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# Continuous Symmetries in Field Theory

B. M. BARBASHOV and V. V. NESTERENKO

Joint Institute for Nuclear Research,
Dubna, USSR

## Contents



## 1. Introduction

The symmetries in quantum field theory can be classified both as *discrete and continuous symmetries*. The well known example of discrete symmetries in the elementary particle theory is the invariance under the $CPT$ transformations.

Continuous symmetries are of two kinds: the symmetry under continuous *coordinate-independent transformations* of field functions and coordinates and the symmetry under *coordinate-dependent transformations*.

Transformations of the first type are specified by a finite set of *numerical parameters*. The invariance of the field theory under these transformations leads to the conservation laws the number of which is equal to the number of transformation parameters.

The parameters of the second kind transformations are *functions of coordinates*. The well known examples of these transformations are the local gauge transformations in electrodynamics, and the general coordinate transformations in the gravitation theory. The invariance of the field theory with respect to these transformations gives rise, in addition to the conservation laws, to certain conditions on the Lagrangian density. Therefore the symmetries of this kind are called sometimes *dynamical symmetries.*

It is convenient to formulate and explore the symmetry properties of the field models in the framework of the Lagrangian method when the dynamical system is specified by its Lagrangian function. The symmetry in the theory is introduced by the requirement that the action integral of the system should be invariant under the corresponding group transformations. Consequences of the action functional invariance with respect to the continuous transformations were studied on the basis of variational methods by D. Hilbert [*1*—*3*], F. Klein [*4*—*6*], H. A. Lorentz [*7*], H. Weil [*8, 9*] in the first quarter of this century. The results in this field were formulated more precisely by E. Noether in the form of two theorems [*10, 11*]. These inevstigations were stimulated to a considerable extent by the general theory of relativity worked out at that time by Einstein, Hilbert, Lorentz, Weil and others.

Attempts to construct the quantum theory of gravitation and investigations of nonabelian gauge fields evoked in the fifties the interest in the study of consequences of the invariance of field theories again.

The invariance of the theory under coordinate-dependent transformations gives rise to *singular Lagrangian densities* in the corresponding action functional (Bergmann [*12*—*18*], Goldberg [*19*]). The generalized Hamiltonian treatment of these systems and their quantization were developed by Dirac [*20, 21*].

On the other hand, the requirement of this invariance leads, as it was mentioned above, to constraints on the Lagrangian-density dependence on field functions. Actually these constraints are the *Noether identities* which follow from the second Noether theorem. In the case of the local nonabelian gauge invariance these problems were considered by Utiyama [*22*], Kibble [*23*], Konopleva [*24*] and others.

In the last time attempts were taken to enlarge the scope of the Noether theorems to a more general class of transformations in order to obtain more reach sets of conservation laws [*25*—*42*].

The survey is dealing with the consequences of the fieldtheories symmetry with respect to the continuous coordinates and fields transformations. The transformations of both kinds, coordinate-dependent and coordinate-independent ones, are treated. The first seven sections of the survey are devoted to the general consideration of this problem without specifying the form of the functional. In the remaining sections some applications of the general approach are given.

In section 2 the variational formalism for transformations acting on functions and independent variables (coordinates) is represented. The proof of both the Noether theorems is given in section 3. Some attempts to generalize the theorems are discussed in section 4. Section 5 is dealing with the Noether classification of conservation laws (proper and improper laws). Strong and weak conservation laws are considered in section 6. The relationship between the action invariance under coordinate-dependent transformations and the singular Lagrangian densities is shown in section 7.

Sections 8—15 are devoted to examples of applications of both the Noether theorems. We consider here an $N$-body system with central forces in classical mechanics, relativistic particles interacting with electromagnetic field, conservation laws and the Noether identities in electrodynamics, Yang-Mills field theory, gravitation theory, and in relativistic string model. In section 14 we discuss the Hilbert procedure of obtaining the symmetric energy-momentum tensor in Minkowski space-time. Some concluding remarks are given in section 16.

It should be noted that in literature there are many review articles dealing with the conservation laws and the first Noether theorem [*43*—*47*]. Therefore we tried to pay more attention to the second Noether theorem and to its applications in field theory models.

## 2. Variation of a Functional under Simultaneous Transformations of Functions and Independent Variables

We shall consider functionals of the form [48]

$$I[u(x)] = \int_\Omega dx \mathcal{L}(x, u, \partial u, \partial^2 u), \tag{2.1}$$

where $u(x)$ is a set of $N$ functions $u_A(x)$, $A = 1, \ldots, N$ depending on $n$ variables $x = (x_1, x_2, \ldots, x_n)$. Symbols $\partial u$ and $\partial^2 u$ denote any partial derivative $\partial u_A/\partial x_\mu \equiv u_{A,\mu}$ or $\partial^2 u_A/\partial x_\mu \partial x_\nu \equiv u_{A,\mu\nu}$, respectively, $\mu, \nu, \varrho, \ldots = 1, 2, \ldots, n$.
The functions $u_A(x)$ and independent variable $x$ will be subjected to the following transformations depending on $r$ parameters $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$

$$\begin{aligned} y_\mu &= \Phi_\mu(x, u, \partial u, \partial^2 u, \ldots; \varepsilon) = x_\mu + \delta x_\mu, \quad \mu = 1, \ldots, n, \\ v_A(y) &= \Psi_A(x, u, \partial u, \partial^2 u, \ldots, \varepsilon) = u_A(x) + \delta u_A, \quad A = 1, \ldots, N. \end{aligned} \tag{2.2}$$

As usual, we suppose that the identical transformations correspond to zero values of the parameters $\varepsilon_i$, $i = 1, \ldots, r$.
The transformations (2.2) may be of two kinds: *coordinate-independent* and *coordinate-dependent* ones. In the first case parameters $\varepsilon_i$ are *numbers* but in the second case they are arbitrary coordinate *functions* $\varepsilon_i(x)$. The groups of coordinate-independent transformations will be denoted by $G_r$ and the group of coordinate-dependent transformations by $G_{\infty r}$.
The variations $\delta x_\mu$ and $\delta u_A$ in Eq. (2.2) in the first order with respect to $\varepsilon_i$ are given by

$$\begin{aligned} \delta x_\mu &= \varepsilon_i \left.\frac{\partial \Phi_\mu}{\partial \varepsilon_i}\right|_{\varepsilon_i = 0} = \varepsilon_i\, \delta x_\mu{}^i, \\ \delta u_A &= \varepsilon_i \left.\frac{\partial \Psi_A}{\partial \varepsilon_i}\right|_{\varepsilon_i = 0} = \varepsilon_i\, \delta u_A{}^i. \end{aligned} \tag{2.3}$$

The repeated indices denote the summation in the corresponding range. In addition to the total variations of functions

$$\delta u_A(x) = v_A(y) - u_A(x)$$

we shall use the form variations of functions

$$\bar{\delta} u_A(x) = v_A(x) - u_A(x).$$

From this definition it follows that *the operation $\bar{\delta}$ commutes with the differentiation* $\bar{\delta} u_{A,\mu}(x) = \partial_\mu \bar{\delta} u_A(x)$. Between these variations there is the following connection

$$\begin{aligned} \delta u_A(x) &= v_A(y) - u_A(x) = v_A(y) - u_A(y) + u_A(y) - u_A(x) \\ &\sim \bar{\delta} u_A(y) + u_{A,\mu} \delta x_\mu \sim \bar{\delta} u_A(x) + u_{A,\mu} \delta x_\mu. \end{aligned} \tag{2.4}$$

The sign $\sim$ means the equality up to the first-order terms with respect to $\varepsilon_i$. Recall that in deriving the equation of motion from the least action principle one uses the form variations of the field functions, the independent variables being unchanged.
The first term in Eq. (2.4) is the variation of $u_A(x)$ due to the variation of the functional dependence but the second term gives the variation of $u_A(x)$ due to the variation of the argument $x$. This rule is valid for the total variations of the partial derivatives of the functions also. In order to prove this we obtain at first some auxiliary formulae.

According to (2.2) the derivative $\partial y_\mu/\partial x_\nu$ can be represented as follows

$$\frac{\partial y_\mu}{\partial x_\nu} = \delta_{\mu\nu} + \frac{\partial \delta x_\mu}{\partial x_\nu}.$$

Further we have

$$\frac{\partial}{\partial x_\nu} = \frac{\partial}{\partial y_\mu}\cdot\frac{\partial y_\mu}{\partial x_\nu} = \frac{\partial}{\partial y_\nu} + \frac{\partial \delta x_\mu}{\partial x_\nu}\cdot\frac{\partial}{\partial y_\mu},$$

therefore

$$\frac{\partial}{\partial x_\nu} - \frac{\partial}{\partial y_\nu} = \frac{\partial \delta x_\mu}{\partial x_\nu}\,\frac{\partial}{\partial y_\mu}. \tag{2.5}$$

Analogously for the second derivatives one obtains

$$\frac{\partial^2}{\partial x_\mu \partial x_\nu} = \left(\frac{\partial}{\partial y_\mu} + \frac{\partial \delta x_\varrho}{\partial x_\mu}\,\frac{\partial}{\partial y_\varrho}\right)\cdot\left(\frac{\partial}{\partial y_\nu} + \frac{\partial \delta x_\sigma}{\partial x_\nu}\,\frac{\partial}{\partial y_\sigma}\right)$$
$$= \frac{\partial^2}{\partial y_\mu\,\partial y_\nu} + \frac{\partial \delta x_\varrho}{\partial x_\mu}\cdot\frac{\partial^2}{\partial y_\varrho\,\partial y_\nu} + \frac{\partial}{\partial y_\mu}\left(\frac{\partial \delta x_\sigma}{\partial x_\nu}\right)\frac{\partial}{\partial y_\sigma} - \frac{\partial \delta x_\sigma}{\partial x_\nu}\cdot\frac{\partial^2}{\partial y_\mu\,\partial y_\sigma}.$$

In the last three terms the derivatives with respect to $x$ can be replaced according to (2.5) by the derivatives with respect to $y$. As a result, we get

$$\frac{\partial^2}{\partial x_\mu\,\partial x_\nu} - \frac{\partial^2}{\partial y_\mu\,\partial y_\nu} = \frac{\partial \delta x_\varrho}{\partial x_\mu}\cdot\frac{\partial^2}{\partial x_\varrho\,\partial x_\nu} + \frac{\partial^2 \delta x_\sigma}{\partial x_\mu\,\partial x_\nu}\,\frac{\partial}{\partial x_\sigma} + \frac{\partial \delta x_\sigma}{\partial x_\nu}\,\frac{\partial^2}{\partial x_\mu\,\partial x_\sigma}. \tag{2.6}$$

Let us consider now the total variation of the first derivative of $u_A(x)$

$$\delta u_{A,\mu}(x) = \frac{\partial v_A(y)}{\partial y_\mu} - \frac{\partial u_A(x)}{\partial x_\mu}$$
$$= \frac{\partial}{\partial y_\mu}\,[v_A(y) - u_A(y)] + \frac{\partial}{\partial y_\mu}\,[u_A(y) - u_A(x)] + \left(\frac{\partial}{\partial y_\mu} - \frac{\partial}{\partial x_\mu}\right) u_A(x)$$
$$= \frac{\partial}{\partial y_\mu}\,\bar{\delta} u_A(y) + \frac{\partial}{\partial y_\mu}\left(\left.\frac{\partial u_A(y)}{\partial y_\sigma}\right|_{y_\mu = x_\mu}\cdot \delta x_\sigma\right) - \frac{\partial \delta x_\sigma}{\partial x_\mu}\,\frac{\partial}{\partial y_\sigma}\,u_A(x).$$

Up to the first-order terms in $\varepsilon$ this expression can be rewritten as follows

$$\delta u_{A,\mu}(x) = \frac{\partial}{\partial x_\mu}\,\bar{\delta} u_A(x) + \frac{\partial^2 u_A(x)}{\partial x_\mu\,\partial x_\sigma}\cdot \delta x_\sigma. \tag{2.7}$$

In the same way using Eq. (2.6) one gets for the second derivatives of $u_A(x)$

$$\delta u_{A,\mu\nu}(x) = \frac{\partial^2 v_A(y)}{\partial y_\mu\,\partial y_\nu} - \frac{\partial^2 u_A(x)}{\partial x_\mu\,\partial x_\nu} = \frac{\partial^2}{\partial y_\mu\,\partial y_\nu}\,[v_A(y) - u_A(y)]$$
$$+ \frac{\partial^2}{\partial y_\mu\,\partial y_\nu}\,[u_A(y) - u_A(x)] + \left(\frac{\partial^2}{\partial y_\mu\,\partial y_\nu} - \frac{\partial^2}{\partial x_\mu\,\partial x_\nu}\right) u_A(x)$$
$$= \frac{\partial^2}{\partial y_\mu\,\partial y_\nu}\,\bar{\delta} u_A(y) + \frac{\partial^2}{\partial y_\mu\,\partial y_\nu}\left(\frac{\partial u_A(x)}{\partial x_\sigma}\,\delta x_\sigma\right)$$
$$- \frac{\partial \delta x_\varrho}{\partial x_\mu}\,\frac{\partial^2 u_A(x)}{\partial x_\varrho\,\partial x_\nu} - \frac{\partial^2 \delta x_\sigma}{\partial x_\mu\,\partial x_\nu}\cdot\frac{\partial u_A(x)}{\partial x_\sigma} - \frac{\partial \delta x_\sigma}{\partial x_\nu}\cdot\frac{\partial^2 u_A(x)}{\partial x_\mu\,\partial x_\sigma}.$$

