k) \otimes A^{-1}_{jl} \sigma_l = \sum \delta_{kl} (\overline{\partial} s_k) \otimes \sigma_l = \sum (\overline{\partial} s_k) \otimes \sigma_k.
\end{aligned}$$

□

# 19 Lecture 19

For the special case of $T^{1,0}$ we have another operator mapping from

$$\Gamma(T^{1,0}) \to \Gamma(\Lambda^{0,1} \otimes T^{1,0}) \tag{19.1}$$

defined as follows. If $X$ is a section of $T^{1,0}$, writing $X = X' - iJX'$ for a real vector field $X'$ then consider $\mathcal{L}_{X'} J$. Since $J^2 = -I$, applying the Lie derivative, we have

$$(\mathcal{L}_{X'} J) \circ J + J \circ (\mathcal{L}_{X'} J) = 0, \tag{19.2}$$

that is, $\mathcal{L}_{X'} J$ anti-commutes with $J$, so using Proposition 18.1 we can we view $\mathcal{L}_{X'} J$ as a section of $\Lambda^{0,1} \otimes T^{1,0}$.

**Proposition 19.1.** *For $X \in \Gamma(T^{1,0})$,*

$$\overline{\partial} X = J \circ \mathcal{L}_{X'} J, \tag{19.3}$$

*where $X' = Re(X)$.*

*Proof.* The proof is similar to the proof of Proposition 17.4 above. For real vector fields $X'$ and $Y'$, we let

$$X = X' - iJX' = \sum X^j \frac{\partial}{\partial z^j},$$
$$Y = Y' - iJY' = \sum Y^j \frac{\partial}{\partial z^j},$$

and we have the formulas

$$X' = \frac{1}{2}(X + \overline{X}) \quad JX' = \frac{i}{2}(X - \overline{X})$$
$$Y' = \frac{1}{2}(Y + \overline{Y}) \quad JY' = \frac{i}{2}(Y - \overline{Y})$$

Expanding the Lie derivative,

$$(\mathcal{L}_{X'}J)(Y') = \mathcal{L}_{X'}(J(Y')) - J(\mathcal{L}_{X'}Y') = [X', JY'] - J[X', Y']. \tag{19.4}$$

In the proof of Proposition 17.4, it was shown that

$$J([X', Y']) = \frac{1}{4}(i[X, Y] - i[\overline{X}, \overline{Y}] + J[X, \overline{Y}] + J[\overline{X}, Y]), \tag{19.5}$$

and

$$[X', JY'] = \frac{i}{4}([X, Y] - [X, \overline{Y}] + [\overline{X}, Y] - [\overline{X}, \overline{Y}]). \tag{19.6}$$

So we have

$$\begin{aligned}
[X', JY'] - J[X', Y'] &= \frac{1}{4}(-i[X, \overline{Y}] + i[\overline{X}, Y] - J[X, \overline{Y}] - J[\overline{X}, Y]) \\
&= -\frac{1}{4}\Big(i(Z - \overline{Z}) + J(Z + \overline{Z})\Big),
\end{aligned}$$

where $Z = [X, \overline{Y}]$. We have that

$$Z = [X, \overline{Y}] = \sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j} + \sum_k X^j (\frac{\partial}{\partial z^j} \overline{Y}^k) \frac{\partial}{\partial \overline{z}^k},$$

which we write as

$$Z = \sum Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j}. \tag{19.7}$$

We next compute

$$\begin{aligned}
i(Z - \overline{Z}) + J(Z + \overline{Z}) &= i\Big(Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} - \overline{W}^j \frac{\partial}{\partial z^j}\Big) \\
&\quad + J\Big(\sum Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} + \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} + \overline{W}^j \frac{\partial}{\partial z^j}) \\
&= i\Big(Z^j \frac{\partial}{\partial z^j} + W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} - \overline{W}^j \frac{\partial}{\partial z^j}\Big) \\
&\quad + i\Big(\sum Z^j \frac{\partial}{\partial z^j} - W^j \frac{\partial}{\partial \overline{z}^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j} + \overline{W}^j \frac{\partial}{\partial z^j}) \\
&= 2i\Big(\sum Z^j \frac{\partial}{\partial z^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j}\Big).
\end{aligned}$$

We have obtained the formula

$$(\mathcal{L}_{X'}J)(Y') = \frac{-i}{2}\Big(\sum Z^j \frac{\partial}{\partial z^j} - \overline{Z}^j \frac{\partial}{\partial \overline{z}^j}\Big) = Im(Z^{1,0}), \tag{19.8}$$

where $Z^{1,0}$ is the $(1,0)$ part of $Z$, which is

$$Z^{1,0} = \sum_j \overline{Y}^k (\frac{\partial}{\partial \overline{z}^k} X^j) \frac{\partial}{\partial z^j}. \tag{19.9}$$

Next, we need to view $\overline{\partial} X$ as a real endomorphism, and from the proof of Proposition 18.1, this is

$$\begin{aligned}
(\overline{\partial} X)(Y') &= Re\big((\overline{\partial} X)(Y' + iJY')\big) \\
&= Re\Big\{\Big(\sum_j \overline{\partial} X^j \otimes \frac{\partial}{\partial z^j}\Big)(Y' + iJY')\Big\} \\
&= Re\Big\{\Big(\sum_j \overline{\partial} X^j\Big)(Y' + iJY')\frac{\partial}{\partial z^j}\Big\}.
\end{aligned}$$

But note that

$$Y' + iJY' = \overline{Y' - iJY'} = \overline{Y} = \sum_j \overline{Y}^j \frac{\partial}{\partial \overline{z}^j}. \tag{19.10}$$

So we have

$$\begin{aligned}
(\overline{\partial} X)(Y') &= Re\Big\{\Big(\sum_j \overline{\partial} X^j\Big)(\overline{Y})\frac{\partial}{\partial z^j}\Big\} \\
&= Re\Big\{\sum_{p,j} \overline{Y}^p \Big(\frac{\partial}{\partial \overline{z}^p} X^j\Big)\frac{\partial}{\partial z^j}\Big\} = Re(Z^{1,0}).
\end{aligned}$$

But since $Z^{1,0}$ is of type $(1,0)$,

$$Im(Z^{1,0}) = -J(Re(Z^{1,0}). \tag{19.11}$$

Finally, we have

$$(\overline{\partial} X)(Y') = Re(Z^{1,0}) = J(Im(Z^{1,0})) = J((\mathcal{L}_{X'}J)(Y')), \tag{19.12}$$

and we are done. $\square$

Letting $\Theta$ denote $T^{1,0}$, there is moreover an entire complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,3} \otimes \Theta) \xrightarrow{\overline{\partial}} \cdots . \tag{19.13}$$

We have that the holomorphic vector fields (equivalently, the automorphisms of the complex structure) are $H^0(M, \Theta)$. The higher cohomology groups $H^1(M, \Theta)$ and $H^2(M, \Theta)$ of this complex play a central role in the theory of deformations of complex structures.

## 19.1 The space of almost complex structures

We define

$$\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\} \tag{19.14}$$

In this subsection, we give some alternative descriptions of this space.

**Proposition 19.2.** *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.*

*Proof.* We note that $GL(2n, \mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n, \mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{19.15}$$

Obviously,

$$(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I, \tag{19.16}$$

and

$$\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J), \tag{19.17}$$

so is indeed a group action. Given $J$ and $J'$, there exists bases

$$\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \ \text{ and } \{e'_1, \ldots, e'_n, J'e'_1, \ldots, J'e'_n\}. \tag{19.18}$$

Define $S \in GL(2n, \mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of $J_0$ is

$$Stab(J_0) = \{A \in GL(2n, \mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{19.19}$$

that is, $A$ commutes with $J_0$. We have seen above in (15.15) that this can be identified with $GL(n, \mathbb{C})$. $\square$

We next give yet another description of this space. Define

$$\begin{aligned}\mathcal{C}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n,\\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.\end{aligned}$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

**Proposition 19.3.** *The space $\mathcal{C}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{19.20}$$

*where*

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-}eigenspace\ of\ J. \tag{19.21}$$

*This an $n$-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.*

*For the converse, given $P \in \mathcal{C}(\mathbb{R}^{2n})$, then $P$ may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is*

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{19.22}$$

*with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{19.23}$$

*Proof.* For the forward direction, we already know this. To see the other direction, consider the projection map $Re$ restricted to $P$

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{19.24}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, $\pi$ has an inverse, which we denote by $J$. Clearly then, (19.22) is satisfied. Since $P$ is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in $P$, so

$$(-JX') + iX' = X'' + iJX'', \tag{19.25}$$

for some real $X''$, which yields the two equations

$$JX' = -X'' \tag{19.26}$$

$$X' = JX''. \tag{19.27}$$

applying $J$ to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{19.28}$$

Since this is true for any $X'$, we have $J^2 = -I_{2n}$. $\square$

**Remark 19.1.** We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$.

**Remark 19.2.** The above propositions embed $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense.

# 20 Lecture 20

## 20.1 Deformations of complex structure

We next let $J(t)$ be a path of complex structures through $J = J(0)$. Such a $J(t)$ is equivalent to a decomposition

$$TM \otimes \mathbb{C} = T^{1,0}(J_t) \oplus T^{0,1}(J_t). \tag{20.1}$$

Note that, for $t$ sufficiently small, this determines an element $\phi(t) \in \Lambda^{0,1}(J) \oplus T^{1,0}(J)$ which we view as a mapping

$$\phi(t) : T^{0,1}(J) \to T^{1,0}(J), \tag{20.2}$$

by writing

$$T^{0,1}(J_t) = \{v + \phi(t)v \mid v \in T^{0,1}(J_0)\}. \tag{20.3}$$

That is, we write $T^{0,1}(J_t)$ as a graph over $T^{0,1}(J_0)$. Conversely, a path $\phi(t)$ in (20.2), corresponds to a path $J(t)$ of almost complex structures. We next show how to write this down. The path $\phi(t)$ corresponds to a path of endomorphisms $I(t)$. These determine a path of almost complex structure $J(t)$ by the following.

**Proposition 20.1.** *Let $J_0$ be a fixed complex structure, and $J$ be another complex structure. Then $J$ has a unique decomposition*

$$J = J^C + J^A, \tag{20.4}$$

*where $J^C J_0 = J_0 J^C$ and $J^A J_0 = -J_0 J^A$. Furthermore, we have the formula*

$$J^C J^A + J^A J^C = 0. \tag{20.5}$$

*Proof.* Given $J$, we define

$$J^C = \frac{1}{2}(J - J_0 J J_0) \tag{20.6}$$

$$J^A = \frac{1}{2}(J + J_0 J J_0). \tag{20.7}$$

Then

$$J^C J_0 = \frac{1}{2}(J J_0 - J_0 J J_0^2) = \frac{1}{2}(J J_0 + J_0 J),$$

and

$$J_0 J^C = \frac{1}{2}(J_0 J - J_0^2 J J_0) = \frac{1}{2}(J_0 J + J J_0).$$

Next,

$$J^A J_0 = \frac{1}{2}(J J_0 + J_0 J J_0^2) = \frac{1}{2}(J J_0 - J_0 J),$$

and

$$J_0 J^A = \frac{1}{2}(J_0 J + J_0^2 J J_0) = \frac{1}{2}(J_0 J - J J_0).$$

To prove uniqueness, if

$$J = J_1^C + J_1^A = J_2^C + J_2^A, \tag{20.8}$$

then

$$J_1^C - J_2^C = J_2^A - J_1^A. \tag{20.9}$$

Denote by $\tilde{J} = J_1^C - J_2^C = J_2^A - J_1^A$. Then $\tilde{J}$ both commutes and anti commutes with $J$, so is then easily seen to vanish identically.

