

The governing equation is

$$\frac{dh}{dt} = h \frac{d^2h}{dz^2} + 2 \left( \frac{dh}{dz} \right)^2 \quad (1)$$

subject to  $h(t, \pm L(t)) = 0$  and  $h(t, 0) = 0$ . Notice

$$h \frac{dh}{dt} = h^2 \frac{d^2h}{dz^2} + 2h \left( \frac{dh}{dz} \right)^2 = \frac{d}{dz} \left( h^2 \frac{dh}{dz} \right) \quad (2)$$

$$\frac{d(h^2)}{dt} = \frac{2}{3} \frac{d^2(h^3)}{dz^2}. \quad (3)$$

Introduce the transformation  $Z = z/L(t)$  into (3). Notice the transform takes differential form as

$$dh = \left( \frac{\partial h}{\partial t} \right)_Z dt + \left( \frac{\partial h}{\partial Z} \right)_t dZ. \quad (4)$$

Applying (4) to the left side of (3) yields

$$\left( \frac{\partial h^2}{\partial t} \right)_z = \left( \frac{\partial h^2}{\partial t} \right)_Z + \left( \frac{\partial h^2}{\partial Z} \right)_t \left( \frac{\partial Z}{\partial t} \right)_z : \quad (5)$$

$$\left( \frac{\partial Z}{\partial t} \right)_z = -\frac{z}{L^2} \frac{dL}{dt} = -\frac{Z}{L} \frac{dL}{dt} \implies \quad (6)$$

$$\left( \frac{\partial h^2}{\partial t} \right)_z = \left( \frac{\partial h^2}{\partial t} \right)_Z - Z \left( \frac{\partial h^2}{\partial Z} \right)_t \frac{1}{L} \frac{dL}{dt}. \quad (7)$$

Similarly, applying (4) to the right side of (3) yields

$$\left( \frac{\partial^2}{\partial z^2} h^3 \right)_t = \frac{\partial}{\partial z} \left( \frac{\partial(h^3)}{\partial Z} \frac{\partial Z}{\partial z} \right)_t : \quad (8)$$

$$\left( \frac{\partial Z}{\partial z} \right)_t = \frac{1}{L} \implies \quad (9)$$

$$\left( \frac{\partial^2}{\partial z^2} h^3 \right)_t = \left( \frac{\partial^2(h^3)}{\partial Z^2} \frac{1}{L^2} \right)_t. \quad (10)$$

Substituting (7) and (10) into (3) yields the following governing equation

$$\frac{\partial h^2}{\partial t} = Z \frac{\partial h^2}{\partial Z} \frac{1}{L} \frac{dL}{dt} + \frac{2}{3L^2} \frac{\partial^2(h^3)}{\partial Z^2} \quad (11)$$

subject to  $h(t, Z = \pm 1) = 0$  and  $\partial_Z h(t, Z = 0) = 0$ . Before a difference equation can be obtained  $L(t)$  must be expressed in known terms; the method follows.

The following argument is valid in the limit as  $Z \rightarrow 1^-$ . Total volume can be expressed as

$$V = \int_{-L}^L h^2(t, z) dz \implies \quad (12)$$

$$0 = \frac{\partial V}{\partial t} = \frac{\partial}{\partial t} \int_{-L}^L h^2 dz = \int_{-L}^L \frac{\partial(h^2)}{\partial t} dz + 2h^2(L, t) \frac{dL}{dt} \quad (13)$$

where  $\partial_t V = 0$  via conservation of mass. The right side of (13) utilizes Leibniz's rule for commutation between the derivative and the integral since  $L = f(t)$ . Flow symmetry has also been used.  $h(t, z = L) = 0$  implies (13) becomes

$$\frac{\partial V}{\partial t} = \int_{-L}^L \left( \frac{\partial h^2}{\partial t} \right) dz \implies \quad (14)$$

$$\frac{\partial V}{\partial t} = \frac{4}{3} \left( \frac{\partial h^3}{\partial z} \Big|_{z=L} - \frac{\partial h^3}{\partial z} \Big|_{z=0} \right) \implies \quad (15)$$

$$0 = \frac{\partial h^3}{\partial z} \Big|_{z=L} \quad (16)$$

via the Fundamental Theorem of Calculus. Notice (16) is a new boundary condition. (15) was established using (3).

Expand  $h$  in a Taylor series about  $Z = 1$ , where the centering is predicated on satisfying the boundary condition  $h = 0$  at  $Z = 1$ . Doing so yields

$$h = \sum_{\mathbb{N}} c_n (Z - 1)^n : c_n := \frac{\partial^n h}{\partial Z^n} \Big|_{Z=1}. \quad (17)$$

Rewriting (11) via (17) yields the following weighted expression:<sup>1</sup>

$$\mathcal{O}(Z - 1) : L \frac{dL}{dt} = -\frac{2}{3} c_1. \quad (18)$$

Notice

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<sup>1</sup>The domain under consideration is  $V_\epsilon(Z = 1) : Z \neq 1$ , as the following is valid in the limit as  $Z \rightarrow 1^-$ .

$$c_1(t) := \left( \frac{\partial h}{\partial Z} \right)_{Z=1} \approx \frac{h(t, 1) - h(t, 1 - \Delta Z)}{\Delta Z} \quad (19)$$

yet  $h(t, Z = 1) = 0$ ; thus we may write (19) as

$$c_1(t) \approx -\frac{h(t, 1 - \Delta Z)}{\Delta Z}. \quad (20)$$

Substituting (20) into (18) yields

$$\int_{L(t)}^{L(t+\Delta t)} L' dL' = \frac{2}{3} \int_t^{t+\Delta t} \frac{h(t', 1 - \Delta Z)}{\Delta Z} dt' \implies \quad (21)$$

$$L^2(t + \Delta t) = L^2(t) + \frac{4}{3} \frac{h(t, 1 - \Delta Z)}{\Delta Z} \Delta t. \quad (22)$$