Substituting $\partial/\partial y$ by $\partial/\partial x$ we obtain up to the first order terms in $\varepsilon$

$$\delta u_{A,\mu\nu}(x) = \bar{\delta} u_{A,\mu\nu} + u_{A,\mu\nu\sigma}\delta x_\sigma . \tag{2.8}$$

Analogously the total variation of the partial derivative of any order $\delta u_{A,\mu\nu\varrho\ldots}$ can be represented as a sum of two terms

$$\delta u_{A,\mu\nu\varrho\ldots} = \bar{\delta} u_{A,\mu\nu\varrho\ldots} + u_{A,\mu\nu\varrho\ldots\sigma}\delta x_\sigma . \tag{2.9}$$

Here the first term is the form variation of $u_{A,\mu\nu\varrho\ldots}$, the second one is the variation of $u_{A,\mu\nu\varrho\ldots}$, due to the variation of the argument $x$.
In what follows we shall use often the *total derivatives with respect to* $x_\mu$. These derivatives take into consideration the *explicit and implicit dependence on* $x$ and they will be denoted by

$$d_\mu = \frac{d}{dx_\mu} = \frac{\partial}{\partial x_\mu} + u_{A,\mu}\frac{\partial}{\partial u_A} + u_{A,\mu\nu}\frac{\partial}{\partial u_{A,\nu}} + u_{A,\mu\nu\varrho}\frac{\partial}{\partial u_{A,\mu\varrho}} .$$

We suppose that under transformation (2.2) the integrand in (2.1) transforms as follows [11]

$$\mathscr{L}(x, u, \partial u, \partial^2 u) \to \mathscr{L}'(y, v, \partial v, \partial^2 v) = \mathscr{L}(y, v, \partial v, \partial^2 v) + \frac{dC_\mu}{dx_\mu} + l(y, v, \partial v, \partial^2 v), \tag{2.10}$$

where $C_\mu$ is linear in parameters $\varepsilon_i$, $i = 1, \ldots, r$ and $l$ contains $\varepsilon^2$ and higher terms. This means that $\mathscr{L}$ is *forminvariant up to divergence*. In the field theory Eq. (2.1) gives the action functional the integrand of which $\mathscr{L}$ is the Lagrangian density. It is the transformation (2.10) that will lead to covariant equations of motion for the field functions $u_A(x)$.
The total variation of $\mathscr{L}$ by the transformations (2.2) can be written in the form

$$\delta\mathscr{L} = \delta_u\mathscr{L} + \frac{d\mathscr{L}}{dx_\mu}\delta x_\mu + \frac{dC_\mu}{dx_\mu}, \tag{2.11}$$

where $\delta_u\mathscr{L}$ is the variation of $\mathscr{L}$ due to the variation of the functional arguments $u$, $\partial u$, $\partial^2 u$

$$\delta_u\mathscr{L} = \frac{\partial\mathscr{L}}{\partial u_A}\bar{\delta} u_A + \frac{\partial\mathscr{L}}{\partial u_{A,\mu}}\cdot\bar{\delta} u_{A,\mu} + \frac{\partial\mathscr{L}}{\partial u_{A,\mu\nu}}\cdot\bar{\delta} u_{A,\mu\nu} . \tag{2.12}$$

The formula (2.11) can be easily proved

$$\begin{aligned}\delta\mathscr{L} &= \mathscr{L}'(y, v, \partial v, \partial^2 v) - \mathscr{L}(x, u, \partial u, \partial^2 u)\\ &\sim \mathscr{L}(y, v, \partial v, \partial^2 v) - \mathscr{L}(x, u, \partial u, \partial^2 u) + \frac{dC_\mu}{dx_\mu}\\ &\sim \frac{\partial\mathscr{L}}{\partial x_\mu}\delta x_\mu + \frac{\partial\mathscr{L}}{\partial u_A}\delta u_A + \frac{\partial\mathscr{L}}{\partial u_{A,\mu}}\cdot\delta u_{A,\mu} + \frac{\partial\mathscr{L}}{\partial u_{A,\mu\nu}}\delta u_{A,\mu\nu} + \frac{dC_\mu}{dx_\mu} .\end{aligned} \tag{2.13}$$

Making use of Eqs. (2.4), (2.7), (2.8) one obtains immediately Eq. (2.11).
Now we return to the functional (2.1). Using the transformations (2.2) we correlate to it a new functional

$$I[v(y)] = \int_{\Omega+\Delta\Omega} dy\mathscr{L}'\big(y, v(y), \partial v(y), \partial^2 v(y)\big), \tag{2.14}$$

where $\Omega + \Delta\Omega$ is a region in the coordinate space into which $\Omega$ transforms by the transformations (2.2). To prove the Noether theorem, we shall calculate the difference $I[v(y)] - I[u(x)]$. It can be shown that the difference in the first order with respect to $\varepsilon_i$, i.e. the variation of the functional $I[u(x)]$, is given by

$$\delta I = \int_\Omega dx \left[\delta_u \mathscr{L} + \frac{d}{dx_\mu}(\mathscr{L}\delta x_\mu + C_\mu)\right], \tag{2.15}$$

where $\delta_u \mathscr{L}$ is defined by Eq. (2.12). The second term of the integrand in Eq. (2.15) has the following full form

$$\frac{d}{dx_\mu}(\mathscr{L}\delta x_\mu + C_\mu) = \frac{\partial \mathscr{L}}{\partial x_\mu}\delta x_\mu + \frac{\partial \mathscr{L}}{\partial u_A} u_{A,\mu}\delta x_\mu + \frac{\partial \mathscr{L}}{\partial u_{A,\nu}} \cdot u_{A,\nu\mu}\delta x_\mu$$
$$+ \frac{\partial \mathscr{L}}{\partial u_{A,\nu\sigma}} \cdot u_{A,\nu\sigma\mu} \cdot \delta x_\mu + \mathscr{L}\frac{\partial \delta x_\mu}{\partial x_\mu} + \frac{\partial C_\mu}{\partial x_\mu} + \frac{\partial C_\mu}{\partial u_A} \cdot u_{A,\mu} + \cdots.$$

Let us prove Eq. (2.15)

$$\delta I = I[v(y)] - I[u(x)] = \int_{\Omega+\Delta\Omega} dy \mathscr{L}'\big(y, v(y), \partial v(y), \partial^2 v(y)\big) - \int_\Omega dx \mathscr{L}\big(x, u(x), \partial u(x), \partial^2 u(x)\big). \tag{2.16}$$

The integration over $y$ in the first term can be replaced by the integration over $x$. The Jacobian of this change of variables according to (2.2) is

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} = \det \left\| \delta_{\mu\nu} + \frac{d\delta x_\mu}{dx_\nu} \right\| = \prod_{\mu=1}^{n} \left(1 + \frac{d\delta x_\mu}{dx_\mu}\right) + o(\varepsilon^2) = 1 + \sum_{\mu=1}^{n} \frac{d\delta x_\mu}{dx_\mu} + o(\varepsilon^2).$$

Equation (2.16) takes now the form

$$\delta \mathscr{L} = \int_\Omega dx \Big[\mathscr{L}\big(y, v(y), \partial v(y), \partial^2 v(y)\big) - \mathscr{L}\big(x, u(x), \partial u(x), \partial^2 u(x)\big)$$
$$+ \mathscr{L}\big(x, u(x), \partial u(x), \partial^2 u(x)\big)\frac{d\delta x_\mu}{dx_\mu} + \frac{dC_\mu}{dx_\mu}\Big] = \int_\Omega dx[\delta\mathscr{L} + \mathscr{L} d_\mu \delta x_\mu + d_\mu C_\mu]. \tag{2.17}$$

Substituting in Eq. (2.17) the total variation $\delta\mathscr{L}$ by $\delta_u\mathscr{L}$ according to (2.11) we get (2.15).

In Eq. (2.15) there appear only the form variations of the functions $u_A(x)$ and their partial derivatives. It is this equation that must be used for obtaining the relation of the total variation $\delta I$ with the variation of $I$ in the least action principle.

If the independent variables $x_\mu$ are not transformed, then the second term in Eq. (2.15) vanishes and we get the usual expression for the functional variation by changing only the form of the functions $u_A(x)$. Supposing that $\bar{\delta} u_A(x)$ vanishes on the boundary of the integration region $\Omega$ and integrating by parts one obtains

$$\delta I = \int_\Omega dx L_A(x, u, \partial u, \ldots)\, \bar{\delta} u_A(x), \tag{2.18}$$

where $L_A$ are the Lagrangian expressions

$$L_A = \frac{\partial \mathscr{L}}{\partial u_A} - \frac{d}{dx_\mu}\left(\frac{\partial \mathscr{L}}{\partial u_{A,\mu}}\right) + \frac{d^2}{dx_\mu\, dx_\nu}\left(\frac{\partial \mathscr{L}}{\partial u_{A,\mu\nu}}\right). \tag{2.19}$$

According to the least action principle $\delta I = 0$ and Eq. (2.18) gives the equation of motion (or the Euler equations) for the field functions $u_A(x)$

$$L_A(x, u, \partial u, \ldots) = 0, \quad A = 1, 2, \ldots, N. \tag{2.20}$$

In what follows we shall use the new quantity $F_\mu(x, u, \partial u, \ldots)$ which we introduce in the following way

$$\delta_u \mathcal{L} = L_A \bar{\delta} u_A - \frac{dF_\mu}{dx_\mu}. \tag{2.21}$$

For the functionals under consideration $F_\mu$ is given by

$$F_\mu = \left(\frac{d}{dx_\nu} \cdot \frac{\partial \mathcal{L}}{\partial u_{A,\nu\mu}} - \frac{\partial \mathcal{L}}{\partial u_{A,\mu}}\right) \bar{\delta} u_A - \frac{\partial \mathcal{L}}{\partial u_{A,\nu\mu}} \delta u_{A,\nu}. \tag{2.22}$$

Substituting (2.22) and (2.19) into (2.21) one easily verifies that these formulae give the right expression (2.12) for $\delta_u \mathcal{L}$. Thus the variation of the functional (2.15) can be represented in the form

$$\delta I = \int_\Omega dx \left[L_A \bar{\delta} u_A - \frac{d}{dx_\mu}(F_\mu - \mathcal{L}\delta x_\mu - C_\mu)\right]. \tag{2.23}$$

It is this equation that will be used for proving Noether's theorems.

Equation (2.23) can be generalized to the Lagrange function $\mathcal{L}$ depending on the derivatives of the $u_A$'s up to an $n$-th order. In this case $\delta I$ is given by (2.23), but $L_A$ and $F_\mu$ should be modified in the following way [*27*, *32*]

$$L_A = \sum_{a=1}^{n} (-1)^a d_{\mu_1} d_{\mu_2} \cdots d_{\mu_a} \frac{\partial \mathcal{L}}{\partial u_{A,\mu_1\mu_2\cdots\mu_a}}, \tag{2.24}$$

$$F_\mu = \sum_{a=1}^{n} \sum_{b=0}^{a-1} (-1)^{b+1} d_{\mu_1} d_{\mu_2} \cdots d_{\mu_b} \left[\frac{\partial \mathcal{L}}{\partial u_{A,\mu_1,\ldots\mu_{a-1}\mu}}\right] d_{\mu_{b+1}} \cdots d_{\mu_{a-1}} \bar{\delta} u_A. \tag{2.25}$$

### 3. The Noether Theorems

We say that the functional $I[u(x)]$ is *invariant*[1]) *under the transformations* (*2.2*) *if* $\delta I = 0$ for an arbitrary integration region $\Omega$. Such transformations will be called the symmetry transformations for the functional (2.1). Thus for these transformations we have from Eq. (2.23)

$$\delta I = \int_\Omega dx \left[L_A \bar{\delta} u_A - \frac{dJ_\mu}{dx_\mu}\right] = 0, \tag{3.1}$$

where $J_\mu = F_\mu - C_\mu - \mathcal{L}\delta x_\mu$. As the integration region $\Omega$ is arbitrary, we obtain here the basic identity

$$L_A \bar{\delta} u_A - \frac{dJ_\mu}{dx_\mu} \equiv 0. \tag{3.2}$$

We shall prove here only the direct statements of the Noether theorems referring to the original Noether paper [*10*, *11*] and to more recent papers [*24*—*42*] where the proof of the converse theorems can be found.

[1]) If $C_\mu$ in Eq. (2.10) does not vanish identically then one says sometimes that the functional $I\,[u(x)]$ is *invariant up to the divergence* under the transformations (2.2).

*The first Noether theorem.* If the functional $I[u(x)]$ is invariant under $r$-parameter group $G_r$ of continnous transformations of coordinates and field functions, then $r$ linear independent combinations of the Lagrangian expressions turn into the divergences. And conversely, from the last condition the invariance of under some transformations group $G_r$ follows.