To prove the formula (20.5), we compute

$$\begin{aligned} J^C J^A &= \frac{1}{4}(J - J_0 J J_0)(J + J_0 J J_0) \\ &= \frac{1}{4}(J^2 + J J_0 J J_0 - J_0 J J_0 J - J_0 J J_0 J_0 J J_0) \\ &= \frac{1}{4}(-I_n + J J_0 J J_0 - J_0 J J_0 J + I_n) \\ &= \frac{1}{4}(J J_0 J J_0 - J_0 J J_0 J). \end{aligned}$$

Next,

$$\begin{aligned} J^C J^A &= \frac{1}{4}(J + J_0 J J_0)(J - J_0 J J_0) \\ &= \frac{1}{4}(J^2 - J J_0 J J_0 + J_0 J J_0 J - J_0 J J_0 J_0 J J_0) \\ &= \frac{1}{4}(-I_n - J J_0 J J_0 + J_0 J J_0 J + I_n) \\ &= \frac{1}{4}(-J J_0 J J_0 + J_0 J J_0 J). \end{aligned}$$

□

**Corollary 20.1.** *Let $I(t)$ be a path of endomorphisms satisfying $I(t)J = -JI(t)$. Then for sufficiently small $t$, $I(t)$ determines a unique almost complex structure $J(t)$ satisfying $J(t)^A = I(t)$.*

*Proof.* We would like to find $J^C(t)$ such that $J(t) = J^C(t) + I(t)$ defines an almost complex structure. In order to do this, we square this equation

$$J(t)^2 = (J^C(t) + I(t))^2 = J^C(t)^2 + J^C(t)I(t) + I(t)J^C(t) + I(t)^2 = -I_n. \tag{20.10}$$

We rewrite this as

$$(J^C(t)^2 + I(t)^2 + I_n) + (J^C(t)I(t) + I(t)J^C(t)) = 0 \tag{20.11}$$

It is easy to see that the first term in parenthesis commutes with $J$ and the second term anti-commutes with $J$. Consequently, both terms are zero. So we have the equation

$$J^C(t)^2 = -I(t)^2 - I_n, \tag{20.12}$$

Conversely, given $I(t)$, if we can find a solution of (20.12) satisfying $J^C(t)J = JJ^C(t)$, then $J(t) = J^C(t)+I(t)$ is an almost complex structure (by Proposition 20.1). Let $\mathcal{J}^C$ denote the space of endomorphisms commuting with $J$. Then then map $F : \mathcal{J}^C \to \mathcal{J}^C$ defined by $J^C \mapsto (J^C)^2$ has surjective differential at $J$. Thus for $t$ sufficiently small there is a a unique solution of (20.12). Note that from the uniqueness in Proposition 20.1, this $J^C$ is unique once $I(t)$ is specified (the other apparent solution $-J^C(t)$ corresponds to $-I(t)$). $\square$

We next return to the Nijenhuis tensor, which we recall is defined by

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\}. \tag{20.13}$$

**Proposition 20.2.** *For any almost complex structure, the Nijenhuis tensor is a section of $\Lambda^{0,2} \otimes T^{1,0}$ as follows. Let $X$ and $Y$ be in $T^{0,1}$, and write $X = X' + iJX'$ and $Y = Y' + iJY'$ for real vectors $X'$ and $Y'$. Then*

$$\Pi_{T^{1,0}}[X,Y] = -\frac{1}{4}(N(X',Y') - iJN(X',Y')). \tag{20.14}$$

*Proof.* To see this, we compute

$$[X,Y] = [X' + iJX', Y' + iJY'] = [X',Y'] - [JX',JY'] + i([X',JY'] + [JX',Y']). \tag{20.15}$$

Notice that $\Pi_{T^{1,0}}(Z) = \frac{1}{2}(Z - iJZ)$, so

$$\begin{aligned}
\Pi_{T^{1,0}}([X,Y]) =& \frac{1}{2}([X',Y'] - [JX',JY'] + i([X',JY'] + [JX',Y'])\\
&- \frac{1}{2}(i(J[X',Y'] - J[JX',JY']) - J[X',JY'] - J[JX',Y'])\\
&= \frac{1}{2}([X',Y'] - [JX',JY'] + J[X',JY'] + J[JX',Y'])\\
&+ \frac{1}{2}iJ([X',Y'] - [JX',JY'] + J[X',JY'] + J[JX',Y'])\\
&= -\frac{1}{4}(N(X',Y') - iJN(X',Y')).
\end{aligned}$$

$\square$

Next, we write the integrability condition for a path of almost complex structures $J(t) = J^C(t) + I(t)$ with corresponding $\phi(t) \in \Lambda^{0,1} \otimes T^{1,0}$.

**Proposition 20.3.** *The complex structure $J(t) = J^C(t) + I(t)$ is integrable if and only if*

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0, \tag{20.16}$$

*where $[\phi(t), \phi(t)] \in \Lambda^{0,2} \otimes T^{1,0}$ is a term which is quadratic in the $\phi(t)$ and its first derivatives, that is,*

$$\|[\phi(t), \phi(t)]\| \leq \|\phi\| \cdot \|\nabla\phi\|, \tag{20.17}$$

*in any local coordinate system.*

*Proof.* By Proposition 20.2, the integrability equation is equivalent to $[T_t^{0,1}, T_t^{0,1}] \subset T_t^{0,1}$. Writing

$$\phi = \sum \phi_{ij} d\overline{z}_i \otimes \frac{\partial}{\partial z^j}, \tag{20.18}$$

if $J(t)$ is integrable, then we must have

$$\Big[\frac{\partial}{\partial \overline{z}^i} + \phi\Big(\frac{\partial}{\partial \overline{z}^i}\Big), \frac{\partial}{\partial \overline{z}^k} + \phi\Big(\frac{\partial}{\partial \overline{z}^k}\Big)\Big] \in T_t^{0,1}. \tag{20.19}$$

This yields

$$\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] \in T_t^{0,1} \tag{20.20}$$

The first two terms are

$$\begin{aligned}\Big[\frac{\partial}{\partial \overline{z}^i}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] + \Big[\phi_{ij}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}\Big] &= \sum_j \Big(\frac{\partial \phi_{kj}}{\partial \overline{z}^i} - \frac{\partial \phi_{ij}}{\partial \overline{z}^k}\Big)\frac{\partial}{\partial z^j}\\ &= (\overline{\partial}\phi)\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial z^j}\Big).\end{aligned}$$

The third term is

$$\begin{aligned}\Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big] &= \phi_{ij}\Big(\frac{\partial}{\partial z^j}\phi_{kl}\Big)\frac{\partial}{\partial z^l} - \phi_{kl}\Big(\frac{\partial}{\partial z^l}\phi_{ij}\Big)\frac{\partial}{\partial z^j}\\ &= [\phi, \phi]\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\Big),\end{aligned}$$

where $[\phi, \phi]$ is defined by

$$[\phi, \phi] = \sum (d\overline{z}^i \wedge d\overline{z}^k)\Big[\phi_{ij}\frac{\partial}{\partial z^j}, \phi_{kl}\frac{\partial}{\partial z^l}\Big], \tag{20.21}$$

and is easily seen to be a well-defined global section of $\Lambda^{0,2} \otimes T^{1,0}$. We have shown that

$$(\overline{\partial}\phi(t) + [\phi(t), \phi(t)])\Big(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^k}\Big) \in T_t^{0,1}. \tag{20.22}$$

But the left hand side is also in $T^{1,0}$. For sufficiently small $t$ however, $T_t^{0,1} \cap T^{1,0} = \{0\}$, and therefore (20.16) holds.

For the converse, if (20.16) is satisfied, then the above argument in reverse shows that the integrability of $T_t^{0,1}$ holds as a distribution, which by Proposition 20.2 is equivalent to integrability of the complex structure $J(t)$. □

Using the above we can identify the $\overline{\partial}$ in the second term of the complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,3} \otimes \Theta) \xrightarrow{\overline{\partial}} \cdots \tag{20.23}$$

with the linearized Nijenhuis tensor at $t = 0$:

**Proposition 20.4.** *Let $J(t)$ be a path of almost complex structures with $J'(0) = I$, corresponding to $\phi \in \Lambda^{0,1} \otimes T^{1,0}$. Then*

$$\overline{\partial}\phi = -\frac{1}{4}(N_J'(I) - iJN_J'(I)). \tag{20.24}$$

*Note we are using the following identification: since the Nijenhuis tensor is $J$ anti-invariant, and skew-symmetric, it is a skew-hermitian 2-form, so by Proposition 16.1, $N + iJN$ is a section of $\Lambda^{0,2} \otimes T^{1,0}$.*

*Proof.* This follows from the above, using the fact that the quadratic term $[\phi, \phi]$ does not contribute to the linearization. □

# 21 Lecture 21

## 21.1 The Kuranishi map

We now have the following theorem.

**Theorem 21.1.** *Let $(M, J)$ be a complex manifold. The space $H^1(M, \Theta)$ is identified with*

$$H^1(M, \Theta) \simeq \frac{Ker(N_J)'}{Im(X \to \mathcal{L}_X J)}, \tag{21.1}$$

*and therefore consists of essential infinitesimal deformations of the complex structure.*

*Furthermore, there is a map*

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta) \tag{21.2}$$

*called the* Kuranishi map *such that the moduli space of complex structures near $J$ is given by the orbit space*

$$\Psi^{-1}(0)/H^0(M, \Theta). \tag{21.3}$$

*Proof.* The identification (21.1) follows from the computations in the previous lecture. The remaining part takes a lot of machinery, so we will only give an outline here.

We consider the three term complex

$$\Gamma(\Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,1} \otimes \Theta) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{0,2} \otimes \Theta), \tag{21.4}$$

and we will abbreviate this as

$$\Gamma(A) \xrightarrow{\overline{\partial}_A} \Gamma(B) \xrightarrow{\overline{\partial}_B} \Gamma(C) \xrightarrow{\overline{\partial}_C} \cdots \tag{21.5}$$

It is not hard to show that this complex is elliptic. We define a map

$$F : \Gamma(B) \to \Gamma(C) \oplus \Gamma(A) \tag{21.6}$$

by

$$F(\phi) = (\Pi_{\Gamma(C)} N_{J_\phi}, \overline{\partial}_A^* \phi). \tag{21.7}$$

where we have fixed a hermitian metric $g$ compatible with $J$, and the adjoint is taken with respect to $g$.

**Claim 21.1.** *For $\phi$ sufficiently small, zeroes of $F$ correspond to integrable complex structures near $\phi$, modulo diffeomorphism.*

For the forward direction, if $\Pi_{\Gamma(C)(J)} N_{J_\phi} = 0$, then $N_{J_\phi} = 0$ if $\phi$ is sufficiently small. For the converse, we have that given any $J_\phi$ near $J$, there exists a diffeomorphism $f : M \to M$ such that $f^* J_\phi = J_{\phi'}$ with $\overline{\partial}_A^* \phi' = 0$. This follows since $\overline{\partial}_A^*$ is the divergence operator with respect to $g$, and then this follows from a version of the Ebin slice theorem. This finishes the claim.

Next, the linearization of $F$ at $\phi = 0$, defined by

$$P(h) = \frac{d}{dt} F(\phi(t)) \Big|_{t=0}, \tag{21.8}$$

where $\phi(t)$ is any path satisfying $\phi(0) = 0$, and $\phi'(0) = h$, is given by

$$P(h) = (\overline{\partial}_B(h), \overline{\partial}_A^*(h)). \tag{21.9}$$

This is an elliptic operator, since the above complex is elliptic. We also know that

$$N_{J_\phi} = \overline{\partial}\phi + [\phi, \phi], \tag{21.10}$$

and the nonlinear term satisfies

$$\|[\phi_1, \phi_1] - [\phi_2, \phi_2]\| \leq C(\|\phi_1\| + \|\phi_2\|) \cdot \|\phi_1 - \phi_2\|. \tag{21.11}$$

Consequently, one can use elliptic theory and this estimate on the nonlinear term together with an infinite-dimensional fixed point theorem to show that the zero set of $F$ is equivalent to the zero set of a map

$$\Psi : Ker(P) \to Coker(P) = Ker(P^*), \tag{21.12}$$

defined between finite-dimensional spaces. Since $M$ is compact, basic Hodge theory shows that

$$Ker(P) \simeq Ker(\overline{\partial}_B) \cap Ker(\overline{\partial}_A^*) \simeq \frac{Ker(\overline{\partial}_B)}{Im(\overline{\partial}_A)} \simeq H^1(M, \Theta), \tag{21.13}$$

and

$$Coker(P) \simeq Ker(\overline{\partial}_B^*) \oplus Ker((\overline{\partial}_A) \simeq \frac{Ker(\overline{\partial}_C)}{Im(\overline{\partial}_B)} \oplus H^0(M, \Theta) \tag{21.14}$$

$$\simeq H^2(M, \Theta) \oplus H^0(M, \Theta). \tag{21.15}$$

So we have

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta) \oplus H^0(M, \Theta) \tag{21.16}$$

Finally, the map $\Psi$ is equivariant with respect to the holomorphic automorphsim group $H^0(M, \Theta)$, so we only need to consider $\Psi$ as a mapping from

$$\Psi : H^1(M, \Theta) \to H^2(M, \Theta), \tag{21.17}$$

and we then obtain the actual moduli space as the orbit space of the action of $H^0(M, \Theta)$ on $\Psi^{-1}(0)$. $\square$

**Corollary 21.1.** *If $H^2(M, \Theta) = 0$, then any such infinitesimal deformation $I$ is* integrable*, that is, $I = J'(0)$ for an actual path of complex structures $J(t)$. If both $H^2(M, \Theta) = 0$ and $H^0(M, \Theta) = 0$ then the moduli space of complex structures near $J$ is smooth of dimension $H^1(M, \Theta)$.*

# 22 Lecture 22

## 22.1 Conformal geometry

Let $u : M \to \mathbb{R}$. Then $\tilde{g} = e^{-2u}g$, is said to be *conformal* to $g$.

