

Fresnel's Equations for Reflection and Refraction

Incident, transmitted, and reflected beams at interfaces

Reflection and transmission coefficients

The Fresnel Equations

Brewster's Angle

Total internal reflection

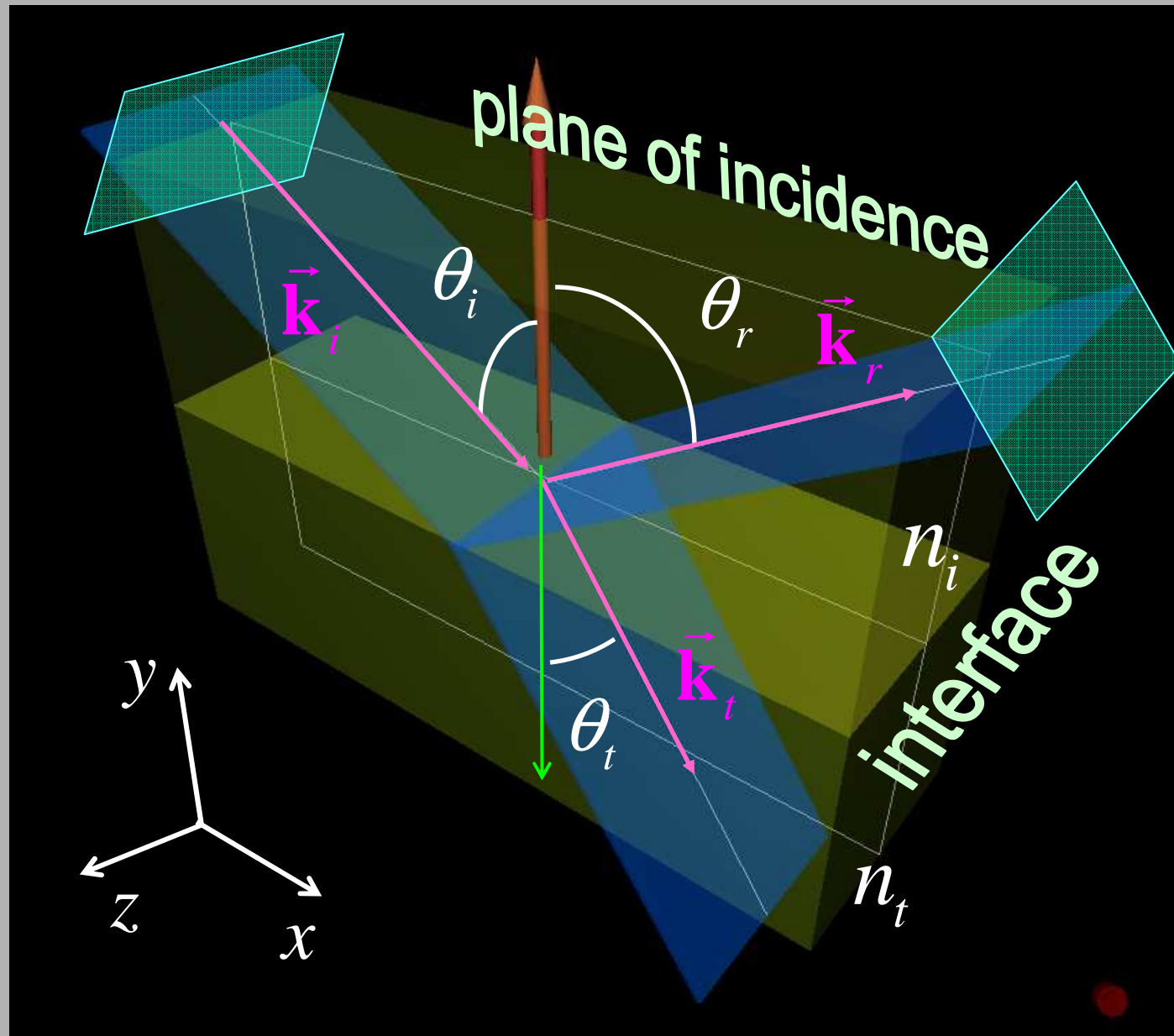
Power reflectance and transmittance

Phase shifts in reflection

