The governing equation is

$$\frac{dh}{dt} = h\frac{d^2h}{dz^2} + 2\left(\frac{dh}{dz}\right)^2 \tag{1}$$

subject to $h(t, \pm L(t)) = 0$ and $h(t, 0) = 0$. Notice

$$h\frac{dh}{dt} = h^2\frac{d^2h}{dz^2} + 2h\left(\frac{dh}{dz}\right)^2 = \frac{d}{dz}\left(h^2\frac{dh}{dz}\right) \tag{2}$$

$$\frac{d(h^2)}{dt} = \frac{2}{3}\frac{d^2(h^3)}{dz^2}. \tag{3}$$

Introduce the transformation $Z = z/L(t)$ into (3). Notice the transform takes differential form as

$$dh = \left(\frac{\partial h}{\partial t}\right)_Z dt + \left(\frac{\partial h}{\partial Z}\right)_t dZ. \tag{4}$$

Applying (4) to the left side of (3) yields

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z + \left(\frac{\partial h^2}{\partial Z}\right)_t \left(\frac{\partial Z}{\partial t}\right)_z : \tag{5}$$

$$\left(\frac{\partial Z}{\partial t}\right)_z = -\frac{z}{L^2}\frac{dL}{dt} = -\frac{Z}{L}\frac{dL}{dt} \implies \tag{6}$$

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z - Z\left(\frac{\partial h^2}{\partial Z}\right)_t \frac{1}{L}\frac{dL}{dt}. \tag{7}$$

Similarly, applying (4) to the right side of (3) yields

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \frac{\partial}{\partial z}\left(\frac{\partial(h^3)}{\partial Z}\frac{\partial Z}{\partial z}\right)_t : \tag{8}$$

$$\left(\frac{\partial Z}{\partial z}\right)_t = \frac{1}{L} \implies \tag{9}$$

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \left(\frac{\partial^2(h^3)}{\partial Z^2}\frac{1}{L^2}\right)_t. \tag{10}$$

Substituting (7) and (10) into (3) yields the following governing equation

$$\frac{\partial h^2}{\partial t} = Z\frac{\partial h^2}{\partial Z}\frac{1}{L}\frac{dL}{dt} + \frac{2}{3L^2}\frac{\partial^2(h^3)}{\partial Z^2} \tag{11}$$

subject to $h(t, Z = \pm 1) = 0$ and $\partial_Z h(t, Z = 0) = 0$. Before a difference equation can be obtained $L(t)$ must be expressed in known terms; the method follows.

The following argument is valid in the limit as $Z \to 1^+$. Total volume can be expressed as

$$V = \int_{-L}^{L} h^2(t, z) dz \implies \tag{12}$$

$$0 = \frac{\partial V}{\partial t} = \frac{\partial}{\partial t} \int_{-L}^{L} h^2 dz = \int_{-L}^{L} \frac{\partial (h^2)}{\partial t} \, dz + 2h^2(L, t) \frac{dL}{dt} \tag{13}$$

where $\partial_t V = 0$ via conservation of mass. The right side of the equality utilizes Leibniz's rule for commutation between the derivative and the integral since $L = f(t)$. Flow symmetry has also been used. $h(t, z = L) = 0$ implies (13) becomes

$$\frac{\partial V}{\partial t} = \int_{-L}^{L} \left(\frac{\partial h^2}{\partial t} \right) dz \implies \tag{14}$$

$$\frac{\partial V}{\partial t} = \frac{4}{3} \left(\left. \frac{\partial h^3}{\partial z} \right|_{z=L} - \left. \frac{\partial h^3}{\partial z} \right|_{z=0} \right) \implies \tag{15}$$

$$0 = \left. \frac{\partial h}{\partial z} \right|_{z=L} \tag{16}$$

via the Fundamental Theorem of Calculus. Notice (16) is a new boundary condition. (15) was established using (3).

