Solving when $f = 1$ (or earlier notation $\phi = 1$)

1- Using integration by parts

(see http://www.people.virginia.edu/~bk5w/home_files/trouble.pdf)

I had:

$$\int_0^{\overline{b}} \left\{ \left(\frac{\gamma(b) * b^2}{2} \right) [1 + \phi * (1 - \gamma(b))] - \left(\frac{[1 - \phi * (1 - \gamma(b))]}{\gamma(b)} \right) \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \right\} dF(b) \tag{1}$$

when $\phi = 1 \Rightarrow$it becomes:

$$\begin{aligned} & \int_0^{\overline{b}} \left\{ \left(\frac{\gamma(b) * b^2}{2} \right) [2 - \gamma(b)] - \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \right\} dF(b) \\ & \int_0^{\overline{b}} \left\{ \left(\frac{\gamma(b) * b^2}{2} \right) [2 - \gamma(b)] \right\} dF(b) - \int_0^{\overline{b}} \left\{ \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \right\} dF(b) \end{aligned} \tag{2}$$

note that the second part of the above equation (equation (2)) can be written as (using integration by parts):

$$\begin{aligned} \int_0^{\overline{b}} u dv &= u * v|_0^{\overline{b}} - \int_0^{\overline{b}} v du \\ \text{Here } u &= \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \text{ and } v = F(b) \\ \int_0^{\overline{b}} \left\{ \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \right\} dF(b) &= F(b) \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \Bigg|_0^{\overline{b}} - \int_0^{\overline{b}} F(b) d \left\{ \int_0^{b} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \right\} \\ &= \int_0^{\overline{b}} \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} - \int_0^{\overline{b}} F(b) \left([\gamma(\widetilde{b})]^2 * \widetilde{b} \right) d\widetilde{b} \\ &= \int_0^{\overline{b}} \left([1 - F(b)] * [\gamma(b)]^2 * b \right) db \end{aligned} \tag{3}$$

Hence equation (2) becomes:

$$\int_0^{\bar{b}} \left\{ \left(\frac{\gamma(b) * b^2}{2} \right) [2 - \gamma(b)] \right\} dF(b) - \int_0^{\bar{b}} \left([1 - F(b)] * [\gamma(b)]^2 * b \right) db$$
$$= \int_0^{\bar{b}} \left\{ \gamma(b) * b^2 - \frac{[\gamma(b)]^2 * b^2}{2} - \frac{[1 - F(b)]}{f(b)} * [\gamma(b)]^2 * b \right\} f(b) db \tag{4}$$

I will assume that b is distributed uniform on $[0, 1]$, so $f(b) = 1$ and $F(b) = b$.

Hence, in order to find optimal $\gamma(b)$, we can do pointwise maximization with respect to $\gamma(b)$.

We get:

$$b^2 = \gamma(b) * b * [2 - b]$$

Hence for $\gamma(b) \neq 0$, we get:

$$\gamma(b) = \frac{b}{2 - b}$$

2- Using exact differential equations:
(see http://www.people.virginia.edu/~bk5w/home_files/trouble.pdf)

Once we use calculus of varitions, we have:

$$-(1 - \phi) * \frac{\sqrt{b}}{\sqrt{z'(b)}} - \phi - \frac{d}{db} \left[\frac{(1 + \phi) * b^{3/2}}{4 * \sqrt{z'(b)}} - \frac{\phi * b}{2} + \frac{(1 - \phi) * \sqrt{b} * z(b)}{2 * [z'(b)]^{3/2}} \right] = 0 \tag{5}$$

when $\phi = 1$, it becomes:

$$-1 - \frac{d}{db} \left[\frac{b^{3/2}}{2 * \sqrt{z'(b)}} - \frac{b}{2} \right] = 0$$

$$1 + \frac{d}{db} \left[\frac{b^{3/2}}{2 * \sqrt{z'(b)}} - \frac{b}{2} \right] = 0$$

$$db + \frac{1}{2 * z'(b)} \left[\left(\frac{3}{2} * \sqrt{z'(b)} * \sqrt{b} \right) db - \left(\frac{b^{3/2}}{2 * \sqrt{z'(b)}} \right) dz'(b) \right] - \frac{1}{2} db = 0$$

$$\left[\frac{1}{2}+\frac{3}{4}\frac{\sqrt{b}}{\sqrt{z'(b)}}\right]db-\left[\frac{1}{4}\left(\frac{b}{z'(b)}\right)^{3/2}\right]dz'(b)=0$$

$$\text{Let } M=-\left[\frac{1}{4}\left(\frac{b}{z'(b)}\right)^{3/2}\right] \text{ and } N=\left[\frac{1}{2}+\frac{3}{4}\frac{\sqrt{b}}{\sqrt{z'(b)}}\right]$$

