The governing equation is

$$\frac{dh}{dt} = h\frac{d^2h}{dz^2} + 2\left(\frac{dh}{dz}\right)^2 \tag{1}$$

subject to $h(t, \pm L(t)) = 0$ and $h(t, 0) = 0$. Notice

$$h\frac{dh}{dt} = h^2\frac{d^2h}{dz^2} + 2h\left(\frac{dh}{dz}\right)^2 = \frac{d}{dz}\left(h^2\frac{dh}{dz}\right) \tag{2}$$

$$\frac{d(h^2)}{dt} = \frac{2}{3}\frac{d^2(h^3)}{dz^2}. \tag{3}$$

Introduce the transformation $Z = z/L(t)$ into (3). The transform takes differential form as

$$dh = \left(\frac{\partial h}{\partial t}\right)_Z dt + \left(\frac{\partial h}{\partial Z}\right)_t dZ. \tag{4}$$

Applying (4) to the left side of (3) yields

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z + \left(\frac{\partial h^2}{\partial Z}\right)_t \left(\frac{\partial Z}{\partial t}\right)_z : \tag{5}$$

$$\left(\frac{\partial Z}{\partial t}\right)_z = -\frac{z}{L^2}\frac{dL}{dt} = -\frac{Z}{L}\frac{dL}{dt} \implies \tag{6}$$

$$\left(\frac{\partial h^2}{\partial t}\right)_z = \left(\frac{\partial h^2}{\partial t}\right)_Z - Z\left(\frac{\partial h^2}{\partial Z}\right)_t \frac{1}{L}\frac{dL}{dt}. \tag{7}$$

Similarly, applying (4) to the right side of (3) yields

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \frac{\partial}{\partial z}\left(\frac{\partial(h^3)}{\partial Z}\frac{\partial Z}{\partial z}\right)_t : \tag{8}$$

$$\left(\frac{\partial Z}{\partial z}\right)_t = \frac{1}{L} \implies \tag{9}$$

$$\left(\frac{\partial^2}{\partial z^2}h^3\right)_t = \left(\frac{\partial^2(h^3)}{\partial Z^2}\frac{1}{L^2}\right)_t. \tag{10}$$

Substituting (7) and (10) into (3) yields the following governing equation

$$\frac{\partial h^2}{\partial t} = Z\frac{\partial h^2}{\partial Z}\frac{1}{L}\frac{dL}{dt} + \frac{2}{3L^2}\frac{\partial^2(h^3)}{\partial Z^2} \tag{11}$$

subject to $h(t, Z = \pm 1) = 0$ and $\partial_Z h(t, Z = 0) = 0$. Before a difference equation can be obtained $L(t)$ must be expressed in known terms; the method follows.

The following argument is valid in the limit as $Z \to 1^-$. Expand $h$ in a Taylor series about $Z = 1$, where the centering is predicated on satisfying the boundary condition $h = 0$ at $Z = 1$. Doing so yields

$$h = \sum_{\mathbb{N}} c_n (Z-1)^n : c_n := \left. \frac{\partial^n h}{\partial Z^n} \right|_{Z=1}. \tag{12}$$

Rewriting (11) via (12) yields the following weighted expression:[1]

$$\mathcal{O}(Z-1) : L\frac{dL}{dt} = -\frac{2}{3}c_1. \tag{13}$$

Notice

$$c_1(t) := \left. \frac{\partial h}{\partial Z} \right|_{Z=1} \approx \frac{h(t,1) - h(t, 1-\Delta Z)}{\Delta Z} \tag{14}$$

yet $h(t, Z = 1) = 0$; thus we may write (14) as

$$c_1(t) \approx -\frac{h(t, 1-\Delta Z)}{\Delta Z}. \tag{15}$$

Substituting (15) into (13) yields

$$\int_{L(t)}^{L(t+\Delta t)} L' dL' = \frac{2}{3} \int_t^{t+\Delta t} \frac{h(t', 1-\Delta Z)}{\Delta Z} dt' \implies \tag{16}$$

$$L^2(t+\Delta t) = L^2(t) + \frac{4}{3}\frac{h(t, 1-\Delta Z)}{\Delta Z}\Delta t. \tag{17}$$