**Proposition 22.1.** *The Christoffel symbols transform as*

$$\tilde{\Gamma}^i_{jk} = g^{il}\Big( - (\partial_j u)g_{lk} - (\partial_k u)g_{lj} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}. \tag{22.1}$$

*Invariantly,*

$$\tilde{\nabla}_X Y = \nabla_X Y - du(X)Y - du(Y)X + g(X,Y)\nabla u. \tag{22.2}$$

*Proof.* Using (1.33), we compute

$$\begin{aligned}
\tilde{\Gamma}^i_{jk} &= \frac{1}{2}\tilde{g}^{il}\Big(\partial_j \tilde{g}_{kl} + \partial_k \tilde{g}_{jl} - \partial_l \tilde{g}_{jk}\Big)\\
&= \frac{1}{2}e^{2u}g^{il}\Big(\partial_j(e^{-2u}g_{kl}) + \partial_k(e^{-2u}g_{jl}) - \partial_l(e^{-2u}g_{jk})\Big)\\
&= \frac{1}{2}e^{2u}g^{il}\Big( - 2e^{-2u}(\partial_j u)g_{kl} - 2e^{-2u}(\partial_k u)e^{-2u}g_{jl} + 2e^{-2u}(\partial_l u)g_{jk}\\
&\qquad\qquad + e^{-2u}\partial_j(g_{kl}) + e^{-2u}\partial_k(g_{jl}) - e^{-2u}\partial_l(g_{jk})\Big)\\
&= g^{il}\Big( - (\partial_j u)g_{kl} - (\partial_k u)g_{jl} + (\partial_l u)g_{jk}\Big) + \Gamma^i_{jk}.
\end{aligned} \tag{22.3}$$

This is easily seen to be equivalent to the invariant expression. ∎

**Proposition 22.2.** *The $(0,4)$-curvature tensor transforms as*

$$\tilde{Rm} = e^{-2u}\Big[Rm + \Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g\Big) \owedge g\Big]. \tag{22.4}$$

*Proof.* Recall the formula (1.54) for the $(1,3)$ curvature tensor

$$\tilde{R}_{ijk}{}^{l} = \partial_i(\tilde{\Gamma}^l_{jk}) - \partial_j(\tilde{\Gamma}^l_{ik}) + \tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik}. \tag{22.5}$$

Take a normal coordinate system for the metric $g$ at a point $x \in M$. All computations below will be evaluated at $x$. Let us first consider the terms with derivatives of

Christoffel symbols, we have

$$\begin{aligned}
&\partial_i(\tilde\Gamma^l_{jk}) - \partial_j(\tilde\Gamma^l_{ik}) = \partial_i\Big[g^{lp}\Big(-(\partial_j u)g_{pk} - (\partial_k u)g_{pj} + (\partial_p u)g_{jk}\Big) + \Gamma^l_{jk}\Big]\\
&-\partial_j\Big[g^{lp}\Big(-(\partial_i u)g_{kp} - (\partial_k u)g_{ip} + (\partial_p u)g_{ik}\Big) + \Gamma^l_{ik}\Big]\\
&= g^{lp}\Big(-(\partial_i\partial_j u)g_{pk} - (\partial_i\partial_k u)g_{pj} + (\partial_i\partial_p u)g_{jk}\Big) + \partial_i(\Gamma^l_{jk})\\
&-g^{lp}\Big(-(\partial_j\partial_i u)g_{kp} - (\partial_j\partial_k u)g_{ip} + (\partial_j\partial_p u)g_{ik}\Big) - \partial_j(\Gamma^l_{ik})\\
&= g^{lp}\Big(-(\partial_i\partial_k u)g_{pj} + (\partial_i\partial_p u)g_{jk} + (\partial_j\partial_k u)g_{ip} - (\partial_j\partial_p u)g_{ik}\Big) + R_{ijk}{}^l.
\end{aligned} \tag{22.6}$$

A simple computation shows this is

$$\partial_i(\tilde\Gamma^l_{jk}) - \partial_j(\tilde\Gamma^l_{ik}) = g^{lp}(\nabla^2 u \owedge g)_{ijpk} + R_{ijk}{}^l. \tag{22.7}$$

Next, we consider the terms that are quadratic Christoffel terms.

$$\begin{aligned}
\tilde\Gamma^l_{im}\tilde\Gamma^m_{jk} - \tilde\Gamma^l_{jm}\tilde\Gamma^m_{ik} = g^{lp}&\Big(-(\partial_i u)g_{mp} - (\partial_m u)g_{ip} + (\partial_p u)g_{im}\Big)\\
\times g^{mr}&\Big(-(\partial_j u)g_{kr} - (\partial_k u)g_{jr} + (\partial_r u)g_{jk}\Big)\\
-g^{lp}&\Big(-(\partial_j u)g_{mp} - (\partial_m u)g_{jp} + (\partial_p u)g_{jm}\Big)\\
\times g^{mr}&\Big(-(\partial_i u)g_{kr} - (\partial_k u)g_{ir} + (\partial_r u)g_{ik}\Big).
\end{aligned} \tag{22.8}$$

Terms in the first product which are symmetric in $i$ and $j$ will cancel with the corresponding terms of the second product, so this simplifies to

$$\begin{aligned}
&\tilde\Gamma^l_{im}\tilde\Gamma^m_{jk} - \tilde\Gamma^l_{jm}\tilde\Gamma^m_{ik}\\
&= g^{lp}g^{mr}\Big((\partial_i u)g_{mp}(\partial_j u)g_{kr} + (\partial_m u)g_{ip}(\partial_k u)g_{jr} + (\partial_p u)g_{im}(\partial_r u)g_{jk}\\
&\quad +(\partial_i u)g_{mp}(\partial_k u)g_{jr} - (\partial_i u)g_{mp}(\partial_r u)g_{jk} + (\partial_m u)g_{ip}(\partial_j u)g_{kr}\\
&\quad -(\partial_m u)g_{ip}(\partial_r u)g_{jk} - (\partial_p u)g_{im}(\partial_j u)g_{kr} - (\partial_p u)g_{im}(\partial_k u)g_{jr}\\
&\quad - \text{same 9 terms with i and j exchanged}\Big)\\
&= g^{lp}\Big((\partial_i u)(\partial_j u)g_{kp} + (\partial_j u)(\partial_k u)g_{ip} + (\partial_p u)(\partial_i u)g_{jk}\\
&\quad +(\partial_i u)(\partial_k u)g_{jp} - (\partial_i u)(\partial_p u)g_{jk} + (\partial_k u)(\partial_j u)g_{ip}\\
&\quad -g^{mr}(\partial_m u)(\partial_r u)g_{ip}g_{jk} - (\partial_p u)(\partial_j u)g_{ik} - (\partial_p u)(\partial_k u)g_{ij}\\
&\quad - \text{same 9 terms with i and j exchanged}\Big)
\end{aligned} \tag{22.9}$$

The first and ninth terms are symmetric in $i$ and $j$. The fourth and sixth terms, taken together, are symmetric in $i$ and $j$. The third and fifth terms cancel, so we have

$$\begin{aligned}
\tilde\Gamma^l_{im}\tilde\Gamma^m_{jk} - \tilde\Gamma^l_{jm}\tilde\Gamma^m_{ik} = g^{lp}&\Big((\partial_j u)(\partial_k u)g_{ip} - (\partial_p u)(\partial_j u)g_{ik} - |\nabla u|^2 g_{ip}g_{jk}\\
&- \text{same 3 terms with i and j exchanged}\Big).
\end{aligned} \tag{22.10}$$

Writing out the last term, we have

$$\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big((\partial_j u)(\partial_k u)g_{ip} - (\partial_i u)(\partial_k u)g_{jp} - (\partial_p u)(\partial_j u)g_{ik} + (\partial_p u)(\partial_i u)g_{jk} - |\nabla u|^2 g_{ip}g_{jk} + |\nabla u|^2 g_{jp}g_{ik}\Big). \tag{22.11}$$

Another simple computation shows this is

$$\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jk} - \tilde{\Gamma}^l_{jm}\tilde{\Gamma}^m_{ik} = g^{lp}\Big[\Big(du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk}. \tag{22.12}$$

Adding together (22.7) and (22.12), we have

$$\tilde{R}_{ijk}{}^l = g^{lp}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} + R_{ijk}{}^l. \tag{22.13}$$

We lower the the index on the right using the metric $\tilde{g}_{lp}$, to obtain

$$\tilde{R}_{ijpk} = e^{-2u}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} = e^{-2u}R_{ijpk}, \tag{22.14}$$

and we are done. □

**Proposition 22.3.** *Let $\tilde{g} = e^{-2u}g$. The $(1,3)$ Weyl tensor is conformally invariant. The $(0,4)$ Weyl tensor transforms as*

$$\tilde{W}_{ijkl} = e^{-2u}W_{ijkl}. \tag{22.15}$$

*The Schouten $(0,2)$ tensor transforms as*

$$\tilde{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g + A. \tag{22.16}$$

*The Ricci $(0,2)$ tensor transforms as*

$$\tilde{Ric} = (n-2)\Big(\nabla^2 u + \frac{1}{n-2}(\Delta u)g + du \otimes du - |\nabla u|^2 g\Big) + Ric. \tag{22.17}$$

*The scalar curvature transforms as*

$$\tilde{R} = e^{2u}\Big(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R\Big). \tag{22.18}$$

*Proof.* We expand (22.13) in terms of Weyl,

$$\tilde{W}_{ijk}{}^l + (\tilde{A} \owedge \tilde{g})_{ijk}{}^l = g^{lp}\Big[\Big(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2\Big) \owedge g\Big]_{ijpk} + W_{ijk}{}^l + (A \owedge g)_{ijk}{}^l. \tag{22.19}$$

Note that

$$
\begin{aligned}
(\tilde{A} \owedge \tilde{g})_{ijk}{}^{l} &= \tilde{g}^{lp}(\tilde{A} \owedge e^{-2u}g)_{ijpk} \\
&= g^{lp}(\tilde{A} \owedge g)_{ijpk}.
\end{aligned}
\tag{22.20}
$$

We can therefore rewrite (22.19) as

$$
\tilde{W}_{ijk}{}^{l} - W_{ijk}{}^{l} = g^{lp}\Big[\Big( -\tilde{A} + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 + A\Big) \owedge g\Big]_{ijpk}. \tag{22.21}
$$

In dimension 2 and 3 the right hand side is zero, so the left hand side is also. In any dimension, recall from Section 7.1, that the left hand side is in $Ker(c)$, and the right hand side is in $Im(\psi)$ (with respect to either $g$ or $\tilde{g}$). This implies that both sides must vanish. To see this, assume $R \in Ker(c) \cap Im(\psi)$. Then $R = h \owedge g$, so $c(R) = (n-2)h + tr(h)g = 0$, which implies that $h = 0$ for $n \neq 2$. This implies conformal invariance of $Weyl$, and also the formula for the conformal transformation of the Schouten tensor. We lower an index of the Weyl,

$$
\tilde{W}_{ijkl} = \tilde{g}_{pk}\tilde{W}_{ijl}{}^{p} = e^{-2u}g_{pk}W_{ijl}{}^{p} = e^{-2u}W_{ijkl}, \tag{22.22}
$$

which proves (22.15). We have the formula

$$
\Big( -\tilde{A} + \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 + A\Big) \owedge g = 0. \tag{22.23}
$$

Recall that $c(A \owedge g) = (n-2)A + tr(A)g = Ric$, so we obtain

$$
-\tilde{Ric} + (n-2)(\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2) + (\Delta u)g + (1 - \frac{n}{2})|\nabla u|^2 + Ric = 0, \tag{22.24}
$$

which implies (22.17). Finally,

$$
\begin{aligned}
\tilde{R} &= \tilde{g}^{-1}\tilde{Ric} = e^{2u}g^{-1}\tilde{Ric} \\
&= (n-2)e^{2u}\Big(\Delta u + \frac{n}{n-2}\Delta u + (1-n)|\nabla u|^2 + R\Big) \\
&= e^{2u}\Big(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R\Big),
\end{aligned}
\tag{22.25}
$$

which is (22.18). $\square$

By writing the conformal factor differently, the scalar curvature equation takes a nice semilinear form, which is the famous Yamabe equation:

**Proposition 22.4.** *If $n \neq 2$, and $\tilde{g} = v^{\frac{4}{n-2}}g$, then*

$$
-4\frac{n-1}{n-2}\Delta v + Rv = \tilde{R}v^{\frac{n+2}{n-2}}. \tag{22.26}
$$

*Proof.* We have $e^{-2u} = v^{\frac{4}{n-2}}$, which is

$$u = -\frac{2}{n-2}\ln v. \tag{22.27}$$

Using the chain rule,

$$\nabla u = -\frac{2}{n-2}\frac{\nabla v}{v}, \tag{22.28}$$

$$\nabla^2 u = -\frac{2}{n-2}\Big(\frac{\nabla^2 v}{v} - \frac{\nabla v \otimes \nabla v}{v^2}\Big). \tag{22.29}$$