The mysterious evanescent wave

Prof. Rick Trebino
Georgia Tech
www.physics.gatech.edu/frog/lectures

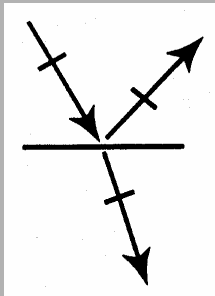
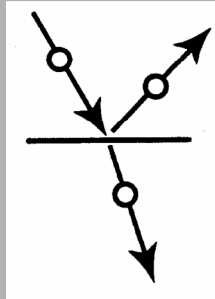
Definitions



More definitions

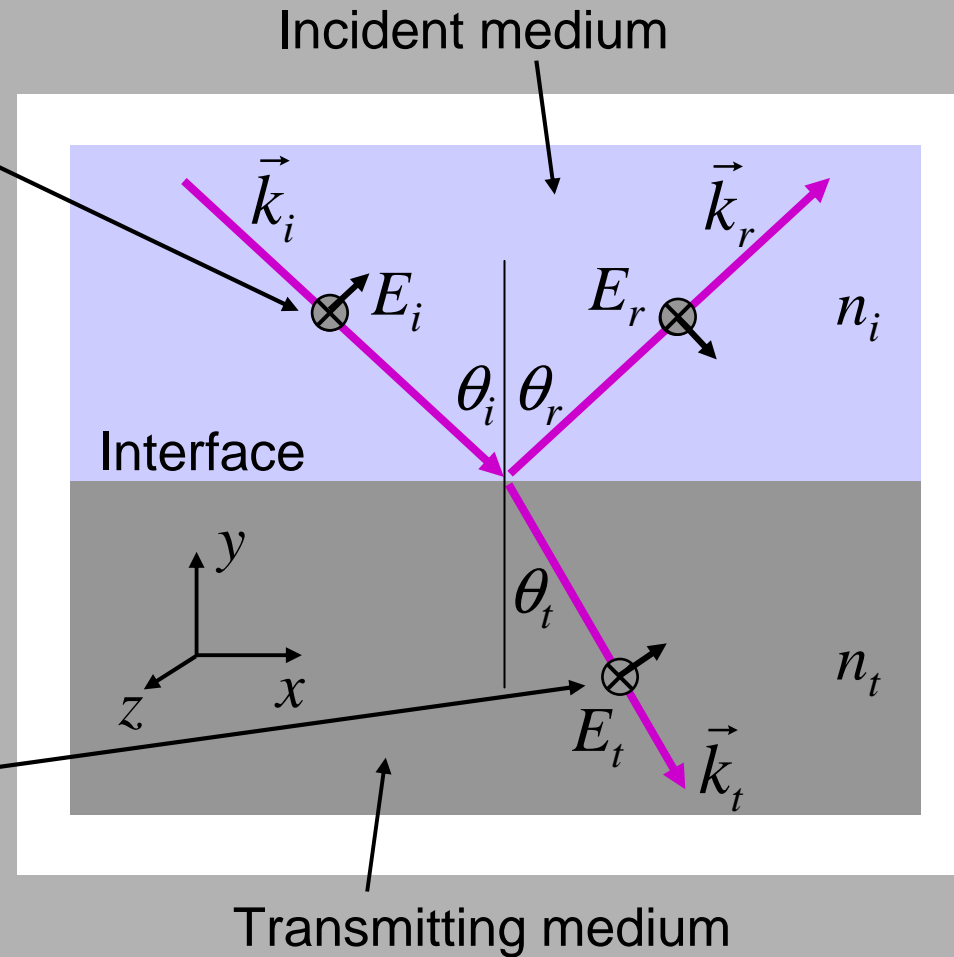
Perpendicular (“S”)

polarization **sticks** out of or into the plane of incidence.