The total variations $\delta u_A$, $\delta u_{A,\mu}$, $\delta x_\mu$, as it was supposed above, are linear in the group parameters $\varepsilon_i$. Therefore by virtue of Eqs. (2.4) and (2.7) the form variations $\bar{\delta} u_A$, $\bar{\delta} u_{A,\mu}$ are linear in $\varepsilon_i$ also

$$\delta u_A = \varepsilon_i \delta u_A{}^i,\ \delta u_{A,\mu} = \varepsilon_i \delta u^i_{A,\mu},\ \delta x_\mu = \varepsilon_i \delta x_\mu{}^i,$$
$$\bar{\delta} u_A = \varepsilon_i \bar{\delta} u_A{}^i,\ \bar{\delta} u_{A,\nu} = \varepsilon_i \bar{\delta} u^i_{A,\nu}. \tag{3.3}$$

In a similar fashion the vector $F_\mu$ in Eq. (2.22) and $C_\mu$ in (2.10) can be represented by expansions

$$F_\mu = \varepsilon_i F_\mu{}^i,\ C_\mu = \varepsilon_i C_\mu{}^i. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) and equating to zero the coefficients of $\varepsilon_i$ one obtains $r$ divergence relations

$$L_A \cdot \bar{\delta} u_A{}^i = \frac{dJ_\mu{}^i}{dx_\mu}, \qquad i = 1, 2, \ldots, r, \tag{3.5}$$

where

$$J_\mu{}^i = \left(d_\nu \frac{\partial \mathcal{L}}{\partial u_{A,\nu\mu}} - \frac{\partial \mathcal{L}}{\partial u_{A,\mu}}\right) \bar{\delta} u_A{}^i - \frac{\partial \mathcal{L}}{\partial u_{A,\mu\nu}} \bar{\delta} u^i_{A,\nu} - \mathcal{L}\, \delta x_\mu{}^i - C_\mu{}^i$$
$$= \left[\left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} - d_\nu \frac{\partial \mathcal{L}}{\partial u_{A,\nu\mu}}\right) u_{A,\sigma} + \frac{\partial \mathcal{L}}{\partial u_{A,\mu\nu}} u_{A,\nu\sigma} - \mathcal{L} \delta_{\mu\sigma}\right] \delta x_\sigma{}^i$$
$$- \left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} - d_\nu \frac{\partial \mathcal{L}}{\partial u_{A,\nu\mu}}\right) \delta u_A{}^i - \frac{\partial \mathcal{L}}{\partial u_{A,\mu\nu}} \delta u^i_{A,\nu} - C_\mu{}^i. \tag{3.6}$$

It should be noted here that we have used the fact that in the first Noether theorem the coordinate-independent transformations are considered. In this case the parameters $\varepsilon_i$ are numbers which can be removed from under the divergence symbol in Eq. (3.2).

In divergence relations (3.5) the functions $u_A(x)$ are not supposed to obey the equations of motion (2.20). If these equations are satisfied, then the left-hand sides in (3.5) vanish and we get $r$ conservation laws

$$\frac{dJ_\mu{}^i}{dx_\mu} = 0, \qquad i = 1, 2, \ldots, r. \tag{3.7}$$

Let us go to the coordinate-dependent transformations forming the general infinite parametric group $G_{\infty r}$. These transformations are defined by $r$ arbitrary functions $\varepsilon_i(x)$, $i = 1, \ldots, r$. In this case there takes place the *second Noether theorem.* If the integral $I[u(x)]$ is invariant under the general group $G_{\infty r}$, that is to say, under transformations of coordinates and field functions which depend on $r$ arbitrary functions $\varepsilon_i(x)$, $i = 1, 2, \ldots, r$, and their derivatives up to a $k$-th order, then there are $r$ identities containing the Lagrangian expressions and their derivatives up to the $k$-th order. The inversion of this theorem is possible also.

For simplicity we consider the case when $k = 1$. The form variation of functions is given now by

$$\bar{\delta} u_A(x) = \gamma_A{}^i \varepsilon_i(x) + \gamma^i_{A\mu} \varepsilon_{i,\mu}(x), \tag{3.8}$$

where the coefficients $\gamma_A{}^i$ and $\gamma^i_{A\mu}$ are functions of $u$, $\partial u$, $\partial^2 u$, ... A similar formula is valid for $\delta x_\mu$ but in what follows we shall not use the manifest form of these variations. Substituting (3.8) into (3.1) and using the equality

$$L_A \gamma^i_{A\mu} \varepsilon_{i,\mu} = \frac{d}{dx_\mu}(L_A \gamma^i_{A\mu} \varepsilon_i) - \frac{d}{dx_\mu}(L_A \gamma^i_{A\mu})\, \varepsilon_i$$

we obtain

$$\delta I = \int dx \left\{\left[L_A \gamma_A{}^i - \frac{d}{dx_\mu}(L_A \gamma^i_{A\mu})\right] \varepsilon_i(x) - \frac{d}{dx_\mu}\left(F_\mu - C_\mu - L_A \gamma^i_{A\mu} \varepsilon_i(x)\right)\right\} = 0. \qquad (3.9)$$

Now we take such functions $\varepsilon_i(x)$, $i = 1, \ldots, r$ that they themselves and their derivatives entering into $F_\mu - C_\mu - L_A \gamma^i_{A\mu} \varepsilon_i$ vanish on the boundary of the integration region $\Omega$, for example, $\varepsilon_i(x) \sim \delta^{(n)}(x - z_i)$, $z \in \Omega$. As a result, one gets

$$\delta I = \int dx \left[L_A \gamma_A{}^i - \frac{d}{dx_\mu}(L_A \gamma^i_{A\mu})\right] \varepsilon_i(x) = 0,$$

whence $r$ identities follow

$$L_A \gamma_A{}^i - \frac{d}{dx_\mu}(L_A \gamma^i_{A\mu}) \equiv 0, \quad i = 1, \ldots, r. \qquad (3.10)$$

These equalities are usually referred to as the *Bianchi identities* [*39*, *40*].
The Bianchi identities (3.10) do not exhaust all the consequences of the invariance of the functional (2.1) under the coordinate-dependent transformations $G_{\infty r}$ [*5*, *49*]. New identities will appear if in the integrand in Eq. (3.9) or (3.1) the coefficients of functions $\varepsilon_i(x)$ and of each their derivatives $\varepsilon_{i,\mu\nu\ldots\varrho\sigma}(x)$ will be equated to zero separately. This procedure corresponds obviously to the following choice of the functions $\varepsilon_i(x)$:

$$\varepsilon_i(x) \sim x_\mu x_\nu \ldots x_\varrho x_\sigma.$$

Let us obtain these identities for the transformations (3.8) that do not affect the coordinates. For simplicity we suppose that $\mathcal{L}$ depends on $x$, $u_A(x)$, $\partial u_A(x)$ and it does not depend on the second derivatives of the field functions $u_A(x)$ and put $C_\mu$ in (2.10) zero. In this case by virtue of (3.6) and (3.8) formula (3.1) takes the form

$$\delta I = \int\limits_\Omega dx \left\{L_A(\gamma_A{}^i \varepsilon_i + \gamma^i_{A\mu} \varepsilon_{i,\mu}) + \frac{d}{dx_\mu}\left[\frac{\partial \mathcal{L}}{\partial u_{A,\mu}}(\gamma_A{}^i \varepsilon_i + \gamma^i_{A\nu} \varepsilon_{i,\nu})\right]\right\} = 0. \qquad (3.11)$$

In the integrand the coefficients of $\varepsilon_i(x)$ and of its first and second derivatives must vanish separately. This gives new identities

$$L_A \gamma_A{}^i + \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma_A{}^i\right) \equiv 0, \quad i = 1, 2, \ldots, r; \qquad (3.12)$$

$$L_A \gamma^i_{A\nu} + \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma^i_{A\nu}\right) + \frac{\partial \mathcal{L}}{\partial u_{A,\nu}} \gamma_A{}^i \equiv 0, \quad i = 1, \ldots, r, \quad \nu, \mu = 1, 2, \ldots, n; \quad (3.13)$$

$$\frac{\partial \mathcal{L}}{\partial u_{A,\nu}} \gamma^i_{A\mu} + \frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma^i_{A\nu} \equiv 0, \quad i = 1, \ldots, r,\ \nu, \mu = 1, 2, \ldots, n. \qquad (3.14)$$

In the last identities (3.14) there appear two terms because $\varepsilon_{i,\mu\nu}$ is symmetric with respect to $\mu$, $\nu$.

The Bianchi identities (3.10) follow from (3.12)—(3.14). Indeed, acting on (3.13) by $d/dx_\nu$ and taking into account that quantities $(\partial\mathcal{L}/\partial u_{A,\mu})\,\gamma^i_{A\nu}$ are skew symmetric with respect to $\mu$, $\nu$, we get

$$\frac{d}{dx_\nu}\left(\frac{\partial\mathcal{L}}{\partial u_{A,\nu}}\,\gamma_A{}^i\right) = -\frac{d}{dx_\nu}\,(L_A\gamma^i_{A\nu}).$$

Substitution of this expression into (3.12) gives (3.10).
The identities (3.14) do not contain the Lagrangian expresions but they impose constraints on the dependence of the Lagrangian density on its arguments (see section 13).
It appears practically that the most important identities are the Bianchi identities (3.10) which are homogeneous and linear in $L_A$.
The identities (3.10), (3.12), (3.13) establish the dependence of the left-hand sides $L_A$ of the Euler equations (2.20). This means that some of these equations are consequences of the other ones. Therefore the number od independent Euler equations is less than the number of unknown functions $u_A(x)$. Thus, in order to obtain $u_A(x)$, one has to supplement the equations of motion (2.20) by some conditions. In the physical literature these conditions are called usually *the gauge fixing conditions.*
In what follows we write the Noether identities for some field models.

## 4. Generalization of the First Noether Theorem

This theorem can be generalized in the following way [*25*].
If the $r$-parameter group of the coordinates and functions transformations (2.2) changes any solution of the Euler equations (2.20) (any extremal) into another solution of these equations again, the functional $I[u(x)]$ considered on the extremals only being invariant under these transformations, then there are $r$ conservation laws (3.7).
In contrast to the Noether formulation of the first theorem the conservation laws obtained here should not be independent because the limited set of functions (only the extremals) on which the functional $I[u(x)]$ is invariant is considered.
This group of transformations of extremals leaving the functional $I[u(x)]$ invariant may be more general than the group in the Noether formulation of the first theorem. And as a consequence, we may obtain here more conservation laws compared to the Noether case.
The proof of this generalized theorem and the corresponding examples of new conserved currents can be found in papers [*25*—*42*].

## 5. Proper and Improper Conservation Laws

Let us consider a special case when the $r$-parameter (Lie) group $G_r$ is obtained from the general group of transformations $G_{\infty r}$ if we suppose that the functions $\varepsilon_i(x)$ in $G_{\infty r}$ are constants. We assume that the functional (2.1) is invariant under coordinate-dependent transformations from $G_{\infty r}$. As a consequence, it will be invariant under $G_r$ group also. In this case both the Noether theorems are valid and there are the divergence relations (3.5) and the Noether indetifies (3.10), (3.12)—(3.14), the divergence relations being consequences of the Noether identities. The conserved currents $J_\mu{}^i(x)$, corresponding to symmetry under $G_r$ group have now a remarkable property: they are linear combinations of the Lagrange expressions plus the divergence of an antisymmetric tensor. Such conservation laws E. Noether [*10*] called *improper* ones in contrast to other *proper conservation laws.*

Consider improper conservation laws for the Lagrangian density $\mathcal{L} = \mathcal{L}(x, u, \partial u)$ the action functional for which is invariant under transformations (3.8). For simplicity we assume that $\delta x_\mu = 0$ and $C_\mu = 0$. By Virtue of Eqs. (2.22) and (3.8) $F_\mu$ can be written in the form

$$F_\mu = -\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \left(\gamma_A{}^i \varepsilon_i(x) + \gamma^i_{A\nu}\varepsilon_{i,\nu}(x)\right) = J_\mu{}^i \varepsilon_i(x) - \frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma^i_{A\nu}\varepsilon_{i,\nu}(x), \tag{5.1}$$

where we have introduced by Eq. (3.6) the conserved currents

$$J_\mu{}^i = -\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma_A{}^i, \qquad i = 1, 2, \ldots, r \tag{5.2}$$

corresponding to the invariance of $I[u(x)]$ under $G_r$ group. Substituting (5.1) and (5.2) into (3.9) and equating to zero the coefficients of $\varepsilon_i(x)$, $\varepsilon_{i,\nu}(x)$ and $\varepsilon_{i,\mu\nu}(x)$ one obtains the identities

$$\frac{dJ_\mu{}^i}{dx_\mu} \equiv L_A \gamma_A{}^i, \tag{5.3}$$

$$J_\mu{}^i \equiv L_A \gamma^i_{A\mu} + \frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}} \gamma^i_{A\mu}\right) \tag{5.4}$$

and identities (3.14) according to which the quantity $(\partial \mathcal{L}/\partial u_{A,\nu})\, \gamma^i_{A\mu}$ is antisymmetric in indices $\mu$ and $\nu$.

The identities (5.3) are the divergence relations (3.5) with $\delta u_A{}^i = \gamma_A{}^i$ corresponding to the invariance under transformation group $G_r$. From Eq. (5.4) it follows that $J_\mu{}^i(x)$ is a linear combination of the Lagrangian expressions $L_A$ plus the divergence of the skew-symmetric quantity $(\partial \mathcal{L}/\partial u_{A,\nu})\, \gamma^i_{A\mu}$.