Substituting these into (22.18), we obtain

$$\begin{aligned}\tilde{R} &= v^{\frac{-4}{n-2}}\Big( -4\frac{n-1}{n-2}\Big(\frac{\Delta v}{v} - \frac{|\nabla v|^2}{v^2}\Big) - 4\frac{n-1}{n-2}\frac{|\nabla v|^2}{v^2} + R\Big)\\ &= v^{\frac{-n+2}{n-2}}\Big( -4\frac{n-1}{n-2}\Delta v + Rv\Big).\end{aligned} \tag{22.30}$$

□

**Proposition 22.5.** *If $n = 2$, and $\tilde{g} = e^{-2u}g$, the conformal Gauss curvature equation is*

$$\Delta u + K = \tilde{K}e^{-2u}. \tag{22.31}$$

*Proof.* This follows from (22.18), and the fact that in dimension 2, $R = 2K$. □

## 22.2 Negative scalar curvature

**Proposition 22.6.** *If $(M, g)$ is compact, and $R < 0$, then there exists conformal metric $\tilde{g} = e^{-2u}g$ with $\tilde{R} = -1$.*

*Proof.* If $n > 2$, we would like to solve the equation

$$-4\frac{n-1}{n-2}\Delta v + Rv = -v^{\frac{n+2}{n-2}}. \tag{22.32}$$

If $n > 2$, let $p \in M$ be a point where $v$ attains a its global maximum. Then (22.26) evaluated at $p$ becomes

$$R(p)v(p) \leq -(v(p))^{\frac{n+2}{n-2}}. \tag{22.33}$$

Dividing, we obtain

$$(v(p))^{\frac{4}{n-2}} \leq -R(p), \tag{22.34}$$

which gives an *a priori* upper bound on $v$. Similarly, by evaluating a a global minimum point $q$, we obtain

$$(v(p))^{\frac{4}{n-2}} \geq -R(q), \tag{22.35}$$

which gives an a priori strictly positive lower bound on $v$. We have shown there exists a constant $C_0$ so that $\|v\|_{C^0} < C_0$. The standard elliptic estimate says that there exists a constant $C$, depending only on the background metric, such that (see [GT01, Chapter 4])

$$\begin{aligned} \|v\|_{C^{1,\alpha}} &\leq C(\|\Delta v\|_{C^0} + \|v\|_{C^0}) \\ &\leq C(\|Rv + v^{\frac{n+2}{n-2}}\|_{C^0} + CC_0 \leq C_1, \end{aligned} \tag{22.36}$$

where $C_1$ depends only upon the background metric. Applying elliptic estimates again,

$$\|v\|_{C^{3,\alpha}} \leq C(\|\Delta v\|_{C^{1,\alpha}} + \|v\|_{C^{1,\alpha}}) \leq C_3, \tag{22.37}$$

where $C_3$ depends only upon the background metric.

In terms of $u$, the equation is

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R = -e^{-2u}. \tag{22.38}$$

Let $t \in [0,1]$, and consider the family of equations

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R = \big((1-t)R - 1\big)e^{-2u}. \tag{22.39}$$

Define an operator $F_t : C^{2,\alpha} \to C^\alpha$ by

$$F_t(u) = 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R - \big((1-t)R - 1\big)e^{-2u}. \tag{22.40}$$

Let $u_t \in C^{2,\alpha}$ satisfy $F_t(u_t) = 0$. The linearized operator at $u_t$, $L_t : C^{2,\alpha} \to C^\alpha$, is given by

$$L_t(h) = 2(n-1)\Delta h - (n-1)(n-2)2\langle \nabla u, \nabla h\rangle + 2\big((1-t)R - 1\big)e^{-2u}h. \tag{22.41}$$

Notice that the coefficient $h$ is strictly negative. The maximum principle and linear theory imply that the linearized operator is invertible. Next, define

$$S = \{t \in [0,1] \mid \text{there exists a solution } u_t \in C^{2,\alpha} \text{ of } F_t(u_t) = 0\}. \tag{22.42}$$

Since the linearized operator is invertible, the implicit function theorem implies that $S$ is open. Assume $u_{t_i}$ is a sequence of solutions with $t_i \to t_0$ as $i \to \infty$. The above elliptic estimates imply there exist a constant $C_4$, independent of $t$, such that $\|u_{t_i}\|_{C^{3,\alpha}} < C_4$. By Arzela-Ascoli, there exists $u_{t_0} \in C^{2,\alpha}$ and a subsequence $\{j\} \subset \{i\}$ such that $u_{t_j} \to u_{t_0}$ strongly in $C^{2,\alpha}$. The limit $u_{t_0}$ is a solution at time $t_0$. This shows that $S$ is closed. Since the interval $[0,1]$ is connected, this implies that $S = [0,1]$, and consequently there must exist a solution at $t = 1$. In the case $n = 2$, the same arugment applied to (22.31) yields a similar a priori estimate, and the proof remains valid. $\square$

# 23 Lecture 23

## 23.1 Uniformization I

**Proposition 23.1.** *A Riemann surface $(M, J)$ is equivalent to an oriented conformal class $(M, [g])$.*

*Proof.* From elementary complex variables, a holomorphic map is equivalent to an orientation preserving conformal map, which implies the proposition. Another way to see this is as follows. A complex structure is a reduction of the structure group of the frame bundle to $GL(1, \mathbb{C}) \subset GL(2, \mathbb{R})$. The explicit map is

$$a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{23.1}$$

An oriented conformal class is a reduction to $CO(2, \mathbb{R}) \subset GL(2, \mathbb{R})$, where

$$CO(2, \mathbb{R}) = \mathbb{R}_+ \times SO(2, \mathbb{R}) = \{\lambda \cdot A | \lambda \in \mathbb{R}_+, A \in SO(2, \mathbb{R})\}, \tag{23.2}$$

and it is easy to see that this is the same as image in (23.1). □

**Theorem 23.1.** *Any compact oriented Riemann surface $M$ of genus $k \geq 1$ is a complex manifold with complex structure given by the Hodge star operator. Furthermore, there is a unique metric $\tilde{g}$ conformal to $g$ having constant curvature.*

*Proof.* First consider the case of genus $k \geq 2$. By the Gauss-Bonnet theorem

$$\int_M K_g dV_g = 2\pi\chi(M) = 2\pi(2 - 2k) < 0. \tag{23.3}$$

We first solve the equation

$$\Delta u = -K + \frac{2\pi}{Vol}(2 - 2k). \tag{23.4}$$

By Fredholm theory, this has a smooth solution since the right hand side has zero mean value. Consider the metric $\tilde{g} = e^{-2u} g$, From (22.31), the Gauss curvature of $\tilde{g}$ is given by

$$\tilde{K} = e^{-2u}(\Delta u + K) = e^{-2u}(\frac{2\pi}{Vol}(2 - 2k)) < 0, \tag{23.5}$$

since $k \geq 2$. We have found a conformal metric with strictly negative curvature, so Proposition 22.6 yields another conformal metric with constant negative curvature. The maximum principle implies this metric is the unique solution in its conformal class.

The universal cover of $M$ is isometric to hyperbolic space. We can therefore find a collection of coordinate charts such that the overlap maps are hyperbolic isometries, that is, they are in $SO(2, 1)$. These maps are orientation preserving and conformal, and therefore holomorphic. Since the Hodge star on 1-forms is conformally invariant it must be integrable.

For the case $k = 1$, the equation to be solved is

$$\Delta u = -K. \tag{23.6}$$

By the Gauss-Bonnet Theorem, the right hand side has integral zero. Elementary Fredholm Theory gives existence of a unique smooth solution. The universal cover of $M$ is isometric to $\mathbb{R}^2$ with the flat metrics. We can therefore find a collection of coordinate charts such that the overlap maps are Euclidean isometries. As above, this proves integrability. $\square$

**Remark 23.1.** Since the Nijenhuis tensor $N \in \Lambda^{0,2} \otimes T^{1,0}$, it vanishes in complex dimension 1, so any almost complex structure on a Riemann surface is integrable. However, the integrability of any smooth surface can easily be proved locally as follows. That is, given any point $p$ on any surface (compact or noncompact), then there is a locally defined function $u : U \to \mathbb{R}$ such that $\tilde{g} = e^{-2u}g$ is flat. This amounts to locally solving $\Delta u = -K$. This can easily be proved by taking the metric in a small neighborhood of any point and extending the metric to a metric on the torus using a cutoff function. The result then follows by the $k = 1$ case in the above theorem.

Notice that the above implies the Uniformization Theorem for genus $k \geq 1$:

**Corollary 23.1** (Uniformization)**.** *Any compact orientable Riemann surface of genus $k \geq 2$ has universal covering biholomorphic to the unit disc. A compact orientable Riemann surface of genus $k = 1$ has universal covering biholomorphic to $\mathbb{C}$.*

The genus $k = 0$ case is slightly more difficult, and we will do this case after some general remarks on conformal geometry.

## 23.2 Constant curvature

Let $g$ denote the Euclidean metric on $\mathbb{R}^n, n \geq 3$, and consider conformal metrics $\tilde{g} = e^{-2u}g$.

**Proposition 23.2.** *If $\tilde{g}$ is Einstein for $n \geq 3$ or constant curvature for $n = 2$, then there exists constant $a, b_i, c$, such that*

$$\tilde{g} = \left(a|x|^2 + b_i x^i + c\right)^{-2} g. \tag{23.7}$$

*Proof.* For the Schouten tensor, we must have

$$\tilde{A} = \nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2 g. \tag{23.8}$$

Let us rewrite the conformal factor as $\tilde{g} = v^{-2}g$, that is $u = \ln v$. The equation is then written

$$v^2 \tilde{A} = v\nabla^2 v - \frac{1}{2}|\nabla v|^2 g. \tag{23.9}$$

Let us assume that $\tilde{g}$ is Einstein, which is equivalent to $\tilde{g}$ having constant curvature. In this case, we have

$$\tilde{A} = \frac{tr(A)}{n}\tilde{g} = \frac{R}{2n(n-1)}v^{-2}g, \tag{23.10}$$

so we obtain

$$\frac{K}{2}g = v\nabla^2 v - \frac{1}{2}|\nabla v|^2 g, \tag{23.11}$$

where $R = n(n-1)K$. The off-diagonal equation is

$$v_{ij} = 0, \ i \neq j, \tag{23.12}$$

implies that we may write $v_i = h_i(x_i)$ for some function $h_i$. The diagonal entries say that

$$\frac{K}{2} = vv_{ii} - \frac{1}{2}|\nabla v|^2. \tag{23.13}$$

Differentiate this in the $x^j$ direction,

$$0 = v_j v_{ii} + vv_{iij} - v_l v_{lj}. \tag{23.14}$$

If $j = i$, then we obtain

$$v_{iii} = 0. \tag{23.15}$$

In terms of $h$,

$$(h_i)_{ii} = 0. \tag{23.16}$$

This implies that

$$h_i = a_i x_i + b_i, \tag{23.17}$$

for some constants $a_i, b_i$. If $j \neq i$, then (23.14) is

$$0 = v_j(v_{ii} - v_{jj}). \tag{23.18}$$

This says that $a_i = a_j$ for $i \neq j$. This forces $v$ to be of the form

$$v = a|x|^2 + b_i x^i + c. \tag{23.19}$$

$\square$

From conformal invariance of the Weyl, we know that $\tilde{W} = 0$, so $\tilde{g}$ being Einstein is equivalent to having constant sectional curvature. The sectional curvature of such a metric is

$$\begin{aligned} K &= 2vv_{ii} - |\nabla v|^2 \\ &= 2(a|x|^2 + b_i x^i + c)2a - |2ax_i + b_i|^2 \\ &= 4ac - |b|^2. \end{aligned} \tag{23.20}$$

If $K > 0$, then the discriminant is negative, so there are no real roots, and $v$ is defined on all of $\mathbb{R}^n$. The metric

$$\tilde{g} = \frac{4}{(1 + |x|^2)^2} g \tag{23.21}$$

represents the round metric with $K = 1$ on $S^n$ under stereographic projection. If $K < 0$ then the solution is defined on a ball, or the complement of a ball, or a half space. The metric

$$\tilde{g} = \frac{4}{(1 - |x|^2)^2} g \tag{23.22}$$

is the usual ball model of hyperbolic space, and

$$\tilde{g} = \frac{1}{x_n^2} g \tag{23.23}$$

is the upper half space model of hyperbolic space. If $K = 0$ and $|b| \neq 0$, the solution is defined on all of $\mathbb{R}^n$.

## 23.3 Conformal transformations

The case $K = 0$ of this proposition implies the follow theorem of Liouville.