Thus we have an expression for  $L^2(t + \Delta t)$ . The time integral was approximated using a left endpoint rule. Next we must find  $dL^2/dt$ . (22) can be rewritten as

$$\frac{L^2(t + \Delta t) - L^2(t)}{\Delta t} = \frac{4}{3} \frac{h(t, 1 - \Delta Z)}{\Delta Z}. \quad (23)$$

Notice the left side term in (23) is approximately  $dL^2/dt$ . The nonlinear  $L$  term in (11) can be expressed as

$$\frac{1}{L} \frac{dL}{dt} = \frac{L}{L^2} \frac{dL}{dt} = \frac{1}{2L^2} \frac{dL^2}{dt} \quad (24)$$

which is approximated numerically from (23) and (22) as

$$\left. \frac{1}{L^2} \frac{dL^2}{dt} \right|_{t=t+\Delta t} = \frac{\frac{4}{3} \frac{h(t, 1 - \Delta Z)}{\Delta Z}}{L^2(t) + \frac{4}{3} \frac{h(t, 1 - \Delta Z)}{\Delta Z} \Delta t}. \quad (25)$$

With the  $L$  terms now represented in (11), the next step is to create a numerical scheme to solve. A finite difference scheme for (11) is computationally draining. In order to alleviate the computational stress we introduce a transform that reduces the nonlinearity from (11)

$$y := h^2. \quad (26)$$

(26) not only expedites the run time, but the finite difference equations that result in this approach automatically conserve volume, unlike the scheme

involving  $h$ , in which volume is conserved only in the limit of vanishing  $\Delta Z$ . Applying this transform to (11) yields

$$\frac{\partial y}{\partial t} = Z \frac{\partial y}{\partial Z} \frac{1}{L} \frac{dL}{dt} + \frac{2}{3L^2} \frac{\partial^2 y^{3/2}}{\partial Z^2}. \quad (27)$$

Transforming (23) and (25) with (26) respectively yields

$$\frac{L^2(t + \Delta t) - L^2(t)}{\Delta t} = \frac{4}{3} \frac{\sqrt{y(t, 1 - \Delta Z)}}{\Delta Z} =: \zeta(t) \quad (28)$$

$$\left. \frac{1}{L^2} \frac{dL^2}{dt} \right|_{t=t+\Delta t} = \frac{\frac{4}{3} \frac{y\sqrt{(t, 1-\Delta Z)}}{\Delta Z}}{L^2(t) + \frac{4}{3} \frac{\sqrt{y(t, 1-\Delta Z)}}{\Delta Z} \Delta t} =: \xi(t). \quad (29)$$

(27) can now be succinctly written as

$$\frac{\partial y}{\partial t} = 2Z \frac{\partial y}{\partial Z} \xi + \frac{2}{3\zeta} \frac{\partial^2 y^{3/2}}{\partial Z^2}. \quad (30)$$

**Claim 1.** *The partial derivative of a function  $f$  lifted to the  $m \in \mathbb{R}$  power can be finitely differenced lucidly as*

$$\frac{\partial f^m}{\partial x} = \frac{f^m|_{x+\Delta x} - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2. \quad (31)$$

*Proof.* Consider a typical method for taking a finite difference of  $f^m$ :

$$\frac{\partial f^m}{\partial x} = m f^{m-1}|_x \frac{f|_{x+\Delta x} - f|_x}{\Delta x} + \mathcal{O}(\Delta x)^2. \quad (32)$$

Now expand  $f|_{x+\Delta x}$  in a Taylor series about  $x$ :

$$f|_{x+\Delta x} = \sum_{\mathbb{N}} \frac{\partial_n f|_x \Delta x^n}{n!} = f|_x + \partial_x f|_x \Delta x + \mathcal{O}(\Delta x)^2. \quad (33)$$

Substituting (33) into (32) and (31) for  $f|_{x+\Delta x}$  demonstrates equality in both methods to  $\mathcal{O}(\Delta x^2)$ . An inductive argument shows this technique holds for the  $n^{\text{th}}$  partial derivative.  $\square$

The validity of Claim 1 implies the following forward time centered space finite difference equation is valid for (30)

$$y_i^{j+1} = \left[ i(y_{i+1}^j - y_{i-1}^j) \zeta^j + \frac{2}{3\zeta^j} \frac{y_{i-1}^{j \ 3/2} - 2 y_i^{j \ 3/2} + y_{i+1}^{j \ 3/2}}{\Delta Z^2} \right] \Delta t + y_i^j. \quad (34)$$

A finite difference scheme for  $\xi$  in (29) and  $\zeta$  in (28) follows:

$$\zeta^j = L^{j-1} + \frac{4}{3} \frac{\sqrt{y_{n-1}^{j-1}}}{\Delta Z} \Delta t \quad (35)$$

$$\xi^j = \frac{4}{3} \frac{\sqrt{y_{n-1}^{j-1}}}{\xi^j \Delta Z} \quad (36)$$

where  $n$  is the number of spacial nodes predefined by the user. Additionally the subscript  $i$  is the  $i^{th}$  spacial node and the superscript  $j$  is the  $j^{th}$  time node. Then it is clear if some initial  $L^0$  and  $h^0$  are defined, all subsequent  $h$  profiles can be found.