When the equations of motion (2.20) are satisfied, then Eq. (5.4) becomes

$$J_\mu{}^i = \frac{dU^i_{\mu\nu}}{dx_\nu}, \tag{5.5}$$

where $U^i_{\mu\nu}$ are the superpotentials [*18, 19, 50*]

$$U^i_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial u_{A,\nu}} \gamma^i_{A\mu}, \qquad i = 1, 2, \ldots, r. \tag{5.6}$$

It should be noted that the superpotentials permit us to write the integral conserved quantities $Q^i$ corresponding to improper conservation laws in Minkowski space-time as a surface two-dimensional integral

$$Q^i = \int_{V_3} J_0{}^i(x)\, d^3x = \int_{V_3} d^3x \sum_{\alpha=1}^{3} \frac{dU^i_{0\alpha}}{dx_\alpha} = \int_\sigma d^2\sigma \sum_{\alpha=1}^{3} n_\alpha U^i_{0\alpha}, \tag{5.7}$$

where $n_\alpha$, $\alpha = 1, 2, 3$ is a unit normal to the surface $\sigma$ which encloses the three-dimensional volume $V_3$. As it will be shown further this possibility corresponds to the Gauss law in electromagnetic theory (see Section 10).

Equation (5.7) enables us to avoid difficulties connected with the field singularities due to the pointlike sources of the fields in $V_3$.

## 6. Weak and Strong Conservation Laws

Weak conservation laws are the usual conservation laws (3.7) that take place when the equations of motion (2.20) are satisfied. Strong conservation laws are laws which are valid whether the field equations are satisfied or not. These laws are combinations of the Noether identities (3.10), (3.12)—(3.14) which have the form of a divergence. From the Bianchi identities (3.10) and (3.12) we obtain $r$ strong conservation laws [*19, 39, 40*]

$$\frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial u_{A,\mu}}\gamma_A{}^i + L_A\gamma^i{}_{A\mu}\right) \equiv 0, \quad i = 1, 2, \ldots, r. \tag{6.1}$$

What is more, we can construct the strong conservation laws containing arbitrary functions $\varepsilon_i(x)$, $i = 1, 2, \ldots, r$ [*40*]. Multiplying the Bianchi identities by $\varepsilon_i(x)$ and substituting them into (3.9) one obtains

$$\frac{d}{dx_\mu}\left(F_\mu - L_A\gamma^i{}_{A\mu}\varepsilon_i(x)\right) \equiv 0, \tag{6.2}$$

where $F_\mu$ by virtue of (2.22) and (3.8) is

$$F_\mu = -\frac{\partial \mathcal{L}}{\partial u_{A,\mu}}\bar{\delta}u_A = -\frac{\partial \mathcal{L}}{\partial u_{A,\mu}}\left(\gamma_A{}^i\varepsilon_i(x) + \gamma^i{}_{A\nu}\varepsilon_{i,\nu}(x)\right). \tag{6.3}$$

## 7. Singular Lagrangian Densities

The Lagrangian density $\mathcal{L}(u, \partial u)$ is called singular if

$$\det \|A_{AB}\| = 0, \tag{7.1}$$

where

$$A_{AB} = \frac{\partial^2 \mathcal{L}}{\partial u_{A,0}\,\partial u_{B,0}}, \quad u_{A,0} = \frac{\partial u_A}{\partial x_0}, \quad x_0 = t. \tag{7.2}$$

The invariance of the action functional under infinite group $G_{\infty r}$ of coordinate-dependent transformations indicates that the corresponding Lagrangian density is singular [*12*]. We show this for the transformations (3.8) with coefficients $\gamma^i_{A\mu}$ depending on $u$ and $\partial u$. Making use of (7.2) one can decompose $L_A$ so as to exhibit $u_{B,00}$, thus

$$L_A = -A_{AB}u_{B,00} + \bar{L}_A = 0, \tag{7.3}$$

where $\bar{L}_A$ does not involve $u_{A,00}$. We assume here that the rank of the matrix $A_{AB}$ is not equal to zero, as in this case Eqs. (7.3) will be the first order with respect to the time derivatives. On substituting (7.3) into the Bianchi identities (3.10) the terms containing second time derivative will lead to terms containing third time derivative coefficients of which must vanish

$$\sum_{B=1}^{N} A_{AB}\gamma^i_{B0} = 0, \quad i = 1, 2, \ldots, r. \tag{7.4}$$

Thus, there are $r$ different sets of $N$ quantities $\gamma^i_{B0}$, $i = 1, \ldots, r$; $B = 1, \ldots, N$, which are zero eigenvectors of the matrix $\Lambda_{AB}$. As a consequence, we get (7.1) and the Lagrangian density $\mathcal{L}(u, \partial u)$ is singular.
The Euler equations (7.3) for singular Lagrangian densities cannot be solved with respect to the second time derivatives of the field functions $u_{A,00}$. Therefore the usual Cauchy problem for these equations cannot be solved uniquely. Indeed for given Cauchy data at the initial moment $t = t_0$

$$u_A(t, x)|_{t=t^0} = \varphi(x), \quad u_{A,0}(t, x)|_{t=t^0} = \psi_A(x), \qquad x = (x_1, x_2, \ldots, x_{n-1}) \tag{7.5}$$

it is impossible to calculate all the derivatives of the function $u_A(x)$ at this moment. More precisely, we do not obtain at $t = t_0$ the values of the derivatives containing two and more differentiations with respect to time $t$. As a result, the field functions $u_A(t, x)$ cannot be obtained at the moment $t$ by summing the Taylor series. Hence the hypersurface $t = t^0$ appears to be *characteristic* for the system (7.3).
It should be noted here that for the singular Lagrangian densities the Cauchy data (7.5) cannot be taken arbitrary because there are $r$ linear combinations of the Euler equations which do not involve the second time derivatives. Multiplying (7.3) by $\gamma^i_{A0}$ and summing over $A$ one gets

$$L_A \gamma^i_{A0} = \bar{L}_A \gamma^i_{A0} = 0. \tag{7.6}$$

The initial data (7.5) must obey these conditions.
For the singular Lagrangian densities there appear the following difficulties in the transition to the Hamiltonian formalism. As is customary, the derivatives of $\mathcal{L}$ with respect to $u_{A,0}$ are designated as the canonical momenta

$$\pi_A = \frac{\partial \mathcal{L}}{\partial u_{A,0}}. \tag{7.7}$$

In view of (7.1) Eqs. (7.7) cannot be solved with respect to the "velocities" $u_{A,0}(x)$ and the momenta $\pi_A(x)$ and "coordinates" $u_A(x)$ obey $r$ relationships. To obtain these constraints on $\pi_A(x)$ and $u_A(x)$, we consider the identities (7.4) substituting into them the definitions (7.2) and (7.7)

$$\frac{\partial \pi_B}{\partial u_{A,0}} \gamma^i_{B0} = 0, \quad i = 1, \ldots, r; \quad A = 1, 2, \ldots, N. \tag{7.8}$$

If the $\gamma^i_{A0}$'s do not depend on the "velocities" $u_{A,0}$ then the straightforward integration of these relationships gives $r$ contraints

$$\pi_A \gamma^i_{A0} - K^i(u_C, u_{C,s}) \equiv 0, \qquad s \neq 0; \quad i = 1, 2, \ldots, r, \tag{7.9}$$

where the $K^i$'s are functions introduced by the integration, but actually determined in any theory.
The Hamiltonian treatment of such systems was considered by Dirac [*20*], Bergmann [*12*—*18*] and in papers [*21, 51, 52*].

## 8. The Classical System of $N$ Massive Points

If the particles are interacting by central pair forces, then the Lagrangian of this system can be taken in the form

$$L = T - U = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i^2 - \sum_{i<j} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|), \quad \dot{\boldsymbol{r}}_i = \frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i. \tag{8.1}$$

The action

$$S[\boldsymbol{r}_i(t)] = \int_{t_1}^{t_2} dt L(t) \tag{8.2}$$

is invariant with respect to the 10-parameter group of the coordinate $\boldsymbol{r}_i$ and time $t$ transformations (the Newton-Galilei group). The infinitesimal transformations are given by

$$t' = t + \delta t, \tag{8.3}$$

$$\boldsymbol{r}_i'(t) = \boldsymbol{r}_i(t) + \delta \boldsymbol{r}, \tag{8.4}$$

$$\boldsymbol{r}_i'(t) = \boldsymbol{r}_i(t) + [\delta \boldsymbol{\varphi} \times \boldsymbol{r}_i(t)], \tag{8.5}$$

$$\boldsymbol{r}_i'(t) = \boldsymbol{r}_i(t) + \delta \boldsymbol{v} \cdot t. \tag{8.6}$$

Under the proper Galilei transformations (8.6) the Lagrangian (8.1) is invariant up to the total divergence only [*11*]. The consequence of this symmetry is the well known 10 integrals of motion in an $N$ point system. The invariance under time translation (8.3) gives rise to the conserved total energy

$$E = T + U = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i^2 + \sum_{i<j} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|).$$

The invariance with respect to the coordinate translations (8.4) results in the conserved total momentum

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i(t).$$

The conservation of the total angular momentum of this system

$$\boldsymbol{M} = \sum_{j=1}^{N} m_j [\boldsymbol{r}_j \times \dot{\boldsymbol{r}}_j]$$

is the consequence of the invariance under rotations (8.5).
The proper Galilei transformations (8.6) lead to the following alteration of the Lagrangian function

$$L'(\boldsymbol{r}_i', \dot{\boldsymbol{r}}_j') = L(\boldsymbol{r}_i, \dot{\boldsymbol{r}}_j) + \frac{d}{dt}\left(\delta \boldsymbol{v} \sum_{j=1}^{N} m_j \boldsymbol{r}_j(t)\right).$$

In this case the quantity $C$ in Eq. (2.13) is given by

$$C = \delta \boldsymbol{v} \sum_{i=1}^{N} m_i \boldsymbol{r}_i. \tag{8.7}$$

Making use of Eqs. (3.6), (3.7) and (8.7) we find

$$\frac{d}{dt}\left(-\sum_{j=1}^{N} m_j \dot{\boldsymbol{r}}_j t + \sum_{j=1}^{N} m_j \boldsymbol{r}_j\right) = 0.$$

This formula can be rewritten in the form

$$\boldsymbol{R}(t) \equiv \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i(t)}{M} = \frac{\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i}{M} t + \boldsymbol{R}_0 = \frac{\boldsymbol{P}}{M} t + \boldsymbol{R}_0,$$

where $M = \sum_{j=1}^{N} m_j$ is the total system mass. Thus the invariance under proper Galilei transformations results in the law of the uniform and rectilinear motion of the system centre of mass $\boldsymbol{R}(t)$.

## 9. The Charged Particles Interacting with Electromagnetic Field

The action in this case is given by

$$S = -\sum_{i=1}^{N} m_i \int \sqrt{\dot{z}_i^2(\tau_i)}\, d\tau_i - \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} - e\sum_{j=1}^{N}\int A_\mu(z_j)\, \dot{z}_j^\mu(\tau_j)\, d\tau_j,$$
$$\dot{z}_i^\mu = dz_i^\mu(\tau_i)/d\tau_i, \qquad \mu = 0, 1, 2, 3, \tag{9.1}$$

where $z_j^\mu(\tau_j)$ are parametric representations of the particle trajectories. This action is invariant under the ten-parameter Poincare group of the coordinate transformations, under the local gauge transformations of the electromagnetic potentials $A_\mu$ and with respect to reparametrizations

$$\tau_j' = f_j(\tau_j), \quad j = 1, 2, \ldots, N, \tag{9.2}$$

where $f_j$ are arbitrary functions. Let us dwell on the consequences of the invariance of this theory under the reparametrizations (9.2). According to the second Noether theorem the equations of particle motion

$$L_\mu{}^i = m_i \frac{d}{d\tau_i}\left(\frac{\dot{z}_{\mu i}(\tau_i)}{\sqrt{\dot{z}_i^2(\tau_i)}}\right) - eF_{\mu\nu}(z_i)\, \dot{z}_i^\nu(\tau_i) = 0, \qquad i = 1, \ldots, N$$

have to satisfy $N$ identities. Taking into account that $\delta z_j^\mu(\tau_j) = -\dot{z}_j^\mu(\tau_j)\, \varepsilon_j(\tau_j)$ we obtain from (3.10)

$$\sum_{\mu=0}^{3} L_\mu{}^j \dot{z}_j^\mu = 0, \quad j = 1, 2, \ldots, N. \tag{9.3}$$

Here there is no summation over $j$. These identities can be easily proved because $F_{\mu\nu}$ is a skewsymmetric tensor and the derivative of the unit Lorentz vector $\dot{z}_i^\mu/\sqrt{\dot{z}_i^2}$ is orthogonal to it.

By virtue of the identities (9.3) we can impose on the dynamical variables $z_i^\mu(\tau_i)$ $N$ conditions, for example $\dot{z}_i^2 = 1$, $i = 1, 2, \ldots, N$. In this case the parameter $\tau_i$ is the proper-time of an $i$-th particle.