**Theorem 23.2** (Liouville)**.** *For $n \geq 3$, then group of conformal transformations of $\mathbb{R}^n$ is generated by rotations, scalings, translations, and inversions.*

*Proof.* Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a conformal transformation. Then $T^* g = v^{-2} g$ for some positive function $v$, which says $v$ is a flat metric which is conformal to the Euclidean metric. By above, we must have $v = a|x|^2 + b_i x^i + c$, with $|b|^2 = 4ac$. If $a = 0$, then $v = c$, so $T$ is a scaling composed with an isometry. If $a \neq 0$, then

$$v = \frac{1}{a} \sum_i (a x_i + \frac{1}{2} b_i)^2. \tag{23.24}$$

From this it follows that $T$ is a scaling and inversion composed with an isometry. □

We note the following fact: the group of conformal transformations of the round $S^n$ is isomorphic to the group of isometric of hyperbolic space $H^{n+1}$. This is proved by showing that in the ball model of hyperbolic space, isometries of $H^{n+1}$ restrict to conformal automorphisms of the boundary $n$-sphere. By identifying $H^{n+1}$ with a component of the unit sphere in $\mathbb{R}^{n,1}$, one shows that $Iso(H^n) = O(n, 1)$. We have some special isomorphisms in low dimensions. For $n = 1$,

$$\begin{aligned} SO(2,1) &= PSL(2, \mathbb{R}), \\ SO(3,1) &= PSL(2, \mathbb{C}) \\ SO(5,1) &= PSL(2, \mathbb{H}). \end{aligned} \tag{23.25}$$

For the first case,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{R}) \tag{23.26}$$

acts upon $H^2$ in the upper half space model by fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}, \tag{23.27}$$

where $z$ satisfies $Im(z) > 0$. The boundary of $H^2$ is $S^1$, which is identified with 1-dimensional real projective space $\mathbb{RP}^1$. The conformal transformations of $S^1$ are

$$[r_1, r_2] \mapsto [ar_1 + br_2, cr_1 + dr_2]. \tag{23.28}$$

It is left as an exercise to find explicit maps from the groups on the right to the isometries of hyperbolic space, and conformal transformations of the sphere in the other two cases.

## 23.4 Uniformization on $S^2$

Since the conformal group of $(S^2, g_S)$, where $g_S$ is the round metric, is noncompact, we cannot hope to prove existence of a constant curvature metric by a compactness argument as in the $k \geq 1$ case. However, there is a trick to solve this case using only linear theory.

**Theorem 23.3.** *If $(M, g)$ is a Riemann surface of genus 0, then $g$ is conformal to $(S^2, g_S)$.*

*Proof.* We remove a point $p$ from $M$, and consider the manifold $(M \setminus \{p\}, g)$. We want to find a conformal factor $u : M \setminus \{p\} \to \mathbb{R}$ such that $\tilde{g} = e^{-2u} g$ is flat. The equation for this is

$$\Delta u = -K. \tag{23.29}$$

However, by the Gauss-Bonnet theorem, the right hand side has integral $4\pi$, so this equation has no smooth solution. But we will find a solution $u$ on $M \setminus \{p\}$ so that $u = O(\log(r))$ and $r \to 0$, where $r(x) = d(p, x)$. Let $\phi$ be a smooth cutoff function satisfying

$$\phi = \begin{cases} 1 & r \leq r_0 \\ 0 & r \geq 2r_0 \end{cases}, \tag{23.30}$$

and $0 \leq \phi \leq 1$, for $r_0$ very small. Consider the function $f = \Delta(\phi \log(r))$. Computing in normal coordinates, near $p$ we have

$$\begin{aligned} \Delta f &= \frac{1}{\sqrt{\det(g)}} \partial_i (g^{ij} u_j \sqrt{\det(g)}) = \frac{1}{\sqrt{\det(g)}} \partial_r (u_r \sqrt{\det(g)}) \\ &= (\log(r))'' + (\log(r))' \frac{(\sqrt{\det(g)})'}{\sqrt{\det(g)}}. \end{aligned}$$

But from Theorem 5.1 above (converting to radial coordinates), $\sqrt{\det(g)} = r + O(r^3)$ as $r \to 0$, so we have

$$\Delta f = -\frac{1}{r^2} + \frac{1}{r}\Big(\frac{1+O(r^2)}{r+O(r^3)}\Big) = -\frac{1}{r^2} + \frac{1}{r^2}\Big(\frac{1+O(r^2)}{1+O(r^2)}\Big) = O(1) \tag{23.31}$$

as $r \to 0$.

Next, we compute

$$\int_M f dV = \lim_{\epsilon\to 0}\int_{M\setminus B(p,r)} \Delta(\phi\log(r))dV = -\lim_{\epsilon\to 0}\int_{S(p,r)} \partial_r(\log(r))d\sigma = -2\pi.$$

Note the minus sign is due to using the *outward* normal of the domain $M \setminus B(p,r)$. Consequently, we can solve the equation

$$\Delta(u) = -2\Delta(\phi\log(r)) - K, \tag{23.32}$$

by the Gauss-Bonnet Theorem and Fredholm Theory in $L^2$. Rewriting this as

$$\Delta\tilde{u} = \Delta(u + 2\phi\log(r)) = -K. \tag{23.33}$$

The space $(M\setminus\{p\}, e^{-2\tilde{u}}g)$ is therefore isometric to Euclidean space, since it is clearly complete and simply connected. By the above, we can write

$$g_S = \frac{4}{(1+|x|^2)^2}e^{-2\tilde{u}}g = e^{-2v}g. \tag{23.34}$$

It is easy to see that $v$ is a bounded solution of

$$\Delta v + K = e^{-2v} \tag{23.35}$$

on $M \setminus \{p\}$ and extends to a smooth solution on all of $M$ by elliptic regularity. □

**Corollary 23.2.** *If $(M, J)$ is a Riemann surface homeomorphic to $S^2$ then it is biholomorphic to the Riemann sphere $(S^2, J_S)$.*

# 24 Lecture 24

## 24.1 Moduli

The Riemann-Roch Theorem for a Riemann surface $(M, J)$ and holomorphic line bundle $\mathcal{E}$ says that

$$dim(H^0(M,\mathcal{E})) - dim(H^1(M,\mathcal{E})) = d + 1 - k, \tag{24.1}$$

where $d$ is the the degree of the line bundle, and $k$ is the genus of $M$. Note the degree is given by counting the zeroes and poles of any meromorphic section.

We apply this to $\mathcal{E} = \Theta$, the holomorphic tangent bundle. The degree of $\Theta$ is $2 - 2g$ which is the Euler characteristic. Note by Serre duality, we have

$$H^1(M, \Theta) = H^0(M, \Theta^* \otimes \Theta^*), \tag{24.2}$$

so the Riemann-Roch formula becomes

$$dim(H^0(M, \Theta)) - H^0(M, \Theta^* \otimes \Theta^*) = d + 1 - k. \tag{24.3}$$

First consider the case of genus 0. In this case, $\Theta^* \otimes \Theta^*$ degree $-4$, so has no holomorphic section. Riemann-Roch gives

$$dim(H^0(M, \Theta)) = 3. \tag{24.4}$$

This is correct because the complex Lie algebra of holomorphic vector fields is isomorphic to the real Lie algebra of conformal vector fields, and the identity component is

$$SO(3,1) = PSL(2, \mathbb{C}), \tag{24.5}$$

which is a 6-dimensional real Lie group.

Next, the case of genus 1. Then the bundles have degree 0, so the space of sections is 1 dimensional, and Riemann-Roch gives $0 = 0$. The moduli space is 1-dimensional.

We get something new for genus $k > 1$. In this case $\Theta$ has negative degree, so has no holomorphic sections. The Riemann-Roch formula yields

$$H^1(M, \Theta) = H^0(M, \Theta^* \otimes \Theta^*) = -(2 - 2k) - 1 + k = 3k - 3, \tag{24.6}$$

thus the moduli space has complex dimension $3k - 3$. Since $H^2(M, \Theta) = 0$ and $H^0(M, \Theta) = 0$, it is a smooth manifold of real dimension $6k - 6$.

## 24.2 Kähler metrics

**Definition 8.** *An almost Hermitian manifold is a triple $(M, g, J)$ such that $g(JX, JY) = g(X, Y)$. The triple is called Hermitian if $J$ is integrable.*

The fundamental 2-form is denoted by $\omega(X, Y) = g(JX, Y)$.

**Proposition 24.1.** *Let $(M, g, J)$ be an almost Hermitian manifold. Then*

$$4g((\nabla_X J)Y, Z) = 6d\omega(X, JY, JZ) - 6d\omega(X, Y, Z) + g(N(Y, Z), JX). \tag{24.7}$$

*Proof.* Computation. $\square$

**Corollary 24.1.** *If $(M, g, J)$ is Hermitian, then $d\omega = 0$ if and only if $J$ is parallel.*

**Definition 9.** *A triple $(M, g, J)$ is Kähler if $J$ is integrable and $d\omega = 0$.*

**Proposition 24.2.** *If $(M, g, J)$ is Kähler, then*

$$Rm(X, Y, Z, W) = Rm(JX, JY, Z, W) = Rm(X, Y, JZ, JW), \tag{24.8}$$
$$Ric(X, Y) = Ric(JX, JY). \tag{24.9}$$

*Proof.* We first claim that

$$R(X, Y)JZ = J(R(X, Y)Z). \tag{24.10}$$

To see this,

$$\begin{aligned} R(X, Y)JZ &= \nabla_X \nabla_Y (JZ) - \nabla_Y \nabla_X (JZ) - \nabla_{[X,Y]} JZ \\ &= J(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = J(R(X, Y)Z), \end{aligned}$$

since $J$ is parallel. Next,

$$\begin{aligned} Rm(JX, JY, U, V) &= -g(R(JX, JY)U, V) = -g(R(U, V)JX, JY) \\ &= -g(JR(JX, JY), -Y) = Rm(X, Y, U, V), \end{aligned}$$

and the others are proved similarly. □

## 24.3 Representations of $U(2)$

Since $U(2) \subset SO(4)$, we can see what happens to the curvature tensor decomposition in dimension 4 when we restrict to $U(2)$. Some representations which are irreducible for $SO(4)$ become reducible when restricted to $U(2)$. Under $SO(4)$, we have

$$\Lambda^2 T^* = \Lambda^2_+ \oplus \Lambda^2_-, \tag{24.11}$$

but under $U(2)$, we have the decomposition

$$\Lambda^2 T^* = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \Lambda^{1,1}. \tag{24.12}$$

Notice that these are the complexifications of real vector spaces. The first is of dimension 2, the second is of dimension 4. Let $\omega$ denote the 2-form $\omega(X, Y) = g(J_0 X, Y)$. This yields the orthogonal decomposition

$$\Lambda^2 T^* = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R} \cdot \omega \oplus \Lambda^{1,1}_0, \tag{24.13}$$

where $\Lambda^{1,1}_0 \subset \Lambda^{1,1}$ is the orthogonal complement of the span of $\omega$, and is therefore 2-dimensional (the complexification of which is the space of *primitive* $(1,1)$-forms).

**Proposition 24.3.** *Under $U(2)$, we have the decomposition*

$$\Lambda^2_+ = \mathbb{R} \cdot \omega \oplus (\Lambda^{2,0} \oplus \Lambda^{0,2}) \tag{24.14}$$
$$\Lambda^2_- = \Lambda^{1,1}_0. \tag{24.15}$$

*Proof.* We can choose an oriented orthonormal basis of the form

$$\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}. \tag{24.16}$$

Let $\{e^1, e^2, e^3, e^4\}$ denote the dual basis. The space of $(1,0)$ forms, $\Lambda^{1,0}$ has generators

$$\theta^1 = e^1 + ie^2, \quad \theta^2 = e^3 + ie^4. \tag{24.17}$$

We have

$$\begin{aligned} \omega &= \frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 + \theta^2 \wedge \overline{\theta}^2) \\ &= \frac{i}{2}\Big((e^1 + ie^2) \wedge (e^1 - ie^2) + (e^3 + ie^4) \wedge (e^3 - ie^4)\Big) \\ &= e^1 \wedge e^2 + e^3 \wedge e^4 = \omega_+^1. \end{aligned} \tag{24.18}$$

Similarly, we have

$$\frac{i}{2}(\theta^1 \wedge \overline{\theta}^1 - \theta^2 \wedge \overline{\theta}^2) = e^1 \wedge e^2 - e^3 \wedge e^4 = \omega_-^1, \tag{24.19}$$

so $\omega_-^1$ is of type $(1,1)$, so lies in $\Lambda_0^{1,1}$. Next,

$$\begin{aligned} \theta^1 \wedge \theta^2 &= (e^1 + ie^2) \wedge (e^3 + ie^4) \\ &= (e^1 \wedge e^3 - e^2 \wedge e^4) + i(e^1 \wedge e^4 + e^2 \wedge e^3) \\ &= \omega_+^2 + i\omega_+^3. \end{aligned} \tag{24.20}$$