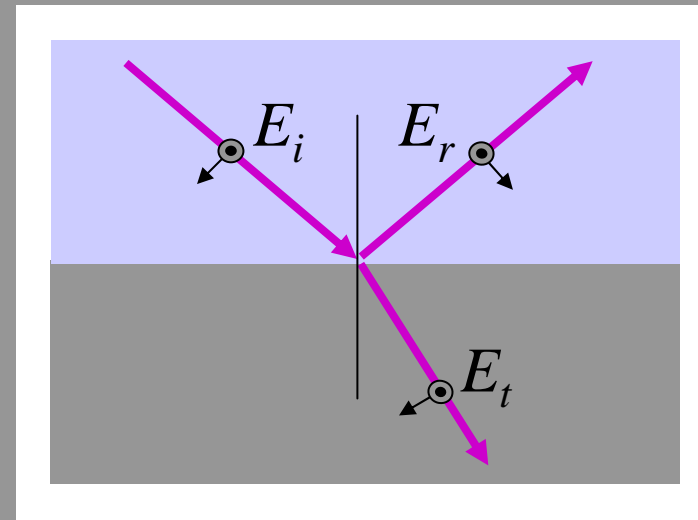
Parallel (“P”)

polarization lies **parallel** to the plane of incidence.



Fresnel Equations

We would like to compute the fraction of a light wave reflected and transmitted by a flat interface between two media with different refractive indices.



$$r_{\perp} = E_{0r} / E_{0i} \quad \text{for the perpendicular polarization}$$
$$t_{\perp} = E_{0t} / E_{0i}$$

$$r_{\parallel} = E_{0r} / E_{0i} \quad \text{for the parallel polarization}$$
$$t_{\parallel} = E_{0t} / E_{0i}$$

where E_{0i} , E_{0r} , and E_{0t} are the field complex amplitudes.

We consider the boundary conditions at the interface for the electric **and magnetic** fields of the light waves.

We'll do the perpendicular polarization first.

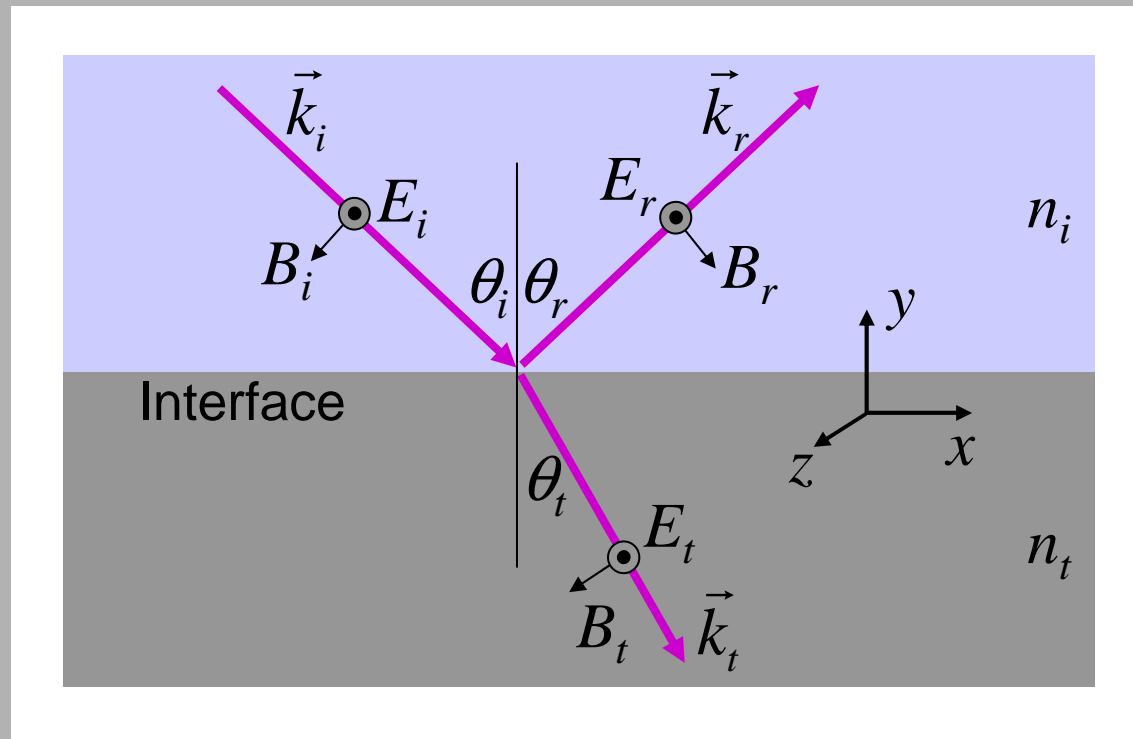
Boundary Condition for the Electric Field at an Interface