Thus we have an expression for $L^2(t + \Delta t)$. The time integral was approximated using a midpoint rule in time at $\Delta t/2$. Before continuing observe that the nonlinear $L$ term in (11) can be expressed as

$$\frac{1}{L}\frac{dL}{dt} = \frac{L}{L^2}\frac{dL}{dt} = \frac{1}{2L^2}\frac{dL^2}{dt}. \tag{18}$$

Thus we seek an expression for $dL^2/dt$. Notice (17) can be rewritten as

[1] The domain under consideration is $V_\epsilon(Z = 1) : Z \neq 1$, as the following is valid in the limit as $Z \to 1^-$.

$$\frac{L^2(t+\Delta t)-L^2(t)}{\Delta t}=\frac{4}{3}\frac{h(t,1-\Delta Z)}{\Delta Z} \tag{19}$$

where the left side term is approximately $dL^2/dt$.

The $L$ terms from (11) are now expressed in terms of known quantities; the next step is to create a numerical scheme to solve for $h$. A finite difference scheme for (11) is onerous and computationally draining. In order to alleviate numerical stress, introduce a transform that rids the nonlinearity in the time derivative in (11) as

$$y := h^2. \tag{20}$$

(20) not only expedites the run time, but the finite difference equations that result in this approach automatically conserve volume, unlike the scheme involving $h$, in which volume is conserved only in the limit of vanishing $\Delta Z$. Applying (20) and (18) to (11) yields

$$\frac{\partial y}{\partial t}=Z\frac{\partial y}{\partial Z}\frac{1}{2L^2}\frac{dL^2}{dt}+\frac{2}{3L^2}\frac{\partial^2 y^{3/2}}{\partial Z^2}. \tag{21}$$

Transforming (17) and (19) with (20) respectively yields

$$L^2(t+\Delta t)=L^2(t)+\frac{4}{3}\frac{\sqrt{y(t,1-\Delta Z)}}{\Delta Z}\Delta t \tag{22}$$

$$\left.\frac{dL^2}{dt}\right|_t=\frac{4}{3}\frac{\sqrt{y(t,1-\Delta Z)}}{\Delta Z}. \tag{23}$$

Note (22) is evaluated at $t+\Delta t$ yet (23) is evaluate at $t$. Before continuing to a finite difference technique for (21) the following claim is stated and proved.

**Claim 1.** *The partial derivative of a function $f$ lifted to the $m \in \mathbb{R}$ power can be finitely differenced lucidly as*

$$\frac{\partial f^m}{\partial x}=\frac{f^m|_{x+\Delta x}-f^m|_x}{\Delta x}+\mathcal{O}(\Delta x)^2. \tag{24}$$

*Proof.* Consider a typical method for taking a finite difference of $f^m$:

$$\frac{\partial f^m}{\partial x}=m\ f^{m-1}\big|_x\frac{f|_{x+\Delta x}-f|_x}{\Delta x}+\mathcal{O}(\Delta x)^2. \tag{25}$$

Now expand $f|_{x+\Delta x}$ in a Taylor series about $x$:

$$f|_{x+\Delta x} = \sum_{\mathbb{N}} \frac{\partial_n f|_x \Delta x^n}{n!} = f|_x + \partial_x f|_x \Delta x + \mathcal{O}(\Delta x)^2. \tag{26}$$

Substituting (26) through $\mathcal{O}(\Delta x)$ into (25) for $f|_{x+\Delta x}$ yields

$$\frac{\partial f^m}{\partial x} = m\ f^{m-1}\big|_x \frac{f|_x + \partial_x f|_x \Delta x - f|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{27}$$

$$= m\ f^{m-1}\big|_x \frac{\partial_x f|_x \Delta x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{28}$$

Substituting (26) through $\mathcal{O}(\Delta x)$ into (24) for $f|_{x+\Delta x}$ yields

$$\frac{\partial f^m}{\partial x} = \frac{(f|_x + \partial_x f|_x \Delta x)^m - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{29}$$

$$= \frac{f^m|_x + m f^{m-1}|_x \partial_x f|_x \Delta x - f^m|_x}{\Delta x} + \mathcal{O}(\Delta x)^2 \tag{30}$$

$$= \frac{m f^{m-1}|_x \partial_x f|_x \Delta x}{\Delta x} + \mathcal{O}(\Delta x)^2. \tag{31}$$