Solving, we obtain

$$\omega_+^2 = \frac{1}{2}(\theta^1 \wedge \theta^2 + \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{24.21}$$

$$\omega_+^3 = \frac{1}{2i}(\theta^1 \wedge \theta^2 - \overline{\theta}^1 \wedge \overline{\theta}^2), \tag{24.22}$$

which shows that $\omega_+^2$ and $\omega_+^3$ are in the space $\Lambda^{2,0} \oplus \Lambda^{0,2}$. Finally,

$$\begin{aligned} \theta^1 \wedge \overline{\theta}^2 &= (e^1 + ie^2) \wedge (e^3 - ie^4) \\ &= (e^1 \wedge e^3 + e^2 \wedge e^4) + i(-e^1 \wedge e^4 + e^2 \wedge e^3) \\ &= \omega_-^2 - i\omega_-^3, \end{aligned} \tag{24.23}$$

which shows that $\omega_-^2$ and $\omega_-^3$ are in the space $\Lambda_0^{1,1}$. $\square$

**Corollary 24.2.** *If $(M, g)$ is Kähler, then*

$$b_2^+ = 1 + 2b^{2,0}, \tag{24.24}$$

$$b_2^- = b^{1,1} - 1, \tag{24.25}$$

$$\tau = b_2^+ - b_2^- = 2 + 2b^{2,0} - b^{1,1}. \tag{24.26}$$

*Proof.* This follows from Proposition 24.3, and Hodge theory on Kähler manifolds, see [GH94]. □

**Proposition 24.4.** *If $(M,g)$ is Kähler, then $W^+$ is determined by the scalar curvature. More explicitly, letting $\omega$ denote the Kähler form, and $\{\omega, \omega^2_+, \omega^3_+\}$ be an ONB of $\Lambda^2_+$, we have*

$$\mathcal{W}^+\omega = \frac{R}{6}\omega, \tag{24.27}$$

$$\mathcal{W}^+\omega^2_+ = -\frac{R}{12}\omega^2_+, \tag{24.28}$$

$$\mathcal{W}^+\omega^3_+ = -\frac{R}{12}\omega^3_+. \tag{24.29}$$

*Equivalently, we may write*

$$\mathcal{W}^+ = \frac{R}{12}(3\omega \odot \omega - I). \tag{24.30}$$

*Proof.* From Proposition 24.2, we have

$$R \in S^2(\Lambda^{1,1}). \tag{24.31}$$

Since $\omega^2_+$ and $\omega^3_+$ are in $\Lambda^{2,0} \oplus \Lambda^{0,2}$, which is orthogonal to the space of $(1,1)$-forms, they must be annihilated by $\mathcal{W}^+ + \frac{R}{12}I$. The first identity then follows since $\mathcal{W}^+$ is traceless. □

**Corollary 24.3.** *In the above basis, the curvature tensor of a Kähler 4-manifold has the form*

$$\mathcal{R} = \left(\begin{array}{c|c} \begin{pmatrix} \frac{R}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} \rho_1 & 0 & 0 \\ \rho_2 & 0 & 0 \\ \rho_3 & 0 & 0 \end{pmatrix} & \mathcal{W}^- \end{array}\right). \tag{24.32}$$

*Proof.* This follows since $Ric$ is a real $(1,1)$-form. □

## 24.4 A Weitzenbock formula

Instead of invoking the traceless condition on $\mathcal{W}^+$, we can directly prove the first identity as follows. Recall the definition of $W$:

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2}(R_{ik}g_{jl} - R_{jk}g_{il} - R_{il}g_{jk} + R_{jl}g_{ik}) \\ &+ \frac{1}{(n-1)(n-2)}R(g_{ik}g_{jl} - g_{jk}g_{il}). \end{aligned} \tag{24.33}$$

Commute covariant derivatives

$$\nabla_i \nabla_j \omega_{kl} = \nabla_j \nabla_i \omega_{kl} - R_{ijk}{}^p \omega_{pl} - R_{ijl}{}^p \omega_{kp}. \tag{24.34}$$

Since the Kähler form is parallel, we have the identity

$$R_{ijk}{}^p \omega_{pl} + R_{ijl}{}^p \omega_{kp} = 0. \tag{24.35}$$

Let us trace this on $i$ and $k$,

$$\begin{aligned} 0 &= g^{ik}(R_{ijk}{}^p \omega_{pl} + R_{ijl}{}^p \omega_{kp}) \\ &= g^{ik} g^{pm} (R_{ijmk} \omega_{pl} + R_{ijml} \omega_{kp} \\ &= -g^{pm} R_{jm} \omega_{pl} + g^{ik} g^{pm} R_{ijml} \omega_{kp}. \end{aligned} \tag{24.36}$$

Skew this identity in $j$ and $l$ to obtain

$$\begin{aligned} 0 &= -g^{pm} R_{jm} \omega_{pl} + g^{pm} R_{lm} \omega_{pj} + g^{ik} g^{pm} R_{ijml} \omega_{kp} - g^{ik} g^{pm} R_{ilmj} \omega_{kp} \\ &= -R_j^p \omega_{pl} + R_l^p \omega_{pj} + g^{ik} g^{pm} (R_{ijml} + R_{iljm}) \omega_{kp}. \end{aligned} \tag{24.37}$$

By the algebraic Bianchi identity,

$$R_{ijml} + R_{imlj} + R_{iljm} = 0, \tag{24.38}$$

so finally we have the identity

$$0 = -R_j^p \omega_{pi} + R_i^p \omega_{pj} - R_{ijkl} \omega^{kl}. \tag{24.39}$$

Let us now work in an ONB. This identity is

$$R_{ijkl} \omega_{kl} = R_{ip} \omega_{pj} - R_{jp} \omega_{pi}. \tag{24.40}$$

Using (24.33), we have

$$\begin{aligned} W_{ijkl} \omega_{kl} &= R_{ip} \omega_{pj} - R_{jp} \omega_{pi} - \frac{1}{n-2} (R_{ip} \omega_{pj} - R_{jp} \omega_{pi} - R_{ip} \omega_{jp} + R_{jp} \omega_{ip}) \\ &\quad + \frac{1}{(n-1)(n-2)} R \omega_{ij} \\ &= \frac{n-4}{n-2} (R_{ip} \omega_{pj} - R_{jp} \omega_{pi}) + \frac{2}{(n-1)(n-2)} R \omega_{ij}. \end{aligned} \tag{24.41}$$

We conclude that for $n = 4$,

$$W_{ijkl} \omega_{kl} = \frac{R}{3} \omega_{ij}, \tag{24.42}$$

which implies that

$$\mathcal{W}^+ \omega = \frac{R}{6} \omega. \tag{24.43}$$

Note that this argument works for any parallel 2-form $\omega$.

## 24.5 Connections

On any Riemannian vector bundle $E$ with a connection, we can consider the following sequence

$$E \overset{\nabla}{\to} T^* \otimes E \overset{d^\nabla}{\to} \Lambda^2(T^*) \otimes E, \tag{24.44}$$

where the first mapping is just covariant differentation, and the second mapping is defined by

$$d^\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma - \alpha \wedge \nabla\sigma. \tag{24.45}$$

It is easy to see that

$$(d^\nabla \circ \nabla)\sigma = \Omega(\sigma), \tag{24.46}$$

where $\Omega$ is the curvature 2-form with values in $End(E)$, defined by

$$\Omega(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma. \tag{24.47}$$

Letting $\Lambda^p(E) = \Gamma(\Lambda^p \otimes E)$, this extends to a mapping $d^\nabla : \Lambda^p(E) \to \Lambda^{p+1}(E)$ by the formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) = &\sum (-1)^j \nabla_{X_j} \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{24.48}$$

This has the property that

$$d^\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge d^\nabla \sigma, \tag{24.49}$$

and we have the property

$$(d^\nabla \circ d^\nabla)\sigma = \Omega \wedge \sigma. \tag{24.50}$$

The differential Bianchi identity takes the form

$$d^\nabla \Omega = 0. \tag{24.51}$$

# 25 Lecture 25

## 25.1 Integration and adjoints

If $T$ is an $(r,s)$-tensor, we define the *divergence* of $T$, div $T$ to be the $(r, s-1)$ tensor

$$(\text{div } T)(Y_1, \dots, Y_{s-1}) = tr\Big(X \to \sharp(\nabla T)(X, \cdot, Y_1, \dots, Y_{s-1})\Big), \tag{25.1}$$

that is, we trace the covariant derivative on the *first* two covariant indices. In coordinates, this is

$$(\text{div } T)^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1}} = g^{ij} \nabla_i T^{i_1 \ldots i_r}_{j j_1 \ldots j_{s-1}}. \tag{25.2}$$

If $X$ is a vector field, define

$$(\text{div } X) = tr(\nabla X), \tag{25.3}$$

which is in coordinates

$$\text{div } X = \delta^i_j \nabla_i X^j = \nabla_j X^j. \tag{25.4}$$

For vector fields and 1-forms, these two are of course closely related:

**Proposition 25.1.** *For a vector field $X$,*

$$\text{div } X = \text{div } (\flat X). \tag{25.5}$$

*Proof.* We compute

$$\begin{aligned} \text{div } X &= \delta^i_j \nabla_i X^j \\ &= \delta^i_j \nabla_i g^{jl} X_l \\ &= \delta^i_j g^{jl} \nabla_i X_l \\ &= g^{il} \nabla_i X_l = \text{div } (\flat X). \end{aligned} \tag{25.6}$$

□

If $M$ is oriented, we define the Riemannian volume element $dV$ to be the oriented unit norm element of $\Lambda^n(T^*M_x)$. Equivalently, if $\omega_1, \ldots \omega_n$ is a positively oriented ONB of $T^*M_x$, then

$$dV = \omega^1 \wedge \cdots \wedge \omega^n. \tag{25.7}$$

In coordinates,

$$dV = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n. \tag{25.8}$$

Recall the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$ defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV_x, \tag{25.9}$$

where $\alpha, \beta \in \Lambda^p$.

**Proposition 25.2.** *(i) The Hodge star is an isometry from $\Lambda^p$ to $\Lambda^{n-p}$.*
*(ii) $*(\omega^1 \wedge \cdots \wedge \omega^p) = \omega^{p+1} \wedge \cdots \wedge \omega^n$ if $\omega_1, \ldots \omega_n$ is a positively oriented ONB of $T^*M_x$. In particular, $*1 = dV$, and $*dV = 1$.*

*(iii) On* $\Lambda^p$, $*^2 = (-1)^{p(n-p)}$.
*(iv) For* $\alpha, \beta \in \Lambda^p$,

$$\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta) = *(\beta \wedge *\alpha). \tag{25.10}$$

*(v) If* $\{e_i\}$ *and* $\{\omega^i\}$ *are dual ONB of* $T_xM$, *and* $T_x^*M$, *respectively, and* $\alpha \in \Lambda^p$, *then*

$$*(\omega^j \wedge \alpha) = (-1)^p i_{e_j}(*\alpha), \tag{25.11}$$

*where* $i_X : \Lambda^p \to \Lambda^{p-1}$ *is* interior multiplication *defined by*

$$i_X\alpha(X_1, \ldots, X_p) = \alpha(X, X_1, \ldots, X_p). \tag{25.12}$$

*Proof.* The proof is left to the reader. □

**Remark 25.1.** *In general, locally there will be two different Hodge star operators, depending upon the two different choices of local orientation. Each will extend to a* global *Hodge star operator if and only if M is orientable. However, one can still construct* global *operators using the Hodge star, even if M is non-orientable, an example of which will be the Laplacian.*

We next give a formula relating the exterior derivative and covariant differentiation.

**Proposition 25.3.** *The exterior derivative* $d : \Omega^p \to \Omega^{p+1}$ *is given by*

$$d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{25.13}$$

*(the notation means that the* $\hat{X}_j$ *term is omitted). That is, the exterior derivative* $d\omega$ *is the skew-symmetrization of* $\nabla\omega$*, we write* $d\omega = Sk(\nabla\omega)$. *If* $\{e_i\}$ *and* $\{\omega^i\}$ *are dual ONB of* $T_xM$, *and* $T_x^*M$, *then this may be written*

$$d\omega = \sum_i \omega^i \wedge \nabla_{e_i}\omega. \tag{25.14}$$

*Proof.* Recall the formula for the exterior derivative [War83, Theorem ?],

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{j=0}^{p} (-1)^j X_j \Big( \omega(X_0, \ldots, \hat{X}_j, \ldots, X_p) \Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned} \tag{25.15}$$

Since both sides of the equation (25.13) are tensors, we may assume that $[X_i, X_j]_x = 0$, at a fixed point $x$. Since the connection is Riemannian, we also have $\nabla_{X_i} X_j(x) = 0$.