Expand h in a Taylor series about $Z = 1$, where the centering is predicated on satisfying the boundary condition $h = 0$ at $Z = 1$. Doing so yields

$$h = \sum_{\mathbb{N}} c_n (Z - 1)^n : c_n := \left. \frac{\partial^n h}{\partial Z^n} \right|_{Z=1}. \tag{17}$$

Rewriting (11) via (17) yields the following weighted expression:[1]

$$\mathcal{O}(Z - 1) : L \frac{dL}{dt} = -\frac{2}{3} c_1. \tag{18}$$

Notice

[1]The domain under consideration is $V_\epsilon(Z = 1) : Z \neq 1$, as the following is valid in the limit as $Z \to 1^+$.

$$c_1(t) := \left(\frac{\partial h}{\partial Z}\right)_{Z=1} \approx \frac{h(t,1) - h(t,1-\Delta Z)}{\Delta Z} \tag{19}$$

yet $h(t, Z = 1) = 0$; thus we may write (19) as

$$c_1(t) \approx -\frac{h(t,1-\Delta Z)}{\Delta Z}. \tag{20}$$

Substituting (20) into (18) yields

$$\int_{L(t)}^{L(t+\Delta t)} L' dL' = \frac{2}{3}\int_t^{t+\Delta t} \frac{h(t',1-\Delta Z)}{\Delta Z} dt' \implies \tag{21}$$

$$L^2(t+\Delta t) = L^2(t) + \frac{4}{3}\frac{h(t,1-\Delta Z)}{\Delta Z}\Delta t := \xi. \tag{22}$$

Thus we have an expression for $L^2(t + \Delta t)$. The time integral was approximated using a left endpoint rule. Next we must find dL^2/dt. (22) can be rewritten as

$$\frac{L^2(t+\Delta t) - L^2(t)}{\Delta t} = \frac{4}{3}\frac{h(t,1-\Delta Z)}{\Delta Z}. \tag{23}$$

Notice the left side term in (23) is approximately dL^2/dt. The nonlinear L term in (11) can be expressed as

$$\frac{1}{L}\frac{dL}{dt} = \frac{L}{L^2}\frac{dL}{dt} = \frac{1}{2L^2}\frac{dL^2}{dt} \tag{24}$$

which is approximated numerically from (23) and (22) as

$$\left.\frac{1}{2L^2}\frac{dL^2}{dt}\right|_{t=t+\Delta t} = \frac{\frac{4}{3}\frac{h(t,1-\Delta Z)}{\Delta Z}}{L^2(t) + \frac{4}{3}\frac{h(t,1-\Delta Z)}{\Delta Z}\Delta t} := \zeta. \tag{25}$$

Notice (11) can be rewritten as

$$\frac{\partial h}{\partial t} = Z\frac{\partial h}{\partial Z}\frac{1}{L}\frac{dL}{dt} + \frac{h}{L^2}\frac{\partial^2 h}{\partial Z^2}. \tag{26}$$

A forward time-centered space finite difference scheme for (26) is

$$h_i^{j+1} = \left[i\left(h_{i+1}^j - h_{i-1}^j\right)\zeta^j + \frac{h_i^j}{\xi^j}\frac{h_{i+1}^j - 2h_i^j + h_{i-1}^j}{\Delta Z^2}\right]\Delta t + h_i^j \tag{27}$$

where i appears since $Z/\Delta Z$ appears and $Z = i\Delta Z$ by definition. A finite difference scheme for ξ and ζ is

$$\xi^j = L^{j-1\,2} + \frac{4}{3}\frac{h_{n-1}^{j-1}}{\Delta Z}\Delta t \tag{28}$$

$$\zeta^j = \frac{4}{3}\frac{h_{n-1}^{j-1}}{\xi^j \Delta Z}. \tag{29}$$

Note that n is the number of spacial nodes predefined by the user. Additionally the subscript i is the i^{th} spacial node and the superscript j is the j^{th} time node. Then it is clear if some initial L^0 and h^0 are defined, all subsequent h profiles can be found.