The equality of (28) and (31) demonstrate equality in both methods to $\mathcal{O}(\Delta x^2)$. An inductive argument shows this technique holds for the $n^{th}$ partial derivative. $\square$

The validity of Claim 1 implies the following forward time centered space finite difference equation is valid for (21)

$$y_i^{j+1} = y_i^j + \left[ i(y_{i+1}^j - y_{i-1}^j) \frac{1}{4\ {L^j}^2} \left( \frac{dL^2}{dt} \right)^j + \frac{2}{3\ {L^j}^2} \frac{{y_{i-1}^j}^{3/2} - 2\ {y_i^j}^{3/2} + {y_{i+1}^j}^{3/2}}{\Delta Z^2} \right] \Delta t. \tag{32}$$

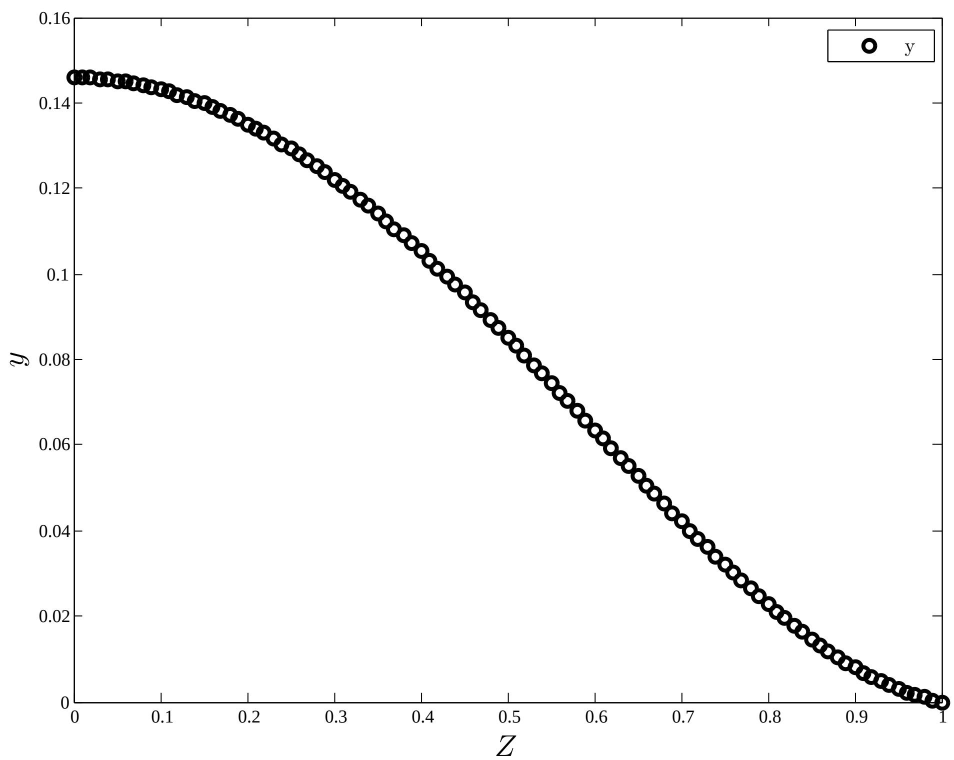
A finite difference scheme for (23) and (22) follows:

$${L^{j+1}}^2 = {L^j}^2 + \frac{4}{3} \frac{\sqrt{y_{n-1}^j}}{\Delta Z} \Delta t \tag{33}$$

$$\left( \frac{dL^2}{dt} \right)^j = \frac{4}{3} \frac{\sqrt{y_{n-1}^j}}{\Delta Z} \tag{34}$$

where $n$ is the number of spacial nodes predefined by the user. Additionally the subscript $i$ is the $i^{th}$ spacial node and the superscript $j$ is the $j^{th}$ time node.

