The governing equation is

$$\frac{dh}{dt} = h\frac{d^2h}{dz^2} + 2\left(\frac{dh}{dz}\right)^2 \tag{1}$$

subject to $h(t, \pm L(t)) = 0$ and $h(t, 0) = 0$. Notice

$$h\frac{dh}{dt} = h^2\frac{d^2h}{dz^2} + 2h\left(\frac{dh}{dz}\right)^2 = \frac{d}{dz}\left(h^2\frac{dh}{dz}\right) \tag{2}$$

$$\frac{d(h^2)}{dt} = \frac{2}{3}\frac{d^2(h^3)}{dz^2}. \tag{3}$$

Introduce the transformation $Z = z/L(t)$ into (3). The transform takes differential form as

$$dh = \left(\frac{\partial h}{\partial t}\right)_Z dt + \left(\frac{\partial h}{\partial Z}\right)_t dZ. \tag{4}$$

Applying (4) to the left side of (3) yields

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z + \left(\frac{\partial h^2}{\partial Z}\right)_t \left(\frac{\partial Z}{\partial t}\right)_z : \tag{5}$$

$$\left(\frac{\partial Z}{\partial t}\right)_z = -\frac{z}{L^2}\frac{dL}{dt} = -\frac{Z}{L}\frac{dL}{dt} \implies \tag{6}$$

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z - Z\left(\frac{\partial h^2}{\partial Z}\right)_t \frac{1}{L}\frac{dL}{dt}. \tag{7}$$

Similarly, applying (4) to the right side of (3) yields

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \frac{\partial}{\partial z}\left(\frac{\partial(h^3)}{\partial Z}\frac{\partial Z}{\partial z}\right)_t : \tag{8}$$

$$\left(\frac{\partial Z}{\partial z}\right)_t = \frac{1}{L} \implies \tag{9}$$

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \left(\frac{\partial^2(h^3)}{\partial Z^2}\frac{1}{L^2}\right)_t. \tag{10}$$

Substituting (7) and (10) into (3) yields the following governing equation

$$\frac{\partial h^2}{\partial t} = Z\frac{\partial h^2}{\partial Z}\frac{1}{L}\frac{dL}{dt} + \frac{2}{3L^2}\frac{\partial^2(h^3)}{\partial Z^2} \tag{11}$$

subject to $h(t, Z = \pm 1) = 0$ and $\partial_Z h(t, Z = 0) = 0$. Before a difference equation can be obtained $L(t)$ must be expressed in known terms; the method follows.

The following argument is valid in the limit as $Z \to 1^-$. Expand h in a Taylor series about $Z = 1$, where the centering is predicated on satisfying the boundary condition $h = 0$ at $Z = 1$. Doing so yields

$$h = \sum_{\mathbb{N}} c_n (Z-1)^n : c_n := \left.\frac{\partial^n h}{\partial Z^n}\right|_{Z=1}. \tag{12}$$

Rewriting (11) via (12) yields the following weighted expression:[1]

$$\mathcal{O}(Z-1) : L\frac{dL}{dt} = -\frac{2}{3}c_1. \tag{13}$$

Notice

$$c_1(t) := \left.\frac{\partial h}{\partial Z}\right|_{Z=1} \approx \frac{h(t,1) - h(t, 1-\Delta Z)}{\Delta Z} \tag{14}$$

yet $h(t, Z = 1) = 0$; thus we may write (14) as

$$c_1(t) \approx -\frac{h(t, 1-\Delta Z)}{\Delta Z}. \tag{15}$$

Substituting (15) into (13) yields

$$\int_{L(t)}^{L(t+\Delta t)} L' dL' = \frac{2}{3}\int_t^{t+\Delta t} \frac{h(t', 1-\Delta Z)}{\Delta Z} dt' \implies \tag{16}$$

$$L^2(t+\Delta t) = L^2(t) + \frac{2}{3}\frac{h(t, 1-\Delta Z)}{\Delta Z}\Delta t. \tag{17}$$

Thus we have an expression for $L^2(t + \Delta t)$. The time integral was approximated using a midpoint rule in time at $\Delta t/2$. Before continuing observe that the nonlinear L term in (11) can be expressed as

$$\frac{1}{L}\frac{dL}{dt} = \frac{L}{L^2}\frac{dL}{dt} = \frac{1}{2L^2}\frac{dL^2}{dt}. \tag{18}$$

Thus we seek an expression for dL^2/dt. Notice (17) can be rewritten as

[1] The domain under consideration is $V_\epsilon(Z = 1) : Z \neq 1$, as the following is valid in the limit as $Z \to 1^-$.

$$\frac{L^2(t+\Delta t) - L^2(t)}{\Delta t} = \frac{2}{3}\frac{h(t, 1-\Delta Z)}{\Delta Z} \tag{19}$$

where the left side term is approximately dL^2/dt.