In addition to the identities (9.3) in the theory under consideration there are the following identities, which are consequences of the invariance of (9.1) under the transformations (9.2)

$$\mathscr{L}_j(z_j) - \frac{\partial \mathscr{L}_j(z_j)}{\partial \dot{z}_{j\mu}} \dot{z}_{j\mu} \equiv 0, \quad j = 1, 2, \ldots, N, \tag{9.4}$$

where $\mathscr{L}_j$ is the Lagrangian function of the $j$-th particle

$$\mathscr{L}_j(z_j) = -m \sqrt{\dot{z}_j^2} - eA_\mu(z_j)\, \dot{z}_j^\mu .$$

If we distract ourselves from the specific form of the Lagrangian function in Eq. (9.1) and demand the invariance of the action $S$ under the transformations (9.2) only, then we obtain a constraint on the form of the admissible Lagrangian $\mathscr{L}_i$. Indeed, according to Eqs. (9.4) the Lagrangian $\mathscr{L}_i$ must be a homogeneous function of the first degree of the particle velocity $\dot{z}_\mu^i(\tau_i)$.
The particle canonical momenta are defined by

$$p_i^\mu(\tau_i) = -\frac{\partial \mathscr{L}}{\partial \dot{z}_\mu^i(\tau_i)} = m_i \frac{\dot{z}_i^\mu(\tau_i)}{\sqrt{\dot{z}_i^2}} + eA^\mu(z_i).$$

We obtain immediately the following constraints for $p_i^\mu$

$$[p_i^\mu(\tau_i) - eA^\mu(z_i)]^2 = m_i^2, \quad i = 1, 2, \ldots, N.$$

Thus, the Lagrangian (9.1) is singular.

## 10. Electrodynamics

The Maxwell-Dirac Lagrangian density is

$$\mathscr{L} = \frac{i}{2} [\overline{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\, \psi - (\partial_\mu + ieA_\mu)\, \overline{\psi}\gamma^\mu\psi] - m\overline{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}, \tag{10.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It gives rise to the following equations of motion

$$\begin{aligned} L_\psi &= (-i\,\partial_\mu + eA_\mu)\, \overline{\psi}\gamma^\mu - m\overline{\psi} = 0, \\ L_{\overline{\psi}} &= (i\gamma^\mu\, \partial_\mu + eA_\mu\gamma^\mu - m)\, \psi = 0, \\ L_{A_\mu} &= \partial^2 A^\mu - \partial^\mu\, \partial_\nu A^\nu + e\overline{\psi}\gamma^\mu\psi = 0. \end{aligned} \tag{10.2}$$

The action with Lagrangian (10.1) is invariant under the gauge transformations of the second kind

$$\begin{aligned} \psi'(x) &= e^{i\alpha(x)}\psi(x), \quad \overline{\psi}'(x) = e^{-i\alpha(x)}\overline{\psi}(x), \\ A_\mu'(x) &= A_\mu(x) - \frac{1}{e}\, \partial_\mu\alpha(x). \end{aligned} \tag{10.3}$$

In this case the form variations of the field functions are defined by

$$\bar{\delta}\psi(x) = i\alpha(x)\, \psi(x), \quad \bar{\delta}\overline{\psi}(x) = -i\alpha(x)\, \overline{\psi}(x), \quad \bar{\delta}A_\mu(x) = -\frac{1}{e}\, \partial_\mu\alpha(x).$$

The coefficients $\gamma_A{}^i$ and $\gamma^i_{A\mu}$ in Eq. (3.8) are

$$\gamma_\psi = i\psi, \quad \gamma_{\psi\mu} = 0, \quad \gamma_{\overline{\psi}} = -i\overline{\psi}, \quad \gamma_{\overline{\psi}\mu} = 0,$$

$$\gamma_{A_\nu} = 0, \quad \gamma_{A_\nu\mu} = -\frac{1}{e}\,\delta_{\mu\nu}.$$

According to (3.10) we obtain the following Noether identity

$$L_\psi\psi(x) - \overline{\psi}(x)\,L_{\overline{\psi}} + \frac{d}{dx^\nu}\,L_{A_\nu} \equiv 0. \tag{10.4}$$

It means that at least one equation in the system (10.2) is a consequence of others. Therefore Eqs. (10.2) do not determine $\overline{\psi}(x)$, $\psi(x)$, $A_\mu(x)$ completely and these functions can be subjected to one condition, for example, the Lorentz condition

$$\partial_\mu A^\mu(x) = 0. \tag{10.5}$$

However, the identity (10.4) does not exhaust all the consequences of the electrodynamics invariance under gauge transformations (10.3). Equations (3.12)—(3.14) give rise to the more general class of identities

$$\left(\frac{\partial\mathscr{L}}{\partial\psi_{,\mu}}\cdot\psi\right)_{,\mu} - \left(\overline{\psi}\,\frac{\partial\mathscr{L}}{\partial\overline{\psi}_{,\mu}}\right)_{,\mu} + L_\psi\overline{\psi} - \overline{\psi}L_{\overline{\psi}} \equiv 0, \tag{10.6}$$

$$L_{A_\nu} + \left(\frac{\partial\mathscr{L}}{\partial A_{\nu,\mu}}\right)_{,\mu} - ie\left(\frac{\partial\mathscr{L}}{\partial\psi_{,\nu}}\,\psi - \overline{\psi}\,\frac{\partial\mathscr{L}}{\partial\overline{\psi}_{,\nu}}\right) \equiv 0, \tag{10.7}$$

$$\frac{\partial\mathscr{L}}{\partial A_{\nu,\mu}} + \frac{\partial\mathscr{L}}{\partial A_{\mu,\nu}} \equiv 0. \tag{10.8}$$

The last identity (10.8) is trivial as it means that $F_{\mu\nu} = -F_{\nu\mu}$. Differentiating with respect to $x_\nu$ Eq. (10.7) and making use of (10.6) and (10.8) one obtains easily the identity (10.4). Therefore we have here only one new identity. Consequently in the system (10.2) actually two equations are consequences of others. And in addition to Eq. (10.5) one may impose one more condition, for example, we can take the Coulomb gauge

$$A_0(x) = 0. \tag{10.9}$$

Only in this case the equation of motion (10.2) together with the conditions (10.5) and (10.9) will define the functions $\overline{\psi}(x)$, $\psi(x)$, and $A_\mu(x)$ to be found.
The invariance of (10.1) under the global gauge transformations with $\alpha(x) = \text{const.}$ results in the conserved current

$$j^\mu(x) = -e\overline{\psi}(x)\,\gamma^\mu\psi(x), \quad \partial_\mu j^\mu(x) = 0.$$

According to the Noether classification it is *improper current*, i.e. it can be expressed in terms of the left-hand sides of the Euler equations and the divergence of a skewsymmetric tensor. It follows directly from (10.2)

$$j^\mu(x) = -L_{A_\mu} + \partial^2 A^\mu - \partial^\mu\,\partial_\nu A^\nu = -L_{A_\mu} - \partial_\nu F^{\mu\nu}.$$

The role of superpotential (5.6) is played here by the tensor of electromagnetic field $F_{\mu\nu}$. The total electric charge is given by the two-dimensional surface integral

(Gauss' law)

$$Q = \int_{V_3} d^3x j_0(x) = -\int_{V_3} d^3x \sum_{\alpha=1}^{3} \partial^\alpha F_{0\alpha} = \int_{V_3} d^3x \operatorname{div} \boldsymbol{E} = \int_\sigma d^2\sigma\, \boldsymbol{En},$$

where $E_\alpha = F_{0\alpha}$, $\alpha = 1, 2, 3$ is electric field and $\boldsymbol{n}$ is unit normal to $\sigma$. Making use of (6.1) we find here *the strong conservation law*

$$\partial_\mu(\partial^2 A^\mu - \partial^\mu\, \partial_\nu A^\nu) = \partial_\mu\, \partial_\nu F^{\nu\mu} \equiv 0:$$

## 11. The Vector Neutral Massive Field Interacting with the Dirac Field

This system is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2} A_{\mu,\nu}A^{\mu,\nu} + \frac{m^2}{2} A_\mu^2 + \frac{i}{2} [\overline{\psi}\gamma^\mu(\partial_\mu - igA_\mu)\, \psi - (\partial_\mu + igA_\mu)\, \overline{\psi}\gamma^\mu\psi] - M\overline{\psi}\psi. \tag{11.1}$$

The action functional in this model is invariant up to divergence under gauge transformations (10.3) with the function satisfying the equation [53]

$$\partial_\mu(\partial^2 + m^2)\, \alpha(x) = 0. \tag{11.2}$$

In this case the Lagrangian (11.1) transforms in the following way

$$\mathcal{L} \to \mathcal{L} + g^{-1}\, \partial_\mu\big(A_\nu(x)\, \partial^\mu\, \partial^\nu\alpha(x)\big).$$

The divergence relation (3.5) by virtue of the equation of motion has the form

$$\frac{d}{dx_\mu}\big(j_\mu(x) - iA_\nu(x)\, \partial_\mu\, \partial^\nu\alpha(x)\big) = 0,$$

where $j_\mu(x) = -g\overline{\psi}\gamma_\mu\psi$. If $\alpha(x) =$ const., then this equation is reduced to the usual conservation law for current $j_\mu(x)$

$$\partial_\mu j^\mu(x) = 0.$$

The Noether identities (3.10) and (3.12)—(3.14) cannot be obtained in this model, as $\alpha(x)$ is not an arbitrary function but it obeys Eq. (11.2). The equations of motion obtained by variation of (12.1) determine the fields $A_\mu(x)$ and $\psi(x)$ completely. The subsidiary condition $\partial_\mu A^\mu(x) = 0$ is imposed in this theory only for physical reasons. It is used for the elimination of the 0-spin particles from the model.

## 12. Nonabelian Gauge Fields

For the invariance of the theory of the multicomponent field $\psi(x)$ under the local transformations

$$\psi'(x) = \omega(x)\psi(x) \tag{12.1}$$

with the matrices $\omega(x)$ from the compact semisimple group $G$, the interaction of $\psi(x)$ with the gauge vector field $W_\mu(x)$ taking values in the Lie algebra of $G$ should be introduced [54—57]. In the Lagrangian this interaction is introduced by the following substitution of the derivatives $\partial_\mu$ acting on field $\psi(x)$

$$\partial_\mu \to \nabla_\mu = \partial_\mu + igW_\mu(x). \tag{12.2}$$

The covariant derivative $\nabla_\mu$ will be transformed by the simple rule

$$\nabla'\psi'(x) = \omega(x)\,\nabla\psi(x) \tag{12.3}$$

if the transformation of the gauge field $W_\mu(x)$ is defined by

$$W_\mu' = \omega W_\mu \omega^{-1} + ig^{-1}(\partial_\mu\omega)\,\omega^{-1}. \tag{12.4}$$

Thus, the field $W_\mu(x)$ is transformed inhomogeneously. Alternatively the strength tensor of the field $W_\mu$

$$G_{\mu\nu}(x) = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu] \tag{12.5}$$

is transformed homogeneously

$$G'_{\mu\nu}(x) = \omega(x)\,G_{\mu\nu}\omega^{-1}(x). \tag{12.6}$$

Further we shall consider the Yang-Mills fields with gauge group $SU(2)$. In this case

$$\begin{gathered} W_\mu = \frac{1}{2}\,\sigma^a W_\mu{}^a, \quad G_{\mu\nu} = \frac{1}{2}\,\sigma^a G^a_{\mu\nu}, \\ G^a_{\mu\nu} = \partial_\mu W_\nu{}^a - \partial_\nu W_\mu{}^a - g\varepsilon^{abc} W_\mu{}^b W_\nu{}^c, \quad a, b, c = 1, 2, 3, \end{gathered} \tag{12.7}$$

where $\sigma^a$ are the Pauli matrices. If $\psi(x)$ is the Dirac field describing the $SU(2)$ doublet, then the total Lagrangian invariant under transformations (12.1), (12.4) is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\,G^a_{\mu\nu}G^{a\mu\nu} + \frac{i}{2}\left\{\overline{\psi}\gamma^\mu\left(\partial_\mu + igW_\mu{}^a\frac{\sigma^a}{2}\right)\psi - \left(\partial_\mu - igW_\mu{}^a\frac{{}^t\sigma^a}{2}\right)\overline{\psi}\gamma^\mu\psi\right\} - m\overline{\psi}\psi \\ &= -\frac{1}{2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} + \frac{i}{2}\,\{\overline{\psi}\gamma^\mu\nabla_\mu\psi - (\nabla_\mu\overline{\psi})\,\gamma^\mu\psi\} - m\overline{\psi}\psi. \end{aligned} \tag{12.8}$$

It leads to the following equations of motion

$$\begin{gathered} L_\nu{}^a \equiv L_{W^{a\nu}} = \partial^\mu G^a_{\mu\nu} - gW^{b\mu}G^c_{\mu\nu}\varepsilon^{abc} - g\overline{\psi}\gamma_\nu\frac{\sigma^a}{2}\psi = 0, \\ L_\psi = \left(i\gamma^\mu\,\partial_\mu - g\frac{\sigma^a}{2}\,W_\nu{}^a\gamma^\nu - m\right)\psi = 0. \end{gathered} \tag{12.9}$$

In the matrix notation these equations can be rewritten in the form

$$\begin{gathered} D^\mu G_{\mu\nu} = j_\nu, \\ (i\gamma^\mu\,\nabla_\mu - m)\,\psi = 0, \end{gathered} \tag{12.10}$$

where $D^\mu = \partial^\mu + ig[W,{}^\mu \ldots]$ is the covariant derivative for the fields with values in Lie algebra of the group $G$ and

$$j_\nu = j_\nu{}^a\cdot\frac{\sigma^a}{2}, \quad j_\nu{}^a = g\overline{\psi}\gamma_\nu\frac{\sigma^a}{2}\,\psi. \tag{12.11}$$

The action with the Lagrangian (12.8) is invariant under the global $U(1)$-transformations

$$\begin{gathered} \psi'(x) = e^{i\alpha}\psi(x), \quad \overline{\psi}'(x) = e^{-i\alpha}\overline{\psi}(x), \quad \alpha = \text{const.}, \\ W_\mu{}'^a(x) = W_\mu{}^a(x) \end{gathered} \tag{12.12}$$

and with respect to the $SU(2)$-transformations (12.1), (12.4) with

$$\omega(x) = \exp\left[i \sum_{a=1}^{3} \frac{\sigma^a}{2} s^a(x)\right]. \tag{12.13}$$

The invariance of the theory under the transformations (12.12) results, as in electrodynamics, in a conserved current

$$Y^\mu(x) = \bar\psi(x)\, \gamma^\mu \psi(x). \tag{12.14}$$

By the infinitesimal $SU(2)$-transformations we have

$$\begin{gathered} W_\mu{}'^a(x) = W_\mu{}^a(x) + \varepsilon^{abc} W_\mu{}^b(x)\, s^c(x) - g^{-1}\, \partial_\mu s^a(x), \\ G'^a_{\mu\nu}(x) = G^a_{\mu\nu}(x) + \varepsilon^{abc} G^b_{\mu\nu}(x)\, s^c(x). \end{gathered} \tag{12.15}$$

Thus, by the global $SU(2)$-transformations when $s^a(x) = \text{const.}$ the Yang-Mills field $W_\mu{}^a(x)$ is transformed as an isotopic vector but by the local ones there appears an additional term $-g^{-1}\, \partial_\mu s^a(x)$. The tensor $G^a_{\mu\nu}(x)$ is transformed in both cases as an isotopic vector.