The Tangential Electric Field is Continuous

In other words:

The total E-field in the plane of the interface is continuous.

Here, all E-fields are in the z-direction, which is in the plane of the interface (xz), so:



$$E_i(x, y = 0, z, t) + E_r(x, y = 0, z, t) = E_t(x, y = 0, z, t)$$

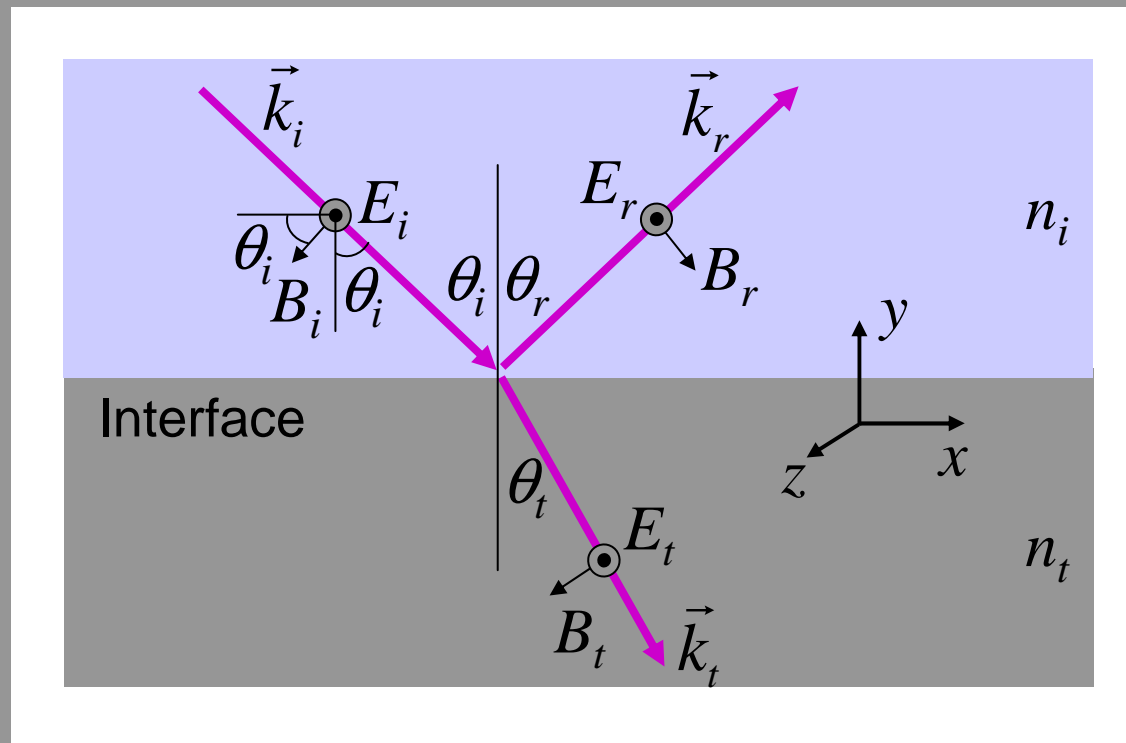
Boundary Condition for the Magnetic Field at an Interface

The Tangential Magnetic Field is Continuous*

In other words:

The total B-field in the plane of the interface is continuous.

