During our exploration of Sobolev spaces, we will have need of the following concepts. In the following definitions, the field of scalars $\mathbb{K}$ is assumed to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

Given $N \in \mathbb{Z}^+$, suppose $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a piecewise smooth boundary $\partial\Omega$. An N-tuple $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \in \mathbb{N}$ is known as a *multi-index*, which has *degree* $|\alpha| := \sum_{i=1}^{N} \alpha_i$. Given a multi-index α and $f \in C^\infty(\Omega)$, we define the partial derivative $D^\alpha f$ as

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}\,.$$

Note that $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_\Omega := \int_\Omega f(x)\overline{g(x)}\,dx$$

and induced norm

$$|f|_\Omega := \sqrt{\langle f, f\rangle_\Omega}\,.$$

Using this inner product, we may equip $C^m(\Omega)$, the collection of m-times continuously differentiable functions, with an inner product by defining

$$\langle f, g\rangle_{m,\Omega} := \sum_{|\alpha|=0}^{m} \langle D^\alpha f, D^\alpha g\rangle_\Omega\,,$$

which induces the norm

$$|f|_{m,\Omega} := \sqrt{\langle f, f\rangle_{m,\Omega}}\,.$$

We further restrict our attention to functions with finite norm, defining

$$\widetilde{C^m}(\Omega) = \{f \in C^m(\Omega) : |f|_{m,\Omega} < \infty\}.$$

Thus $\widetilde{C^m}(\Omega)$ is an inner product space. However, unlike $C^\infty(\Omega)$, $\widetilde{C^m}(\Omega)$ is *not* a Hilbert space, as it is not complete with respect to the metric induced by this norm. To illustrate this fact, consider the following examples over $\mathbb{R}$ (i.e. $N = 1$).

Let $\Omega = (-2, 2)$ and define the sequence of functions

$$f_n(x) = \sqrt{x^2 + \frac{1}{2^n}}$$