If in Eq. (12.15) $s^a(x) = \text{const.}$, then we have, according to the first Noether theorem, the following conserved isotopic current

$$I_\mu{}^a(x) = g\bar\psi(x)\, \gamma_\mu \frac{\sigma^a}{2}\, \psi(x) + g\varepsilon^{abc} G^b_{\mu\nu}(x)\, W^{c\nu}(x) \tag{12.16}$$

or in the matrix notation

$$I_\mu(x) = I_\mu{}^a(x)\, \frac{\sigma^a}{2} = j_\mu(x) + ig[W^\nu(x), G_{\mu\nu}(x)], \tag{12.17}$$

where $j_\mu(x)$ is defined in Eq. (12.11).

The gauge field $W_\mu{}^a(x)$ carries the isotopic charge and gives a contribution into the total isotopic current (the second term in Eq. (12.16)). It should be noted that the quantity $g\varepsilon^{abc} G^b_{\mu\nu}(x)\, W^{c\nu}(x)$ and, as a consequence, the isotopic current density (12.16), are not isotopic vectors under local $SU(2)$-transformations. The total isotopic spin of the system

$$T^a = \int_{V_3} I_0{}^a(x)\, d^3x \tag{12.18}$$

is, however, the isotopic vector with respect to the local gauge transformations that at the spatial infinity become constant independent on the coordinate transformations, i.e. $s^a(x) \xrightarrow[|x| \to \infty]{} \text{const.}$ Indeed, using the equations of motion (12.9) and taking into account that $G^a_{00}(x) = 0$ we obtain

$$T^a = \int_{V_3} I_0{}^a(x)\, d^3x = \int_{V_3} \partial^\mu G^a_{\mu 0}(x)\, d^3x = -\int_{V_3} d^3x \sum_{j=1}^{3} \frac{\partial G^a_{j0}}{\partial x^j}. \tag{12.19}$$

By Gauss' theorem this integral can be reduced to the integral of $G^a_{j0}(x)$ on the surface containing the three-dimensional space $V_3$. If on this surface $s^a(x) = \text{const.}$, then $T^a$ is a vector with respect to the transformations (12.13).

The action with Lagrangian (12.8) is invariant under global $U(1)$-transformations (12.12) and under global and local $SU(2)$-transformations. With respect to the local-transformations it is not invariant. Thus, according to the Noether classification the current $Y_\mu(x)$ in Eq. (12.13) is *a proper current*. Alternatively, the isotopic current $I_\mu{}^a(x)$ (12.16) is *an improper current*. As a consequence the current $I_\mu{}^a(x)$ is reduced, by virtue of the equations of motion, to the divergence of the antisymmetric tensor $G^a_{\mu\nu}(x)$. Indeed, the first equation in (12.9) can be rewritten by making use of (12.16) as follows

$$\partial^\mu G^a_{\mu\nu}(x) = I_\nu{}^a(x). \tag{12.20}$$

Both the currents $Y^\mu(x)$ (12.14) and $I_\mu{}^a(x)$ (12.16) are conserved separately, so their linear combinations will be conserved also. In the Weinberg model [55], for example, as the electric current one takes the sum

$$J_\mu{}^{\mathrm{em}}(x) = \frac{1}{2}\, Y_\mu(x) + I_\mu{}^3(x). \tag{12.21}$$

We note here the common properties of the Yang-Mills model and the Einstein gravitattion theory (see sections 13, 14). The isotopic current density (12.16) consists of two parts. The first term is caused by the "matter" fields (all the fields except the Yang-Mills field $W_\mu{}^a(x)$) and it is the isotopic vector. The second term as it was noted above, describes the isotopic charge of the Yang-Mills field and it is not the isotopic vector.
The similar situation holds in the gravitation theory. The total energy-momentum density $\Theta_{\mu\nu}(x)$ consists of two terms also

$$\Theta_{\mu\nu}(x) = T^{\mathrm{sym}}_{\mu\nu}(x) + t_{\mu\nu}(x), \tag{12.22}$$

where $T^{\mathrm{sym}}_{\mu\nu}(x)$ is the symmetric energy-momentum tensor of the "matter" fields (all fields except the gravitation field), $t_{\mu\nu}(x)$ is the density of the energy-momentum of the gravitation field. Under the general coordinate transformations (13.2) $T^{\mathrm{sym}}_{\mu\nu}(x)$ is transformed as a tensor, while $t_{\mu\nu}(x)$ is not a general covariant tensor. At the best $t_{\mu\nu}(x)$ is transformed as a tensor only under linear (affine) coordinate transformations. Therefore it is called the *pseudotensor or the complex of the energy-momentum of the gravitation field.*
Now we shall go to the Noether identities in the Yang-Mills theory (12.8). Making use of Eqs. (3.10) we obtain three identities

$$(D_\mu L^\mu)^c + i\,\frac{g}{2}\,(L_\psi \sigma^c \psi - \bar{\psi}\sigma^c L_{\bar{\psi}}) \equiv 0, \tag{12.23}$$

where $L^\mu$, $L_{\bar{\psi}}$ and $L_\psi$ are defined in Eqs. (12.9), (12.10). Here we have put as usual, $L_\mu = L_\mu{}^a\, \sigma^a/2$.
By virtue of the identities (12.23) the Euler equations (12.9) or (12.10) do not determine completely the dynamical variables $W_\mu{}^a(x)$ and $\psi(x)$. Therefore these equations can be supplemented by gauge conditions. For example, the Lorentz gauge can be imposed

$$\partial^\mu W_\mu{}^a(x) = 0, \quad a = 1, 2, 3. \tag{12.24}$$

At the end of this section devoted to the Yang-Mills fields it should be noted that in this theory besides the Noether concerved currents there are conserved quantities of quite a different *topological origin*. It is the so called topological charge in the Euclidean formulation of the Yang-Mills theory

$$q = \frac{g^2}{32\pi^2} \int d^4x G^a_{\mu\nu}(x)\, \tilde{G}^a_{\mu\nu}(x),$$

where $\tilde{G}^a_{\mu\nu}(x)$ is a tensor dual to the $G^a_{\mu\nu}(x)$

$$\tilde{G}^a_{\mu\nu}(x) = \frac{1}{2} \varepsilon_{\mu\nu\varrho\sigma} G^a_{\varrho\sigma}(x).$$

This property of the Yang-Mills fields was investigated intensively in the last years (see, for example [55—58]).

### 13. The Gravitation Theory

The Einstein theory of gravitation, called the general theory of relativity, is a new method of description of the interaction transmission. This method is very different from the usual one. In the general theory of relativity it is postulated that the gravitating masses alter the geometrical properties of the space-time. The space-time is now not flat but it acquires a nonzero curvature. This curvature alters the flow of all the physical processes, thereby the existence of gravitating sources displays at the physical level.

In general theory of relativity the space-time is considered to be the four-dimensional Riemannian manifold with the metric tensor $g_{\mu\nu}(x)$. The curvature of the space-time is defined by the curvature tensor of the fourth rank $R_{\mu\nu\lambda\varrho}(x)$ (the Riemann-Christoffel tensor). This tensor is constructed by differential geometry formulae from the metric tensor $g_{\mu\nu}(x)$ and its first and second derivatives. The space-time is flat only in the case when $R_{\mu\nu\lambda\varrho}(x) = 0$. At first sight the curvature tensor $R_{\mu\nu\lambda\varrho}(x)$ should be taken as a dynamical variable in the gravitation theory. But in Einstein's theory this role is played by the metric tensor $g_{\mu\nu}(x)$ the form of which depends not only on the space-time curvature but also on the chosen coordinate system. It is important that in general case one cannot pick out the part of $g_{\mu\nu}(x)$ determined by space-time curvature and the part caused by the coordinate system.

This situation is analogous in some sense to that in the theory of the electromagnetic field. In this theory the quantities directly measured in experiment are the intensities of electric and magnetic fields (the strength tensor $F_{\mu\nu}(x)$). On the other hand the electromagnetic field theory is constructed by means of the electromagnetic potential $A_\mu(x)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The action functional in Einstein's theory is taken in the form

$$S = S_g + S_m, \quad S_g = -\varkappa^{-1} \int R \sqrt{-g}\, d^4x, \quad S_m = \int \mathcal{L}_m \sqrt{-g}\, d^4x, \tag{13.1}$$

where $R(x)$ is the scalar curvature: $R = g^{\mu\nu} R_{\mu\nu}$, $R_{\mu\nu} = g^{\varrho\sigma} R_{\varrho\mu\sigma\nu}$, $\mathcal{L}_m$ is the Lagrangian density of all the fields but the gravitational field, $\varkappa$ is the gravitational constant. The function $\mathcal{L}_m$ must be chosen so that the action functional $S_m$, and as a consequence $S$, be invariant under the general coordinate transformations

$$x'^\mu = f^\mu(x), \quad \mu = 0, 1, 2, 3. \tag{13.2}$$

The variation of the gravitational field $g_{\mu\nu}(x)$ gives the Einstein-Hilbert equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \varkappa T_{\mu\nu}, \tag{13.3}$$

where $(1/2)\sqrt{-g}\, T_{\mu\nu} = \delta S_m / \delta g^{\mu\nu}$ is the energy-momentum tensor of all the fields except for the gravitational field.

According to the second Noether theorem the invariance of the functional (13.1) under the transformations (13.2) gives rise to four Bianchi identities (3.10) which must be satisfied by the left-hand sides of Eqs. (13.3). It is interesting to obtain these identities anew, instead of the substitution into (3.10) of concrete expressions for $\gamma_A{}^i$ and $\gamma^i_{A\mu}$. We shall not specify the form of $S$ but suppose for simplicity that only one vector field $W_\mu(x)$ interacts with the gravitational field.

The Euler equations, obtained by the variation of $g^{\mu\nu}(x)$ and $W^\mu(x)$ in $S$, have the form

$$\mathcal{S}_{\mu\nu}(x) = 0, \tag{13.4}$$

$$L_\nu = 0, \tag{13.5}$$

where $\mathcal{S}_{\mu\nu}$ us a symmetric tensor $\mathcal{S}_{\mu\nu} = \mathcal{S}_{\nu\mu}$ and

$$\mathcal{S}_{\mu\nu} = G_{\mu\nu} + T_{\mu\nu}, \tag{13.6}$$

$$\frac{1}{2}\sqrt{-g}\,G_{\mu\nu} = \frac{\partial S_g}{\delta g^{\mu\nu}},\quad \frac{1}{2}\sqrt{-g}\,T_{\mu\nu} = \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{13.7}$$

For Einstein's theory

$$G_{\mu\nu} = -\varkappa^{-1}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right). \tag{13.8}$$

By the infinitesimal transformations

$$x'^\mu = x^\mu + \varepsilon^\mu(x)$$

the form variations of $W^\mu(x)$ and the metric tensor $g^{\mu\nu}(x)$ are given by

$$\bar\delta W^\mu(x) = -W^\mu_{,\nu}\varepsilon^\nu(x) + W^\nu\varepsilon^\mu_{,\nu}(x), \tag{13.9}$$

$$\bar\delta g^{\mu\nu}(x) = \nabla^\nu\varepsilon^\mu(x) + \nabla^\mu\varepsilon^\nu(x), \tag{13.10}$$

where $\nabla^\mu$ denotes the covariant differentiation. The variation $\delta S$ by virtue of (3.1) can be written as

$$\delta S = \frac{1}{2}\int \mathcal{G}_{\mu\nu}\bar\delta g^{\mu\nu}\sqrt{-g}\,d^4x + \int L_\mu\bar\delta W^\mu\sqrt{-g}\,d^4x - \int \frac{dJ^\mu}{dx^\mu}\,d^4x. \tag{13.11}$$

We choose functions $\varepsilon^\mu(x)$ so that they and their derivatives in div $J$ in Eq. (13.11) vanish on the boundary of integration region. Substituting (13.9) and (13.10) into (13.11) one gets

$$\begin{aligned}\delta S &= 2\int \mathcal{S}_{\mu\nu}\nabla^\nu\varepsilon^\mu(x)\sqrt{-g}\,d^4x + \int L_\mu\big(W^\nu\varepsilon^\mu_{,\nu}(x) - W^\mu_{,\nu}\varepsilon^\nu(x)\big)\sqrt{-g}\,d^4x\\ &= \int [(2\mathcal{S}_\mu{}^\nu + L_\mu W^\nu)\,\varepsilon^\mu_{;\nu}(x) + (2\mathcal{S}_\mu{}^\nu\,\Gamma^\mu_{\sigma\nu} - L_\mu W^\mu_{,\sigma})\,\varepsilon^\sigma(x)]\sqrt{-g}\,d^4x\\ &= \int [-2\nabla_\nu\mathcal{S}_\mu{}^\nu - \partial_\nu(L_\mu W^\nu) - L_\sigma W^\sigma_{,\mu}]\,\varepsilon^\mu(x)\sqrt{-g}\,d^4x = 0.\end{aligned} \tag{13.12}$$

The functions $\varepsilon^\mu(x)$ inside the integration region are arbitrary, from Eq. (13.12) four identities follow

$$2\nabla_\nu\mathcal{S}_\mu{}^\nu - \partial_\nu(L_\mu W^\nu) + L_\nu W^\nu_{,\mu} \equiv 0, \tag{13.13}$$

$$\mu = 0, 1, 2, 3.$$

These identities are the Bianchi identities (3.10) written now for the gravitational field. As in electrodynamics one can obtain for the Euler equations in GTR a more general class of identities corresponding to Eqs. (3.12)—(3.14). We shall not consider them referring to Klein's paper [5].