Then it is clear if some initial $L^0$ and $h^0$ are defined, all subsequent $h$ profiles can be found.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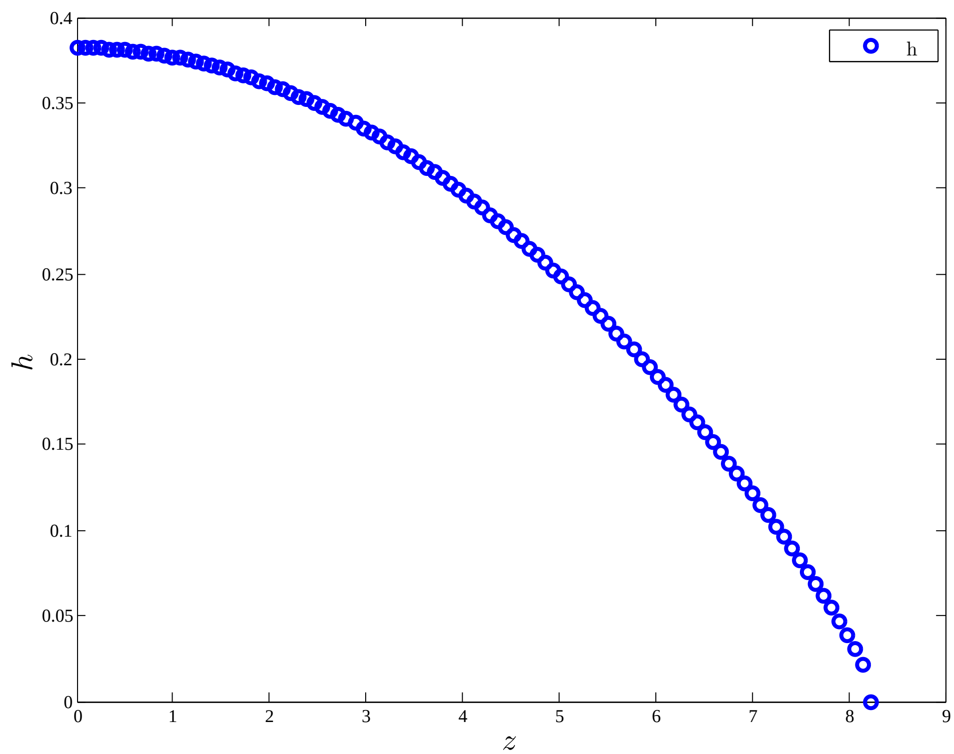
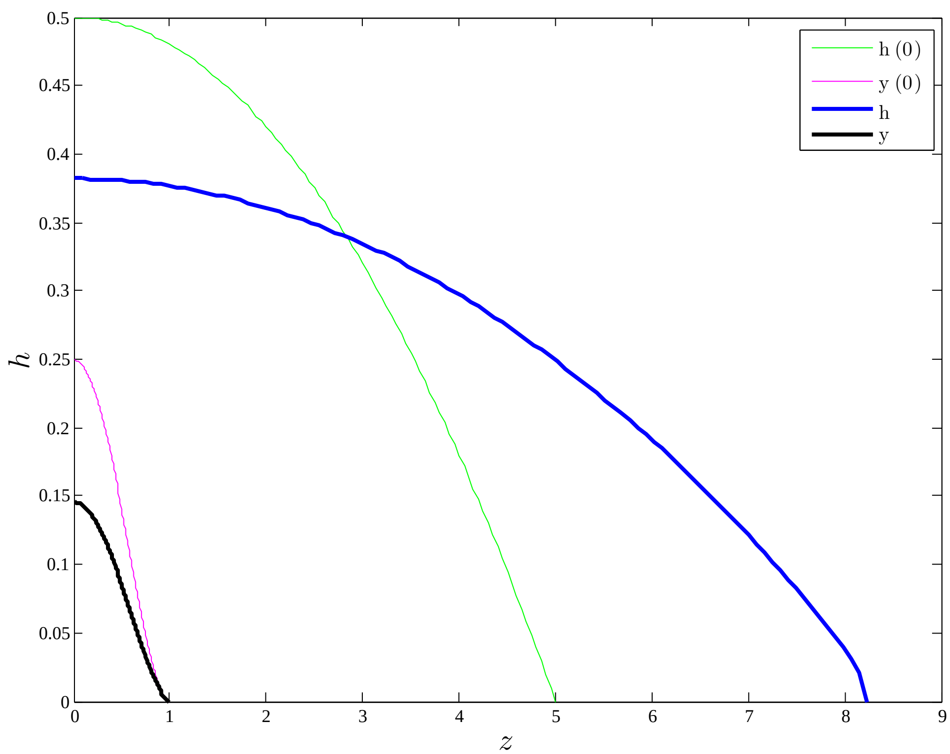
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Z
y



