A new sieve

The starting point of the sieve is $(Pi(n)-2) + C(n) = n/3$ for an integer n with C(n) giving the number of composites up to n but within the progressions (6k+1) and (6k-1) and Pi(n) is the prime counting function. Why n/3? Simply because we know from the start that 2/3 of numbers are multiple of 2 and 3 and therefore do not need to be considered. The minus 2 accounts for the primes 2 and 3 outside the arithmetic progressions.

This equation simply states that if we calculated C(n), we can determine Pi(n) by using the above equation. It is much easier to calculate C(n) than it is Pi(n).

It is known that all the primes, except 2 and 3, fall in one of the following two arithmetic progressions:

$S1=\{6k-1\}=5,11,17,23,29,35,41,47,53,59....$and

$S2=\{6k+1\}=7,13,19,25,31,37,43,49,55,61....$

The composites within these two sets are all generated by multiplication in theory. In practice, we will see below that we do not need the multiplications. The matrices for these products are given by:

M1= S1*S1 for numbers of the form $(6i-1)*(6j-1)$ (1)

M2= S2*S2 for numbers of the form $(6i+1)*(6j+1)$ (2)

M3= S3*S3 for numbers of the form $(6i+1)*(6j-1)$ (3)

where (i,j) are the indices giving the position of the matrix elements in the matrices M1,M2 and M3. For example, we can say that M2 is the product of: $\{7,13,19,25,31,...\}*\{7,13,19,25,31,...\}=49,...$We can say that $M2(1,1)=49$ or we can represent 49 by $(49-1)/6=8$ and M2(1,1) becomes equal to 8.

We prefer to work in the latter representation and we get the following equations for M1, M2 and M3 in terms of

the indices i,j with i=(1,2,3,4...) and j=(1,2,3,4...).

$M1(i,j)=6ij-(i+j)$ (4)

M2(i,j)=6ij+(i+j) (5)

M3(i,j)=6ij+(i-j) (6)

with M(i,j)<=N/6 (since any M(i,j)>N/6 will produce numbers larger than N when converted from indices to integers). The elements of M2(i,1) are 8,15,22,29,36...and are generated either from equation (5) or from each other starting from the first element 8 and adding 7 to the (i-1,1) element for i>1 at basically no cost. To generate the first element of the second column of the matrix M2, we use the following fact: we can calculate M2(2,2) from 13*13=169 or (169-1)/6=28 or from the fact that: 13*13=(7+6)*(7+6)=7*7 + 2*6*7 + 36 but also M2(2,2)=8 + 2*7 + 6 = M2(1,1)+ 2*previous element(7) + 6. Once we have M2(2,2), we just add 13 to get the other elements until we reach the maximum element. We see that to calculate M2(i+1,j+1), we do not need to do all the multiplications, except for the one in 2*6*7. We do not need to store matrices either as explained further below.

If we are interested in the numbers that 8,15,22,29... represent, we can always convert them using N(i)=6*M2(i,1)+1 to get 49,91...numbers which belongs to the (6k+1) series. Note that M1 and M2 are symmetric but M3 is not.

We associate an index for every element of S1 and S2 and we get:

Ix=1,2,3,4,5,6,7,8,9,19,11,....

S1=5,11,17,23,29,35,41,47,53,59....and

Ix=1,2,3,4,5,6,7,8,9,19,11,....

S2=7,13,19,25,31,37,43,49,55,61...

The central idea of the new sieve is to realize that the matrix elements of M1 and M2 produce the indices associated with S2={6k+1} and those of M3 the ones associated with S1={6k-1}. The index of a number gives its position in the series S1 or S2. We notice that the indices for the primes will not be produced.

For example M2(1,1)=8 tells us that the 8th element of S2 is a composite whose value is 6*8+1=49.

Essentially, by considering only N/3 numbers and producing the matrices M1, M2 and M3 of the indices, we will be able to determine the primes below N. A numerical example will make things clear:

N=900

we only need to find the indices for N/3=300 numbers equally divided between S1 and S2 (150 each).

M1=4,9,14,19,20,24,...

M2=8,15,22,28,29,36,....

M3=6,11,13,16,20,21,....

We take the indices produced in these matrices and overwrite the original indices for S1 and S2 (keeping in mind to use M1 and M2 for S2 and M3 for S1). The original indices of S1 and S2 that will not be overwritten will simply point to the primes (since primes are not produced by multiplication).

We need to find i(max) and j(max) for the three matrices M1, M2 and M3

	imax	jmax
M1	30	5
M2	21	4
M3	21	29

Remember the condition M(i,j)<=N/6 when you are generating the M(i,j). You do not want to do more work than you really have to. Note that some indices will be produced more than once because of the associativity of the multiplication but it does not affect the final result. Some original indices will be overwritten more than once.