Thus, in general theory of relativity the equations of motion can be supplemented at least by 4 conditions. For example, one may choose such a coordinate system which satisfies harmonic conditions [59]

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu}(x)\right) = 0.$$

The reasoning that leads us to the identities (13.13) is applicable obviously to the gravitational action $S_g$ separately. This functional depends on the metric tensor $g_{\mu\nu}(x)$ and its derivatives but not on the "matter" fields. As a result we obtain the following identities

$$\nabla_\nu G_\mu{}^\nu \equiv 0, \quad \mu = 0, 1, 2, 3, \tag{13.14}$$

where the tensor $G_{\mu\nu}$ is given by (13.7). In Einstein's theory $G_{\mu\nu}$ has the form (13.8) and the identities (13.14) in this case are consequences of the Bianchi identity for the Rimann-Christoffel tensor $R_{\mu\nu\lambda\sigma}$. In the literature there are considered other Lagrangian densities for the gravitational field which are quadratic in the curvature tensor [60]

$$\mathcal{L}_g = aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma}. \tag{13.15}$$

In this case Eqs. (13.14) give rise to new identities.

We are going now to the consideration of the identities (13.13). Suppose that the equations of motion for the matter fields (13.5) are satisfied. Then by virtue of (13.6) and (13.14) one obtains from (13.13)[2])

$$\nabla_\nu T_\mu{}^\nu = 0, \quad \mu = 0, 1, 2, 3. \tag{13.16}$$

Thus, the covariant divergence of the energy-momentum tensor obtained by variation of $g^{\mu\nu}$ in $S_m$ vanishes by virtue of the equations of motion of the "matter" fields only. Using this fact Hilbert proposed [4] a method for obtaining the symmetric energy-momentum tensor for field theories in a flat space-time. This problem will be considered in the next section.

Now we examine in the framework of the gravitation theory the identities (3.14). As it was noted, these formulae do not contain the Lagrangian expressions, i.e. they are constraints on the Lagrangian density form. Let us assume that the Lagrangian density $\mathcal{L}_W$ of the vector field $W_\mu(x)$ interacting with the gravitational field has to be constructed [1]. For simplicity we suppose that $\mathcal{L}_W$ depends on $W_\mu(x)$, $\partial_\nu W_\mu(x)$ and $g_{\mu\nu}(x)$. By the infinitesimal transformations (13.7) Eq. (3.8) for the vector field $W_\mu(x)$ has the form

$$\bar{\delta} W_A(x) = W_{A,i}\varepsilon^i - W_B\varepsilon^B_{.A}, \quad A, B, i = 0, 1, 2, 3. \tag{13.17}$$

Thus

$$\gamma^i_{A\mu} = -\delta_{A\mu}W^i. \tag{13.18}$$

Substituting (13.17) into (3.14) we get

$$\frac{\partial \mathcal{L}_W}{\partial W_{\mu,\nu}} + \frac{\partial \mathcal{L}_W}{\partial W_{\nu,\mu}} \equiv 0. \tag{13.19}$$

This means that the derivatives of $W_\mu$ can occur in $\mathcal{L}_W$ in the combination

$$W_{\mu,\nu} - W_{\nu,\mu}$$

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[2]) On the other hand this equality can be considered as a consequence of the equations of the gravitational field (13.4) alone.

only. It is a natural requirement if one takes into account that

$$W_{\mu,\nu} - W_{\nu,\mu} = \nabla_\nu W_\mu - \nabla_\mu W_\nu .$$

Many applications of the second Noether theorem in Yang-Mills theory and in gravitation theory can be found in papers [*61, 62*].

## 14. The Energy-Momentum Tensor

In Lorentz-invariant field theories there are used a few different energy-momentum tensors: canonical, symmetric, and improved ones.
The canonical energy-momentum tensor $T^c_{\mu\nu}$ is constructed according to the first Noether theorem as a consequence of the invariance under space-time translations

$$x'^\mu = x^\mu + a^\mu, \quad a^\mu = \text{const.} \tag{14.1}$$

If the field theory in Minkowski space-time is defined by the action functional

$$S = \int \mathcal{L}(u_A, \partial u_A)\, d^4x, \tag{14.2}$$

where $u_A(x)$ is a set of field functions, then *the canonical energy-momentum tensor* has the form

$$T^c_{\mu\nu} = \Pi_\mu{}^A\, \partial_\nu u_A - \vartheta_{\mu\nu}\mathcal{L}, \quad \partial^\mu T^c_{\mu\nu} = 0, \tag{14.3}$$

$$\Pi_\mu{}^A = \frac{\partial \mathcal{L}}{\partial\, \partial^\mu u_A}, \quad \vartheta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1). \tag{14.4}$$

The name of this tensor is due to its component $T^c_{00}$ is the Hamiltonian density $\mathcal{H}$, constructed according to the law

$$\mathcal{H} = \Pi_0{}^A\, \partial_0 u_A - \mathcal{L} = T^c_{00}. \tag{14.5}$$

In general $T^c_{\mu\nu}$ is not symmetric, $T^c_{\mu\nu} \neq T^c_{\nu\mu}$, for example, in models with vector fields. The method of obtaining the symmetric energy-momentum tensor was proposed by Hilbert [*4*].
For this purpose we generalize the action functional $S$ in Eq. (14.2) for a curved space-time with metric tensor $g_{\mu\nu}(x)$. This is made by substituting the usual derivatives $\partial_\mu$ by the covariant ones $\nabla_\mu$ and by introducing manifestly $g^{\mu\nu}(x)$ into all the sums over the Lorentz indices. As a result, we obtain the general invariant action

$$\bar{S} = \int \mathcal{L}(u_A, \nabla_\mu u_A) \sqrt{-g}\, d^4x. \tag{14.6}$$

The variation of $g^{\mu\nu}(x)$ in $S$ gives rise to the symmetric tensor

$$\frac{1}{2} \sqrt{-g}\, \bar{T}_{\mu\nu} = \frac{\delta \bar{S}}{\delta g^{\mu\nu}}, \tag{14.7}$$

and the variations of $u_A(x)$ result in the equations of motion for the fields $u_A(x)$

$$\bar{L}_A = \frac{\delta \bar{S}}{\delta u_A} = 0. \tag{14.8}$$

The reasoning of the preceding section is applicable to the tensor $\bar{T}_{\mu\nu}$ also (see Eqs. (13.13) and (13.16)). Therefore when equations (14.8) hold, the covariant divergence of $\bar{T}_{\mu\nu}$ vanishes

$$\nabla^{\nu}\bar{T}_{\nu\mu}(x) = 0. \tag{14.9}$$

Now we put in $\bar{T}_{\mu\nu}$

$$g_{\mu\nu}(x) = \mathring{g}_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1). \tag{14.10}$$

This gives rise to the tensor

$$T^{s}_{\mu\nu} = \bar{T}_{\mu\nu}\big|_{g_{\mu\nu}(x)=\mathring{g}_{\mu\nu}}, \tag{14.11}$$

which is symmetric obviously and the usual divergence of which vanishes

(14.12)

if the equations of motion

$$L_A = \frac{\delta S}{\delta u_A} = \bar{L}_A\bigg|_{g_{\mu\nu}(x)=\mathring{g}_{\mu\nu}} = 0 \tag{14.13}$$

hold. The manifest formulae which enable us to construct in this way *the symmetric energy-momentum tensor* $T^{s}_{\mu\nu}$ for a given Lagrangian density were written by Rosenfeld [*63*] and Belinfante [*64*]

$$T^{s}_{\mu\nu} = T^{c}_{\mu\nu} + \frac{1}{2}\,\partial^{\lambda}[\Pi_{\lambda}{}^{A}(\Sigma_{\mu\nu}u)_{A} - \Pi_{\mu}{}^{A}(\Sigma_{\lambda\nu}u)_{A} - \Pi_{\nu}{}^{A}(\Sigma_{\mu\lambda}u)_{A}] = T^{c}_{\mu\nu} + \partial^{\lambda}f_{\lambda\mu\nu}, \tag{14.14}$$

where $T^{c}_{\mu\nu}$ is given by (14.3) and

$$\Pi_{\mu}{}^{A} = \frac{\partial\mathcal{L}}{\partial u_{A,\mu}}, \quad f_{\lambda\mu\nu} = -f_{\mu\lambda\nu}.$$

The spin operator $\Sigma_{\mu\nu}$ for the scalar, Dirac, and vector field respectively has the form

$$\Sigma_{\mu\nu} = 0, \quad \Sigma_{\mu\nu} = \frac{1}{2}\,[\gamma_{\mu}, \gamma_{\nu}],$$
$$(\Sigma_{\mu\nu})_{\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta}. \tag{14.15}$$

A simple method for derivation of the symmetric energy-momentum tensor (14.14) without using the variation procedure was proposed in paper [*65*]. This method is analogous to that used in the classical mechanics [*66*] for derivation of the energy and momentum conservation laws without Noether's theorem.
The tensors $T^{s}_{\mu\nu}$ and $T^{c}_{\mu\nu}$ differ by a divergence. As a consequence, the conserved integral quantities corresponding to them are identical obviously

$$P_{\mu} = \int T^{c}_{0\mu}(x)\,d^{3}x = \int T^{s}_{0\mu}(x)\,d^{3}x. \tag{14.16}$$

By means of the symmetric energy-momentum tensor $T^{s}_{\mu\nu}$ the angular momentum tensor $M_{\mu\nu}$ is written in a compact form

$$M_{\mu\nu} = -\int (x_{\mu}T^{s}_{0\nu} - x_{\nu}T^{s}_{0\mu})\,d^{3}x. \tag{14.17}$$

The canonical energy-momentum tensor $T^{c}_{\mu\nu}$ used for this purpose leads to the more complicated formula

$$M_{\mu\nu} = -\int [x_{\mu}T^{c}_{0\nu} + x_{\nu}T^{c}_{0\mu} + \Pi_{0}{}^{A}(\Sigma_{\mu\nu}u)_{A}]\,d^{3}x. \tag{14.18}$$

The improved energy-momentum tensor $\Theta_{\mu\nu}^{\text{Imp}}$ is introduced in quantum field theory and is given by [67]

$$\Theta_{\mu\nu}^{\text{Imp}} = T_{\mu\nu}^{s} - \frac{1}{6} \sum_{A} (\partial_\mu \partial_\nu - g_{\mu\nu} \Box)\, u_A^2(x), \tag{14.19}$$

where the sum involves all the scalar fields $u_A(x)$.

At the end of this section we discuss shortly the problem of construction of the energy-momentum tensor and the corresponding conservation laws in general theory of relativity. If the equations of the gravitational field are obtained by variation of the action $S_g$ invariant under general coordinate transformations (13.2), then this requires that in the righthand sides of these equations the Hilbert symmetric energy-momentum tensor $T_{\mu\nu}^{s}$ of matter fields must be used. Therefore it is this tensor by means of which the matter fields have to be introduced into the energy-momentum conservation law. As it was shown in the preceding section, only the covariant divergence of this tensor instead of the usual one vanishes by virtue of the equations of motion of the matter fields

$$\nabla^\mu T_{\mu\nu}^{s} = 0. \tag{14.20}$$

Generally from this equation one cannot construct any conservation laws[3].

In order to formulate the energy-momentum conservation law in the gravitation theory the energy-momentum density of the gravitational field has to be defined. There are different definitions of this density [7, 70–76] but unfortunately each of them has shortcomings.