0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
h
0
1
2
3
4
5
6
7
8
9
z
h


h (0)
y (0)
h
y
h
z
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9

# 1 MATLAB Code

```
clear all; close all; clc
%% Constants
Hi = 1;% initial maximum height
L  = 5;% initial max length of tip

% Intervals
M=100;% amount of space nodes
tf = 14;% final time to run code;
N=M^2;% amount of time nodes
dz = L/M;
dt = tf/(N-1);
zvec = 0:dz:L;% z vector
ndt = tf/dt;% amount of time iterations

%% Initial conditions
    h = -(Hi/L^2)*zvec.^2+Hi;% initial height profile, which is quadratic by guess
    plot(zvec,h,'g')% original plot over h and z
    hold on

    % Initial Volume
    iv = 0;
    for i=1:length(h)-1
        iv = iv+((h(i)+h(i+1))/2)^2*dz;
    end% for i

    %% y and Z transform
    y = h.^2;% transform h-->y
    ZVEC = zvec./L;% transform z-->Z
    dz = ZVEC(2)-ZVEC(1);% redefine dz from Z transform
    L2 = L^2;% initial L^2 value
    plot(ZVEC,y,'m')% plot transform y over Z

    ynew = y;% new y storage vector for finite difference

%% Time loop and spacial loop for finite differencing
    for jj = 2:ndt; % Solve h profile during next dt (ndt should be placed here)
        y3 = y.^(3/2);% calculates y^1.5
        ysq = y.^(1/2);% calculates square root of y
        L2 = L2+4/3*ysq(end-1)/(dz)*dt;% Calculates new L^2
        DL2 = 4/3*ysq(end-1)/dz;% calculates derivative
```

```
of L^2

    %% Finite difference in space
        ynew(1) = 4/(3*L2)*(y3(2)-y3(1))/dz^2*dt+y(1);% reflective boundary condition for evaluating y(1)
        for ii = 2:length(ZVEC)-1;  % move along z and solve
            ynew(ii) = y(ii)+(ii*(y(ii+1)-y(ii-1))/(4*L2)*DL2...
                +2/(3*L2)*(y3(ii-1)-2*y3(ii)+y3(ii+1))/(dz^2))*dt;% forward time, centered space
        end% for ii
        y = ynew;% overright previous y with new profile
    end % for jj

    h = ynew.^(1/2);% transform y-->h
    zvec = ZVEC.*L2^(1/2);% transform Z-->z
    plot(zvec,h,'linewidth',2)

    % Final Volume
    dz = zvec(2)-zvec(1);% new dz from y-->h transform
    fv = 0;
    for i=1:length(h)-1
        fv = fv+((h(i)+h(i+1))/2)^2*dz;
    end% for i

%% Plots
%% all plots
figure(1)
plot(ZVEC,ynew,'k','linewidth',2)
xlabel('$z$','Interpreter','latex','fontsize', 15);
ylabel('$h$','Interpreter','latex','fontsize', 15);
hh = legend('h(0)','y(0)','h','y');
set(hh,'Interpreter','latex','fontsize', 10);
set(gca,'Units','normalized','FontUnits','points','FontWeight','normal',...
    'FontSize',9,'FontName','Times')

%% h over z
figure(2)
plot(zvec,h,'o','linewidth',2)
xlabel('$z$','Interpreter','latex','fontsize', 15);
ylabel('$h$','Interpreter','latex','fontsize', 15);
hh = legend('h');
```

```
set(hh,'Interpreter','latex','fontsize', 10);
set(gca,'Units','normalized','FontUnits','points','FontWeight','normal',...
    'FontSize',9,'FontName','Times')

%% y over Z
figure(3)
plot(ZVEC,ynew,'ok','linewidth',2)
xlabel('$Z$','Interpreter','latex','fontsize', 15);
ylabel('$y$','Interpreter','latex','fontsize', 15);
hh = legend('y');
set(hh,'Interpreter','latex','fontsize', 10);
set(gca,'Units','normalized','FontUnits','points','FontWeight','normal',...
    'FontSize',9,'FontName','Times')
```