The L terms from (11) are now expressed in terms of known quantities; the next step is to create a numerical scheme to solve for h. A finite difference scheme for (11) is onerous and computationally draining. In order to alleviate numerical stress, introduce a transform that rids the nonlinearity in the time derivative in (11) as

$$y := h^2. \tag{20}$$

(20) not only expedites the run time, but the finite difference equations that result in this approach automatically conserve volume, unlike the scheme involving h, in which volume is conserved only in the limit of vanishing ΔZ. Applying (20) and (18) to (11) yields

$$\frac{\partial y}{\partial t} = Z\frac{\partial y}{\partial Z}\frac{1}{2L^2}\frac{dL^2}{dt} + \frac{2}{3L^2}\frac{\partial^2 y^{3/2}}{\partial Z^2}. \tag{21}$$

Transforming (17) and (19) with (20) respectively yields

$$L^2(t+\Delta t) = L^2(t) + \frac{2}{3}\frac{\sqrt{y(t, 1-\Delta Z)}}{\Delta Z}\Delta t \tag{22}$$

$$\left.\frac{dL^2}{dt}\right|_t = \frac{2}{3}\frac{\sqrt{y(t, 1-\Delta Z)}}{\Delta Z}. \tag{23}$$

Note (22) is evaluated at $t + \Delta t$ yet (23) is evaluate at t. Before continuing to a finite difference technique for (21) the following claim is stated and proved.

Claim 1. *The partial derivative of a function f lifted to the $m \in \mathbb{R}$ power can be finitely differenced lucidly as*

$$\frac{\partial f^m}{\partial x} = \frac{f^m|_{x+\Delta x} - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{24}$$

Proof. Consider a typical method for taking a finite difference of f^m:

$$\frac{\partial f^m}{\partial x} = m\ f^{m-1}\big|_x \frac{f|_{x+\Delta x} - f|_x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{25}$$

Now expand $f|_{x+\Delta x}$ in a Taylor series about x:

$$f|_{x+\Delta x} = \sum_{\mathbb{N}} \frac{\partial_n f|_x \Delta x^n}{n!} = f|_x + \partial_x f|_x \Delta x + \mathcal{O}(\Delta x)^2. \tag{26}$$

Substituting (26) through $\mathcal{O}(\Delta x)$ into (25) for $f|_{x+\Delta x}$ yields

$$\frac{\partial f^m}{\partial x} = m\ f^{m-1}\big|_x \frac{f|_x + \partial_x f|_x \Delta x - f|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{27}$$

$$= m\ f^{m-1}\big|_x \frac{\partial_x f|_x \Delta x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{28}$$

Substituting (26) through $\mathcal{O}(\Delta x)$ into (24) for $f|_{x+\Delta x}$ yields

$$\frac{\partial f^m}{\partial x} = \frac{(f|_x + \partial_x f|_x \Delta x)^m - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{29}$$

$$= \frac{f^m|_x + m f^{m-1}|_x \partial_x f|_x \Delta x - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{30}$$

$$= \frac{m f^{m-1}|_x \partial_x f|_x \Delta x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{31}$$

The equality of (28) and (31) demonstrate equality in both methods to $\mathcal{O}(\Delta x^2)$. An inductive argument shows this technique holds for the n^{th} partial derivative. □

The validity of Claim 1 implies the following forward time centered space finite difference equation is valid for (21)

$$y_i^{j+1} = y_i^j + \left[i(y_{i+1}^j - y_{i-1}^j) \frac{1}{2\ L^{j\ 2}} \left(\frac{dL^2}{dt} \right)^j + \frac{2}{3\ L^{j\ 2}} \frac{{y_{i-1}^j}^{3/2} - 2\ {y_i^j}^{3/2} + {y_{i+1}^j}^{3/2}}{\Delta Z^2} \right] \Delta t. \tag{32}$$

A finite difference scheme for (23) and (22) follows:

$$L^{j\ 2} = L^{j-1\ 2} + \frac{2}{3} \frac{\sqrt{y_{n-1}^{j-1}}}{\Delta Z} \Delta t \implies \tag{33}$$

$$L^j = \sqrt{L^{j-1\ 2} + \frac{2}{3} \frac{\sqrt{y_{n-1}^{j-1}}}{\Delta Z} \Delta t} \tag{34}$$

$$\left(\frac{dL^2}{dt} \right)^j = \frac{2}{3} \frac{\sqrt{y_{n-1}^{j-1}}}{\Delta Z} \tag{35}$$

where n is the number of spacial nodes predefined by the user. Additionally the subscript i is the i^{th} spacial node and the superscript j is the j^{th} time node. Then it is clear if some initial L^0 and h^0 are defined, all subsequent h profiles can be found.