[3]) There is a very special case when Eqs. (14.20) give rise to integral conserved quantities. It is the case when the spacetime manifold has some symmetry that leads to the existence of *the Killing vectors* $\overset{(n)}{\xi}_\mu(x)$ [59]. These vectors are defined by the equation

$$\nabla_\mu \overset{(n)}{\xi}_\nu + \nabla_\nu \overset{(n)}{\xi}_\mu = 0. \tag{14.21}$$

The conserved quantities $Q_n$ are constructed in the following way

$$Q_n = \int \overset{(n)}{\xi}{}^\mu T_{\mu\nu}^{s} \sqrt{-g}\, d^3\sigma^\nu, \tag{14.22}$$

where the integration is spread out to the three-dimensional hypersurface $\sigma$. Let us show that $Q_n$ does not depend on the choice of the hypersurface $\sigma$. For this purpose we consider the integral

$$\begin{aligned} I &= \int_\Omega \frac{\partial}{\partial x_\nu} \left(\sqrt{-g}\, \overset{(n)}{\xi}{}^\mu T_{\mu\nu}^{s}\right) d^4x = \int_\Omega \sqrt{-g}\, \nabla^\nu \left(\overset{(n)}{\xi}{}^\mu T_{\mu\nu}^{s}\right) d^4x \\ &= \int_\Omega \sqrt{-g} \left(\nabla^\nu \overset{(n)}{\xi}{}^\mu\right) T_{\mu\nu}^{s}\, d^4x + \int_\Omega \sqrt{-g}\, \overset{(n)}{\xi}{}^\mu \nabla^\nu T_{\mu\nu}^{s}\, d^4x. \end{aligned} \tag{14.23}$$

The integration is spread out here to the four-dimensional "cylindrical" region $\Omega$, bounded by two three-dimensional space-like hypersurface $\sigma_1$ and $\sigma_2$ (the "ends" of the cylinder) connected by the side surface $\Sigma$. As usual, we suppose that $T_{\mu\nu}^{s}|_\Sigma = 0$. Making use of Gauss' theorem in (14.23) and taking into account (14.20), (14.21), we find:

$$Q_n(\sigma_1) = Q_n(\sigma_2).$$

As an example of the application of this method of constructing the conserved integral quantities one may consider the field models in the de Sitter universe [24, 68, 69]. In this case the space-time has a constant curvature and there are ten Killing vectors [24].

The Hilbert method can be spread out to the gravitational field also, i.e. one can put [7, 70–72]

$$T^g_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_g}{\delta g^{\mu\nu}}. \tag{14.24}$$

n this case the total energy-momentum tensor of matter and gravitational fields

$$\Theta_{\mu\nu} = T^m_{\mu\nu} + T^g_{\mu\nu} = \frac{2}{\sqrt{-g}} \cdot \frac{\delta S}{\delta g^{\mu\nu}} \tag{14.25}$$

vanishes by virtue of the equations of motion of gravitational field (13.3) or (13.4). Einstein's criticism of this definition and objections to it can be found in [72].

For the gravitational field the canonical energy-momentum tensor may be constructed also. Here there appear two possibilities. One may use the Lagrangian density $\mathcal{R} = \sqrt{-g}\, R$ containing the second derivatives of the metric tensor which leads to the general invariant action functional $S_g = -\varkappa^{-1} \int R \sqrt{-g}\, d^4x$ or the noninvariant density without the second derivatives of $g_{\mu\nu}(x)$

$$\mathcal{G} = \sqrt{-g}\, G = \sqrt{-g}\, g^{\mu\nu}(\Gamma^\varrho_{\mu\sigma}\Gamma^\sigma_{\nu\varrho} - \Gamma^\varrho_{\mu\nu}\Gamma^\sigma_{\varrho\sigma}) = \sqrt{-g}\, R - \partial_\mu \omega^\mu, \tag{14.26}$$

where

$$\omega^\mu = -\frac{1}{\sqrt{-g}}\, \partial_\lambda(-g^{\lambda\mu} \cdot g).$$

The difference of these two densities is the total divergence. Hence, they lead to the same equations of motion (13.3) but the canonical energy-mometum tensors will be different. The first case was considered by LORENTZ [7], the second one by EINSTEIN [75]

$$\sqrt{-g}\, t_\nu^{L\mu} = \left[\frac{\partial \mathcal{R}}{\partial g_{\lambda\varrho,\mu}} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial \mathcal{R}}{\partial g_{\lambda\varrho,\mu\sigma}}\right)\right] g_{\lambda\varrho,\nu} + \frac{\partial \mathcal{R}}{\partial g_{\lambda\varrho,\mu\sigma}}\, g_{\lambda\varrho,\nu\sigma} - \delta_\nu{}^\mu \mathcal{R}, \tag{14.27}$$

$$\sqrt{-g}\, t_\nu^{E\mu} = \frac{\partial \mathcal{G}}{\partial g_{\lambda\varrho,\mu}}\, g_{\lambda\varrho,\nu} - \mathcal{G} \cdot \delta_\nu{}^\mu = \sqrt{-g} \cdot \left(\frac{\partial G}{\partial g_{\lambda\varrho,\mu}} \cdot g_{\lambda\varrho,\nu} - \delta_\nu{}^\mu G\right). \tag{14.28}$$

The quantities $t_\nu^{L\mu}$ and $t_\nu^{E\mu}$, introduced here, are not general covariant tensors obviously. Therefore they are called pseudotensors or the complexes of energy-momentum density of the gravitational field. There are other methods of definition of this density [73, 74, 76].

It follows from the equations of motion (13.3) that

$$\frac{\partial}{\partial x^\nu} \sqrt{-g}\, (T_\mu{}^{s\nu} + t_\mu{}^{L\nu}) = \frac{\partial}{\partial x^\nu} \sqrt{-g}\, (T_\mu{}^{s\nu} + t_\mu{}^{E\nu}) = 0. \tag{14.29}$$

Using Gauss' theorem, one can construct straightforward the integral conserved quantities.

The main shortcoming of Eq. (14.29) is that it is covariant under affine coordinate transformations only. Obviously this contradicts the general covariance principle that is the basis of Einstein's theory.

Here there appear the following difficulties. In the first place the introduction of the energy-momentum density of the gravitational field $t_\mu{}^\nu$ is not unique. Second, due to the nontensorial behaviour of $t_\mu{}^\nu$ one can choose such a coordinate system that all components of $t_\mu{}^\nu$ vanish in any given point of space [77, 78]. On the other hand, in a

flat space-time without sources of the gravitational field one can obtain nonzero values of $t_\mu{}^\nu$ by an appropriate choice of the curvilinear coordinates [79, 80].
The definition of energy in Einstein's theory of gravitation is discussed until now [81, 82].

## 15. Relativistic String Model

The relativistic string is the one-dimensional object the action of which is proportional to the area of the world sheet traced out by the string during its motion in Minkowski space [83—85]. If $x^\mu(\tau, \sigma)$ is the parametric representation of the string world surface, then the action of the relativistic string is given by

$$S = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{(\dot{x}\acute{x})^2 - \dot{x}^2\acute{x}^2} = -\gamma \iint d^2\xi \sqrt{-g(\xi)}, \tag{15.1}$$

where $\dot{x} = \partial x/\partial\tau$, $\acute{x} = \partial x/\partial\sigma$, $\tau = \xi^1$, $\sigma = \xi^2$, $g = \det \|g_{ij}\|$, $g_{ij} = (\partial x^\mu/\partial\xi^i)(\partial x_\mu/\partial\xi^j)$, $i, j = 1, 2$. The parameter $\sigma$ specifies points along the string, $\tau$ plays the role of the evolution parameter.
The relativistic string model can be considered as a system of four fields $x^\mu(\xi^1, \xi^2)$, $\mu = 0, 1, 2, 3$ in the two-dimensional space $\{\xi^1, \xi^2\}$. The action (14.1) is invariant under the reparametrization

$$\bar{\xi}^i = f^i(\xi^1, \xi^2), \quad i = 1, 2. \tag{15.2}$$

These transformations are defined by two arbitrary functions and as a consequence the left hand sides of the Euler equations for the string must satisfy two identities. Functions $x^\mu(\tau, \sigma)$ are scalars under the transformations (15.2)

$$\bar{x}^\mu(\bar{\xi}^1, \bar{\xi}^2) = x^\mu(\xi^1, \xi^2).$$

So

$$\delta x_\mu(\xi) = 0, \quad \bar{\delta} x_\mu(\xi) = -x_{\mu,i}\varepsilon^i(\xi),$$

where $\varepsilon^i(\xi)$ are infinitesimal transformations of parameters $\xi^i$

$$\bar{\xi}^i = \xi^i + \varepsilon^i(\xi). \tag{15.3}$$

The variation $\delta S$, which is equal to zero, has the form

$$\delta S = \int d^2\xi[\delta_x \mathcal{L} + d_i(\mathcal{L}\varepsilon^i)] = 0,$$

where $\delta_x\mathcal{L}$ is a form variation of the Lagrangian density

$$\delta_x\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x_\mu}\bar{\delta}x_\mu + \frac{\partial\mathcal{L}}{\partial x_{\mu,i}}\cdot\bar{\delta}x_{\mu,i}.$$

Denoting the left hand sides of the Euler equations by $\mathcal{L}_\mu$

$$L_\mu = \frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\xi^i}\left(\frac{\partial\mathcal{L}}{\partial x^\mu_{,i}}\right),$$

we write Eq. (14.3) in the form

$$\delta S = \int d^2\xi \left\{\left[\left(\mathcal{L}\delta_{ij} - \frac{\partial\mathcal{L}}{\partial x_{\mu,i}}\bar{\delta}x_{\mu,j}\right)\varepsilon^j\right]_{,i} - L^\mu x_{\mu,j}\varepsilon^j\right\} = 0. \tag{15.4}$$

At first we choose the variations $\varepsilon^j(\xi)$ of the independent variables $\xi^j$ so that they vanish on the boundary of the integration region. Then the expression in square brackets in Eq. (15.4) gives no contribution to $\delta S$, and as a consequence, we find two identities

$$L_\mu \dot{x}^\mu \equiv 0, \quad L_\mu \acute{x}^\mu \equiv 0. \tag{15.5}$$

Actually, these equations are generalizations of the identities (9.3) which take place in the relativistic mechanics of point particles.

Taking into account the identities (15.5) one can supplement the Euler equations in the theory under consideration by two conditions on $x_\mu(\tau, \sigma)$. Usually, the isometric coordinate system on the world sheet of the string is chosen [86]. In this case we have

$$(\dot{x} \pm \acute{x})^2 = 0.$$

The equations of motion reduce now to the D'Alembert equation for $x_\mu(\tau, \sigma)$

$$\ddot{x}_\mu - x''_\mu = 0.$$

The expression in parentheses in Eq. (15.4) is the twodimensional energy-momentum tensor corresponding to the invariance of the string theory under translations in space $\xi^i$

$$t_{ij} = \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} \cdot x^\mu_{,j} - \delta_{ij}\mathcal{L}.$$

We set now the functions $\varepsilon^i(\xi)$ in Eq. (15.4) at first to constants and after that we substitute $\varepsilon^i$ by functions linear in $\xi^i$. Taking into account identities (15.5) one obtains new identities

$$t_{ij} \equiv 0, \; i, j = 1, 2. \tag{15.6}$$

These equations can be derived by equating to zero in Eq. (15.4) the coefficients of $\varepsilon^i(\xi)$ and its first derivatives $\varepsilon^i_{,j}(\xi)$

$$\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \dot{x}_\mu \equiv 0, \quad \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \acute{x}_\mu} \acute{x}_\mu \equiv 0,$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \cdot \acute{x}_\mu \equiv 0, \quad \frac{\partial \mathcal{L}}{\partial \acute{x}_\mu} \dot{x}_\mu \equiv 0.$$

The first identity means that the Hamiltonian constructed by the canonical law vanishes here identically. The third identity is a constraint on the canonical variables $x_\mu$ and $\pi_\mu = -\partial\mathcal{L}/\partial\dot{x}^\mu$. It can be written in the form $\acute{x}_\mu \pi^\mu = 0$. Thus, the Lagrangian density in the theory of the relativistic string is singular.

The vanishing of the parametric energy-momentum tensor in the string theory does not mean that there are no notions of the string energy and string momentum. As this theory is invariant under inhomogeneous Lorentz transformations

$$\bar{x}_\mu = \Lambda_{\mu\nu} x^\nu(\tau, \sigma) + a_\mu,$$

the usual energy-momentum tensor can be constructed in the same fashion as for the point relativistic particle. And this tensor does not vanish identically.

## 16. Conclusion

Application of Noether's theorems requires to determine the symmetry of the problem under consideration. To find the symmetry, i.e. the group of coordinate and function transformations under which the action is invariant may be in some cases a more difficult task than the construction of the conserved currents. There are many examples of that.

Long ago the additional integral has been known in the Kepler problem with the potential $\alpha/r$. This is the Laplace-Runge-Lenz vector [87–89]

$$\boldsymbol{R} = [\boldsymbol{v} \times \boldsymbol{M}] + \alpha \frac{\boldsymbol{r}}{r} = \text{const}.$$

where $\boldsymbol{M}$ is the angular momentum. This integral is additional to the set of ten integrals discussed in section 8. The connected with this integral $SO(4)$ or $SO(3, 1)$ symmetry of the classical and quantum Kepler problem has been discovered only recently [90–92]. Another example here is the infinite series of conservation laws in nonlinear evolution equations integrable by the inverse scattering method [93, 94] (the sine-Gordon equation, the nonlinear Schrödinger equation, the Korteweg-de Vries equation). The connection of these conservation laws with the symmetries of the corresponding nonlinear equations was shown only after the explicit derivation of the conserved currents. For example, the infinite series of conservation laws for the sine-Grodon equation is the consequence of its symmetry under the infinite series of one-parameter transformation groups [95]. These comments do not depreciate the role of Noether's theorems in the field theory because the field models are constructed usually on the basis of some symmetries. In particular, the Weinberg-Salam model is based on the $SU(2) \times U(1)$ local symmetry [55].

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