You can check that the above equation is exact. Hence:

$$\begin{aligned}
F(z'(b),b) &= \int M dz'(b)+\psi(b) \\
F(z'(b),b) &= -\frac{1}{4}\int\left(\frac{b}{z'(b)}\right)^{3/2} dz'(b)+\psi(b) \\
F(z'(b),b) &= \frac{b^{3/2}}{2\sqrt{z'(b)}}+\psi(b) \\
\frac{\partial F}{\partial b} &= N \text{ (exact differential stuff)} \\
&\Rightarrow \frac{3}{4}\frac{\sqrt{b}}{\sqrt{z'(b)}}+\psi'(b)=\frac{1}{2}+\frac{3}{4}\frac{\sqrt{b}}{\sqrt{z'(b)}} \\
&\Rightarrow \psi'(b)=\frac{1}{2}\Rightarrow\psi(b)=\frac{1}{2}*b+\text{constant} \\
F(z'(b),b) &= \frac{b^{3/2}}{2\sqrt{z'(b)}}+\frac{1}{2}*b+\text{constant} \\
F(z'(b),b) &= \text{constant (since it is exact differential equation)} \\
&\Rightarrow \frac{b^{3/2}}{2\sqrt{z'(b)}}+\frac{1}{2}*b=k \text{ where } k \text{ is a constant} \\
z'(b) &= \frac{b^3}{(2*k-b)^2} \\
\text{since } z'(b) &= [\gamma(b)]^2*b \\
\textbf{(see http} &: \textbf{//www.people.virginia.edu/\textasciitilde bk5w/home_files/trouble.pdf)} \quad (6) \\
\gamma(b) &= \frac{b}{2*k-b} \text{ for } b\neq 0 \quad (7)
\end{aligned}$$

One thing in this solution is there is still the unknown parameter k in there. I know form the integration by parts solution that $k=1$ and my hunch is that I think I can obtain that if I can plug in $\gamma(b)=\frac{b}{2*k-b}$ into equation (1) and numerically find the optimal value of k, which has to be 1 for $\phi=1$.

3- Another solution:

I have in hand:

$$\begin{aligned}
z''(b) &= 3 * \frac{z'(b)}{b} + 2 * \left[\frac{z'(b)}{b}\right]^{3/2} \quad \text{for } b \neq 0. \\
z''(b) &= \frac{z'(b)}{b}\left[3 + 2 * \left(\frac{z'(b)}{b}\right)^{1/2}\right] \\
\text{let } u &= z'(b) \text{ then} \\
u'(b) &= \frac{u(b)}{b}\left[3 + 2 * \left(\frac{u(b)}{b}\right)^{1/2}\right] \\
\text{let } p &= \left(\frac{u(b)}{b}\right)^{1/2} \quad \text{then} \\
b * p'(b) &= -\frac{1}{2} * p + \frac{1}{2} * p * (3 + 2 * p) \\
p'(b) &= -\frac{1}{2} * \frac{p}{b} + \frac{1}{2} * \frac{p}{b} * (3 + 2 * p) \text{ for } b \neq 0 \\
\frac{dp}{db} &= \frac{p(1+p)}{b} \\
\int \frac{dp}{p(1+p)} &= \int \frac{db}{b} \\
\ln(\frac{p}{1-p}) + k_1 &= \ln b + k_2 \text{ where } k_1 \text{ and } k_2 \text{ are constants.} \\
&\Rightarrow \quad p = \frac{b * \exp(k)}{1 - b * \exp(k)}
\end{aligned}$$

Note that the variable $p(b)$ is actually equal to $\gamma(b)$ (Since $z'(b) = u(b)$ and $p(b) = \left(\frac{u(b)}{b}\right)^{1/2} \Rightarrow p(b) = \left(\frac{z'(b)}{b}\right)^{1/2}$. Furthermore, $z'(b) = [\gamma(b)]^2 * b$, so $p(b) = \left(\frac{[\gamma(b)]^2 * b}{b}\right)^{1/2} \Rightarrow p(b) = \gamma(b)$).

Again I need to find the constant (which is k). Since I know it has to be $\gamma(b) = \frac{b}{2-b}$, the constant $k = -\ln 2$. Again my hunch is that I can plug in $\gamma(b) = \frac{b*\exp(k)}{1-b*\exp(k)}$ in equation (1) and find the optimal cosntant k numerically (which in this case has to be $-\ln 2$).

Answer to edit-1: I agree that the equation can have (I think it has) multiple solutions when $b = 0$.

Answer to edit-2: It is good news since I need $\gamma(b)$ nondecreasing in b. But do you know a formal proof for that.

Thanks for your interest and time you spend on this problem.