We then compute at the point $x$.

$$\begin{aligned} d\omega(X_0,\dots,X_p)(x) &= \sum_{j=0}^{p}(-1)^j X_j\Big(\omega(X_0,\dots,\hat{X}_j,\dots,X_p)\Big)(x) \\ &= \sum_{j=0}^{p}(-1)^j(\nabla_{X_j}\omega)(X_0,\dots,\hat{X}_j,\dots,X_p)(x), \end{aligned} \tag{25.16}$$

using the definition of the covariant derivative. This proves the first formula. For the second, note that

$$\nabla_{X_j}\omega = \nabla_{(X_j)^i e_i}\omega = \sum_{i=1}^{n}\omega^i(X_j)\cdot(\nabla_{e_i}\omega), \tag{25.17}$$

so we have

$$\begin{aligned} d\omega(X_0,\dots,X_p)(x) &= \sum_{j=0}^{p}(-1)^j\sum_{i=1}^{n}\omega^i(X_j)\cdot(\nabla_{e_i}\omega)(X_0,\dots,\hat{X}_j,\dots,X_p)(x) \\ &= \sum_i(\omega^i\wedge\nabla_{e_i}\omega)(X_0,\dots,X_p)(x). \end{aligned} \tag{25.18}$$

□

**Proposition 25.4.** *For a vector field $X$,*

$$*(\mathrm{div}\ X) = (\mathrm{div}\ X)dV = d(i_X dV) = \mathcal{L}_X(dV). \tag{25.19}$$

*Proof.* Fix a point $x \in M$, and let $\{e_i\}$ be an orthonormal basis of $T_xM$. In a small neighborhood of $x$, parallel translate this frame along radial geodesics. For such a frame, we have $\nabla_{e_i}e_j(x) = 0$. Such a frame is called an *adapted* moving frame field at $x$. Let $\{\omega^i\}$ denote the dual frame field. We have

$$\begin{aligned} \mathcal{L}_X(dV) &= (di_X + i_X d)dV = d(i_X dV) \\ &= \sum_i \omega^i\wedge\nabla_{e_i}\big(i_X(\omega^1\wedge\cdots\wedge\omega^n)\big) \\ &= \sum_i \omega^i\wedge\nabla_{e_i}\Big((-1)^{j-1}\sum_{j=1}^{n}\omega^j(X)\omega^1\wedge\cdots\wedge\hat{\omega}^j\wedge\cdots\wedge\omega^n\Big) \\ &= \sum_{i,j}(-1)^{j-1}e_i\big(\omega^j(X)\big)\omega^i\wedge\omega^1\wedge\cdots\wedge\hat{\omega}^j\wedge\cdots\wedge\omega^n \\ &= \sum_i \omega^i(\nabla_{e_i}X)dV \\ &= (\mathrm{div}\ X)dV = *(\mathrm{div}\ X). \end{aligned} \tag{25.20}$$

□

**Corollary 25.1.** *Let $(M,g)$ be compact, orientable and without boundary. If $X$ is a $C^1$ vector field, then*

$$\int_M (\text{div } X) dV = 0. \tag{25.21}$$

*Proof.* Using Stokes' Theorem and Proposition 25.4,

$$\int_M (\text{div } X) dV = \int d(i_X dV) = \int_{\partial M} i_x dV = 0. \tag{25.22}$$

$\square$

Using this, we have an integration formula for $(r,s)$-tensor fields.

**Corollary 25.2.** *Let $(M,g)$ be as above, $T$ be an $(r,s)$-tensor field, and $S$ be a $(r,s+1)$ tensor field. Then*

$$\int_M \langle \nabla T, S \rangle dV = -\int_M \langle T, \text{div } S \rangle dV. \tag{25.23}$$

*Proof.* Let us view the inner product $\langle T, S \rangle$ as a 1-form $\omega$. In coordinates

$$\omega = \langle T, S \rangle = T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s} dx^j. \tag{25.24}$$

Note the indices on $T$ are reversed, since we are taking an inner product. Taking the divergence, since $g$ is parallel we compute

$$\begin{aligned} \text{div } (\langle T, S \rangle) &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r} S^{i_1 \dots i_r}_{j j_1 \dots j_s}) \\ &= \nabla^j (T^{j_1 \dots j_s}_{i_1 \dots i_r}) S^{i_1 \dots i_r}_{j j_1 \dots j_s} + T^{j_1 \dots j_s}_{i_1 \dots i_r} \nabla^j S^{i_1 \dots i_r}_{j j_1 \dots j_s} \\ &= \langle \nabla T, S \rangle + \langle T, \text{div } S \rangle. \end{aligned} \tag{25.25}$$

The result then follows from Proposition 25.1 and Corollary 25.1. $\square$

**Remark 25.2.** *Some authors define $\nabla^* = -div$, for example [Pet06].*

Recall the adjoint of $d$, $\delta : \Omega^p \to \Omega^{p-1}$ defined by

$$\delta \omega = (-1)^{n(p+1)+1} * d * \omega. \tag{25.26}$$

**Proposition 25.5.** *The operator $\delta$ is the $L^2$ adjoint of $d$,*

$$\int_M \langle \delta \alpha, \beta \rangle dV = \int_M \langle \alpha, d\beta \rangle dV, \tag{25.27}$$

*where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$.*

*Proof.* We compute

$$\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big( d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, \delta\alpha \rangle dV.
\end{aligned} \tag{25.28}$$

□

**Proposition 25.6.** *On $\Omega^p$, $\delta = -\mathrm{div}$.*

*Proof.* Let $\omega \in \Omega^p$. Fix $x \in M$, and dual ONB $\{e_i\}$ and $\{\omega^i\}$. We compute at $x$,

$$\begin{aligned}
(\mathrm{div}\ \omega)(x) &= \sum_j i_{e_j} \nabla_{e_j} \omega \\
&= \sum_j (-1)^{p(n-p)} \Big( i_{e_j} \big( * * (\nabla_{e_j} \omega) \big) \Big) \\
&= (-1)^{p(n-p)} \sum_j (-1)^{n-p} * (\omega^j \wedge * \nabla_{e_j} \omega) \\
&= (-1)^{(p+1)(n-p)} \sum_j * \big( \omega^j \wedge \nabla_{e_j} (*\omega) \big) \\
&= (-1)^{n(p+1)} (*d * \omega)(x).
\end{aligned} \tag{25.29}$$

□

**Remark 25.3.** *Formula (25.6) requires a bit of explanation. The divergence is defined on tensors, while $\delta$ is defined on differential forms. What we mean is defined on the first line of (25.29), where the covariant derivative is the induced covariant derivative on forms.*

An alternative proof of the proposition could go as follows.

$$\begin{aligned}
\int_M \langle \alpha, \delta\beta \rangle dV &= \int_M \langle d\alpha, \beta \rangle dV \\
&= \int_M \langle Sk(\nabla \alpha), \beta \rangle dV \\
&= \int_M \langle \nabla \alpha, \beta \rangle dV \\
&= \int_M \langle \alpha, -\mathrm{div}\ \beta \rangle dV.
\end{aligned} \tag{25.30}$$

Thus both $\delta$ and $-\mathrm{div}$ are $L^2$ adjoints of $d$. The result then follows from uniqueness of the $L^2$ adjoint.

# 26 Lecture 26

## 26.1 Bochner and Weitzenböck formulas

For $T$ an $(r,s)$-tensor, the rough Laplacian is given by

$$\Delta T = \text{div } \nabla T. \tag{26.1}$$

For $\omega \in \Omega^p$ we define the *Hodge laplacian* $\Delta_H : \Omega^p \to \Omega^p$ by

$$\Delta_H \omega = (d\delta + \delta d)\omega. \tag{26.2}$$

We say a $p$-form is *harmonic* if it is in the kernel of $\Delta_H$.

**Proposition 26.1.** *For $T$ and $S$ both $(r,s)$-tensors,*

$$\int_M \langle \Delta T, S \rangle dV = -\int_M \langle \nabla T, \nabla S \rangle dV = \int_M \langle T, \Delta S \rangle dV. \tag{26.3}$$

*For $\alpha, \beta \in \Omega^p$,*

$$\int_M \langle \Delta_H \alpha, \beta \rangle dV = \int_M \langle \alpha, \Delta_H \beta \rangle dV. \tag{26.4}$$

*Proof.* Formula (26.3) is an application of (26.1) and Corollary (25.2). For the second, from Proposition 25.5,

$$\begin{aligned}
\int_M \langle \Delta_H \alpha, \beta \rangle dV &= \int_M \langle (d\delta + \delta d)\alpha, \beta \rangle dV \\
&= \int_M \langle d\delta\alpha, \beta \rangle dV + \int_M \langle \delta d\alpha, \beta \rangle dV \\
&= \int_M \langle \delta\alpha, \delta\beta \rangle dV + \int_M \langle d\alpha, d\beta \rangle dV \\
&= \int_M \langle \alpha, d\delta\beta \rangle dV + \int_M \langle \alpha, \delta d\beta \rangle dV \\
&= \int_M \langle \alpha, \Delta_H \beta \rangle dV.
\end{aligned} \tag{26.5}$$

□

Note that $\Delta$ maps alternating $(0,p)$ tensors to alternating $(0,p)$ tensors, therefore it induces a map $\Delta : \Omega^p \to \Omega^p$ (note that on [Poo81, page 159] it is stated that the rough Laplacian of an $r$-form is in general not an $r$-form, but this seems to be incorrect). On $p$-forms, $\Delta$ and $\Delta_H$ are two self-adjoint linear second order differential operators. How are they related? Consider the case of 1-forms.

**Proposition 26.2.** *Let $\omega \in \Omega^1(M)$. If $d\omega = 0$, then*

$$\Delta\omega = -\Delta_H(\omega) + Rc^T(\omega). \tag{26.6}$$

*Proof.* In Proposition **??** above, we showed that on functions,

$$\Delta df = d\Delta f + Rc^T(df). \tag{26.7}$$

But on functions, $\Delta f = -\Delta_H f$. Clearly $\Delta_H$ commutes with $d$, so we have

$$\Delta(df) = -\Delta_H(df) + Rc^T(df). \tag{26.8}$$

Given any closed 1-form $\omega$, by the Poincaré Lemma, we can locally write $\omega = df$ for some function $f$. This proves the formula. $\square$

**Corollary 26.1.** *If $(M, g)$ has non-negative Ricci curvature, then any harmonic 1-form is parallel. In this case $b_1(M) \leq n$. If, in addition, Rc is positive definite at some point, then any harmonic 1-form is trivial. In this case $b_1(M) = 0$.*

*Proof.* Formula (26.6) is

$$\Delta\omega = Rc^T(\omega). \tag{26.9}$$

Take inner product with $\omega$, and integrate

$$\int_M \langle \Delta\omega, \omega \rangle = -\int_M |\nabla\omega|^2 dV = \int_M Ric(\sharp\omega, \sharp\omega) dV \tag{26.10}$$

This clearly implies that $\nabla\omega \equiv 0$, thus $\omega$ is parallel. If in addition $Rc$ is strictly positive somewhere, $\omega$ must vanish identically. The conclusion on the first Betti number follows from the Hodge Theorem. $\square$

We next generalize this to $p$-forms.

**Definition 10.** *For $\omega \in \Omega^p$, we define a $(0, p)$-tensor field $\rho_\omega$ by*

$$\rho_\omega(X_1, \dots, X_p) = \sum_{i=1}^{n} \sum_{j=1}^{p} \Big(\mathcal{R}_{\Lambda^p}(e_i, X_j)\omega\Big)(X_1, \dots, X_{j-1}, e_i, X_{j+1}, \dots, X_p), \tag{26.11}$$

*where $\{e_i\}$ is an ONB at $x \in M$.*

**Remark 26.1.** *Recall what this means. The Riemannian connection induces a metric connection in the bundle $\Lambda^p(T^*M)$. The curvature of this connection therefore satisfies*

$$\mathcal{R}_{\Lambda^p} \in \Gamma\Big(\Lambda^2(T^*M) \otimes \mathfrak{so}(\Lambda^p(T^*M))\Big). \tag{26.12}$$

*We leave it to the reader to show that (26.11) is well-defined.*

The relation between the Laplacians is given by

**Theorem 26.1.** *Let $\omega \in \Omega^p$. Then*

$$\Delta_H \omega = -\Delta\omega + \rho_\omega. \tag{26.13}$$

*We also have the formula*

$$\langle \Delta_H \omega, \omega \rangle = \frac{1}{2}\Delta_H |\omega|^2 + |\nabla \omega|^2 + \langle \rho_\omega, \omega \rangle. \tag{26.14}$$

*Proof.* Take $\omega \in \Omega^p$, and vector fields $X, Y_1, \dots, Y_p$. We compute

$$\begin{aligned}(\nabla\omega - d\omega)(X, Y_1, \dots, Y_p) &= (\nabla_X \omega)(Y_1, \dots, Y_p) - d\omega(X, Y_1, \dots, Y_p) &(26.15)\\ &= \sum_{j=1}^{p} (\nabla_{Y_j}\omega)(Y_1, \dots, Y_{j-1}, X, Y_{j+1}, \dots, Y_p), &(26.16)\end{aligned}$$

using Proposition 25.3. Fix a point $x \in M$. Assume that $(\nabla Y_j)_x = 0$, by parallel translating the values of $Y_j$ at $x$. Also take $e_i$ to be an adapted moving frame at $p$. Using Proposition 25.6, we compute at $x$

$$\begin{aligned}(\text{div } \nabla\omega + \delta d\mu)(Y_1, \dots, Y_p) &= \text{div } (\nabla\omega - d\omega)(Y_1, \dots, Y_r)\\ &= \sum_{i=1}^{n} \Big(\nabla_{e_i}(\nabla\omega - d\omega)\Big)(e_i, Y_1, \dots, Y_p)\\ &= \sum_{i=1}^{n} \Big(e_i(\nabla\omega - d\omega)\Big)(e_i, Y_1, \dots, Y_p)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p} e_i\Big((\nabla_{Y_j}\omega)(Y_1, \dots, Y_{j-1}, e_i, Y_{j+1}, \dots, Y_p)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{p} (\nabla_{e_i}\nabla_{Y_j}\omega)(Y_1, \dots, Y_{j-1}, e_i, Y_{j+1}, \dots, Y_p)\end{aligned} \tag{26.17}$$

We also have

$$\begin{aligned}d\delta\omega(Y_1, \dots, Y_p) &= \sum_{j=1}^{p} (-1)^{j+1} (\nabla_{Y_j}\delta\omega)(Y_1, \dots, \hat{Y}_j, \dots, Y_p)\\ &= \sum_{j=1}^{p} (-1)^j Y_j \left( \sum_{i=1}^{n} (\nabla_{e_i}\omega)(e_i, Y_1, \dots, \hat{Y}_j, \dots, Y_p) \right)\\ &= -\sum_{i=1}^{n}\sum_{j=1}^{p} (\nabla_{Y_j}\nabla_{e_i}\omega)(Y_1, \dots, Y_{j-1}, e_i, Y_{j+1}, \dots, Y_p).\end{aligned} \tag{26.18}$$

The commutator $[e_i, Y_j](x) = 0$, since $\nabla_{e_i} Y_j(x) = 0$, and $\nabla_{Y_j} e_i(x) = 0$, by our choice. Consequently,

$$(\Delta_H\omega + \Delta\omega)(Y_1, \dots, Y_p) = (\Delta_H\omega + \text{div } \nabla\omega)(Y_1, \dots, Y_p) = \rho_\omega(Y_1, \dots, Y_p). \tag{26.19}$$

This proves (26.13). For (26.14), we compute at $x$

$$\begin{aligned}
\text{div } \nabla\omega(Y_1, \ldots, Y_p) &= \sum_i \nabla_{e_i}(\nabla\omega)(e_1, Y_1, \ldots, Y_p) \\
&= \sum_i e_i(\nabla_{e_i}\omega)(Y_1, \ldots, Y_p) \\
&= \sum_i (\nabla_{e_i}\nabla_{e_i}\omega)(Y_1, \ldots, Y_p).
\end{aligned} \tag{26.20}$$

Next, again at $x$,

$$\begin{aligned}
\langle -\text{div}\nabla\omega, \omega\rangle &= -\sum_i \langle \nabla_{e_i}\nabla_{e_i}\omega, \omega\rangle \\
&= -\sum_i e_i\left(\langle \nabla_{e_i}\omega, \omega\rangle - \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle\right) \\
&= -\frac{1}{2}\sum_i (e_i e_i|\omega|^2) + |\nabla\omega|^2 \\
&= \frac{1}{2}\Delta_H|\omega|^2 + |\nabla\omega|^2.
\end{aligned} \tag{26.21}$$

□

**Remark 26.2.** *The rough Laplacian is therefore "roughly" the Hodge Laplacian, up to curvature terms. Note also in (26.14), the norms are tensor norms, since the right hand side has the term $|\nabla\omega|^2$ and $\nabla\omega$ is not a differential form. We are using (1.19) to identify forms and alternating tensors.*

# 27 Lecture 27

## 27.1 Manifolds with positive curvature operator

We begin with a general property of curvature in exterior bundles.

**Proposition 27.1.** *Let $\nabla$ be a connection in a vector bundle $\pi : E \to M$. As before, extend $\nabla$ to a connection in $\Lambda^p(E)$ by defining it on decomposable elements*

$$\nabla_X(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \nabla_X s_i \wedge \cdots \wedge s_p. \tag{27.1}$$

*For vector fields $X, Y$, $\mathcal{R}_{\Lambda^p(E)}(X,Y) \in End(\Lambda^p(E))$ acts as a derivation*

$$\mathcal{R}_{\Lambda^p(E)}(X,Y)(s_1 \wedge \cdots \wedge s_p) = \sum_{i=1}^{p} s_1 \wedge \cdots \wedge \mathcal{R}_\nabla(X,Y)(s_i) \wedge \cdots \wedge s_p. \tag{27.2}$$

*Proof.* We prove for $p = 2$, the case of general $p$ is left to the reader. Since this is a tensor equation, we may assume that $[X, Y] = 0$. We compute

$$
\begin{aligned}
&\mathcal{R}_{\Lambda^2(E)}(X,Y)(s_1 \wedge s_2) = \nabla_X \nabla_Y (s_1 \wedge s_2) - \nabla_Y \nabla_X (s_1 \wedge s_2)\\
&= \nabla_X \Big( (\nabla_Y s_1) \wedge s_2 + s_1 \wedge (\nabla_Y s_2) \Big) - \nabla_Y \Big( (\nabla_X s_1) \wedge s_2 + s_1 \wedge (\nabla_X s_2) \Big)\\
&= (\nabla_X \nabla_Y) s_1 \wedge s_2 + \nabla_Y s_1 \wedge \nabla_X s_2 + \nabla_X s_1 \wedge \nabla_Y s_2 + s_1 \wedge (\nabla_X \nabla_Y) s_2\\
&- (\nabla_Y \nabla_X) s_1 \wedge s_2 - \nabla_X s_1 \wedge \nabla_Y s_2 - \nabla_Y s_1 \wedge \nabla_X s_2 - s_1 \wedge (\nabla_Y \nabla_X) s_2\\
&= \Big( \mathcal{R}_\nabla (X,Y)(s_1) \Big) \wedge s_2 + s_1 \wedge \Big( \mathcal{R}_\nabla (X,Y)(s_2) \Big).
\end{aligned}
\tag{27.3}
$$

□

We apply this to the bundle $E = \Lambda^p(T^*M)$. Recall for a 1-form $\omega$,

$$
\nabla_i \nabla_j \omega_l = \nabla_j \nabla_i \omega_l - R_{ijl}{}^k \omega_k. \tag{27.4}
$$

In other words,

$$
(\mathcal{R}(\partial_i, \partial_j)\omega)_l = -R_{ijl}{}^k \omega_k, \tag{27.5}
$$

where the left hand side means the curvature of the connection in $T^*M$, but the right hand side is the Riemannian curvature tensor. For a $p$-form $\omega \in \Omega^p$, with components $\omega_{i_1 \dots i_p}$, Proposition 27.1 says that

$$
\Big( \mathcal{R}_{\Lambda^p}(e_\alpha, e_\beta)\omega \Big)_{i_1 \dots i_p} = - \sum_{k=1}^{p} R_{\alpha\beta i_k}{}^l \omega_{i_1 \dots i_{k-1} l i_{k+1} \dots i_p}, \tag{27.6}
$$

where the left hand side now means the curvature of the connection in $\Lambda^p(T^*M)$.

Next, we look at $\rho_\omega$ in coordinates. It is written

$$
(\rho_\omega)_{i_i \dots i_p} = g^{\alpha l} \sum_{j=1}^{p} \big( \mathcal{R}_{\Lambda^p}(\partial_\alpha, \partial_{i_j})\omega \big)_{i_1 \dots i_{j-1} l i_{j+1} \dots i_p}. \tag{27.7}
$$

Using (27.6), we may write $\rho_\omega$ as

$$
\begin{aligned}
(\rho_\omega)_{i_i \dots i_p} = &- g^{\alpha l} \sum_{j=1}^{p} \sum_{k=1, k \neq j}^{p} R_{\alpha i_j i_k}{}^m \omega_{i_1 \dots i_{j-1} l i_{j+1} \dots i_{k-1} m i_{k+1} \dots i_p}\\
&- g^{\alpha l} \sum_{j=1}^{p} R_{\alpha i_j l}{}^m \omega_{i_1 \dots i_{j-1} m i_{j+1} \dots i_p}
\end{aligned}
\tag{27.8}
$$

Let us rewrite the above formula in an orthonormal basis,

$$
\begin{aligned}
(\rho_\omega)_{i_i \dots i_p} = &- \sum_{l,m=1}^{n} \sum_{j=1}^{p} \sum_{k=1, k \neq j}^{p} R_{l i_j m i_k} \omega_{i_1 \dots i_{j-1} l i_{j+1} \dots i_{k-1} m i_{k+1} \dots i_p}\\
&+ \sum_{m=1}^{n} \sum_{j=1}^{p} R_{i_j m} \omega_{i_1 \dots i_{j-1} m i_{j+1} \dots i_p}.
\end{aligned}
\tag{27.9}
$$

Using the algebraic Bianchi identity (1.51), this is

$$R_{li_jmi_k} + R_{lmi_ki_j} + R_{li_ki_jm} = 0, \tag{27.10}$$

which yields

$$R_{li_jmi_k} - R_{mi_jli_k} = R_{lmi_ji_k}. \tag{27.11}$$

Substituting into (27.9) and using skew-symmetry,

$$\begin{aligned}
(\rho_\omega)_{i_i\dots i_p} &= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p}(R_{li_jmi_k} - R_{mi_jli_k})\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_{k-1}mi_{k+1}\dots i_p}\\
&\quad+\sum_{m=1}^{m}\sum_{j=1}^{p}R_{i_jm}\omega_{i_1\dots i_{j-1}mi_{j+1}\dots i_p}\\
&= -\frac{1}{2}\sum_{l,m=1}^{n}\sum_{j=1}^{p}\sum_{k=1,k\neq j}^{p}R_{lmi_ji_k}\omega_{i_1\dots i_{j-1}li_{j+1}\dots i_{k-1}mi_{k+1}\dots i_p}\\
&\quad+\sum_{m=1}^{m}\sum_{j=1}^{p}R_{i_jm}\omega_{i_1\dots i_{j-1}mi_{j+1}\dots i_p}.
\end{aligned} \tag{27.12}$$

**Theorem 27.1.** *If $(M^n, g)$ is closed and has non-negative curvature operator, then any harmonic form is parallel. In this case, $b_1(M) \leq \binom{n}{k}$. If in addition, the curvature operator is positive definite at some point, then any harmonic $p$-form is trivial for $p = 1 \dots n-1$. In this case, $b_p(M) = 0$ for $p = 1 \dots n-1$.*

*Proof.* Let $\omega$ be a harmonic $p$-form. Integrating the Weitzenböck formula (26.14), we obtain

$$0 = \int_M |\nabla\omega|^2 dV + \int_M \langle \rho_\omega, \omega\rangle dV. \tag{27.13}$$

It turns out the the second term is positive if the manifold has positive curvature operator [Poo81, Chapter 4], [Pet06, Chapter 7]. Thus $|\nabla\omega| = 0$ everywhere, so $\omega$ is parallel. A parallel form is determined by its value at a single point, so using the Hodge Theorem, we obtain the first Betti number estimate. If the curvature operator is positive definite at some point, then we see that $\omega$ must vanish at that point, which implies the second Betti number estimate. Note this only works for $p = 1 \dots n-1$, since $\rho_\omega$ is zero in these cases. □

This says that all of the real cohomology of a manifold with positive curvature operator vanishes except for $H^n$ and $H^0$. We say that $M$ is a rational homology sphere (which necessarily has $\chi(M) = 2$). If $M$ is simply-connected and has positive curvature operator, then is $M$ diffeomorphic to a sphere? In dimension 3 this was answered affirmatively by Hamilton in [Ham82]. Hamilton also proved the answer is yes in dimension 4 [Ham86]. Very recently, Böhm and Wilking have shown that the answer is yes in all dimensions [BW06]. The technique is using the Ricci flow, which we will discuss shortly.

We also mention that recently, Brendle and Schoen have shown that manifolds with 1/4-pinched curvature are diffeomorphic to space forms, again using the Ricci flow. If time permits, we will also discuss this later [BS07].

**Remark 27.1.** *On 2-forms, the Weitzenböck formula is*

$$(\Delta_H \omega)_{ij} = -(\Delta w)_{ij} - \sum_{l,m} R_{lmij}\omega_{lm} + \sum_m R_{im}\omega_{mj} + \sum_m R_{jm}\omega_{im}. \tag{27.14}$$

*Through a careful analysis of the curvature terms, M. Berger was able to prove a vanishing theorem for $H^2(M,\mathbb{R})$ provided that the sectional curvature is pinched between* $1$ *and* $2(n-1)/(8n-5)$ *[Ber60].*

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