Here, all B-fields are in the xy-plane, so we take the x-components:



$$-B_i(x, y=0, z, t) \cos(\theta_i) + B_r(x, y=0, z, t) \cos(\theta_r) = -B_t(x, y=0, z, t) \cos(\theta_t)$$

*It's really the tangential B/μ , but we're using $\mu = \mu_0$

Reflection and Transmission for Perpendicularly (S) Polarized Light

Canceling the rapidly varying parts of the light wave and keeping only the complex amplitudes:

$$\begin{aligned} E_{0i} + E_{0r} &= E_{0t} \\ -B_{0i} \cos(\theta_i) + B_{0r} \cos(\theta_r) &= -B_{0t} \cos(\theta_t) \end{aligned}$$

But $B = E / (c_0 / n) = nE / c_0$ and $\theta_r = \theta_i$:

$$n_i (E_{0r} - E_{0i}) \cos(\theta_i) = -n_t E_{0t} \cos(\theta_t)$$

Substituting for E_{0t} using $E_{0i} + E_{0r} = E_{0t}$:

$$n_i (E_{0r} - E_{0i}) \cos(\theta_i) = -n_t (E_{0r} + E_{0i}) \cos(\theta_t)$$

Reflection & Transmission Coefficients for Perpendicularly Polarized Light

Rearranging $n_i(E_{0r} - E_{0i})\cos(\theta_i) = -n_t(E_{0r} + E_{0i})\cos(\theta_t)$ yields:

$$E_{0r} [n_i \cos(\theta_i) + n_t \cos(\theta_t)] = E_{0i} [n_i \cos(\theta_i) - n_t \cos(\theta_t)]$$

Solving for E_{0r} / E_{0i} yields the reflection coefficient :

$$r_{\perp} = E_{0r} / E_{0i} = [n_i \cos(\theta_i) - n_t \cos(\theta_t)] / [n_i \cos(\theta_i) + n_t \cos(\theta_t)]$$

Analogously, the transmission coefficient, E_{0t} / E_{0i} , is

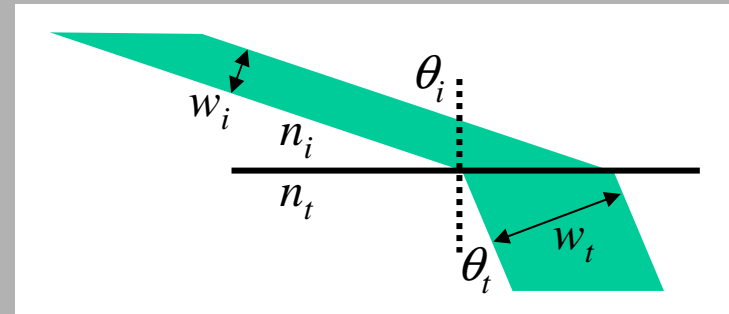
$$t_{\perp} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / [n_i \cos(\theta_i) + n_t \cos(\theta_t)]$$

These equations are called the **Fresnel Equations** for **perpendicularly** polarized light.

Simpler expressions for r_{\perp} and t_{\perp}

Recall the magnification at an interface, m :

$$m = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)}$$



Also let ρ be the ratio of the refractive indices, n_t / n_i .

Dividing numerator and denominator of r and t by $n_i \cos(\theta_i)$:

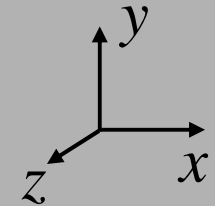
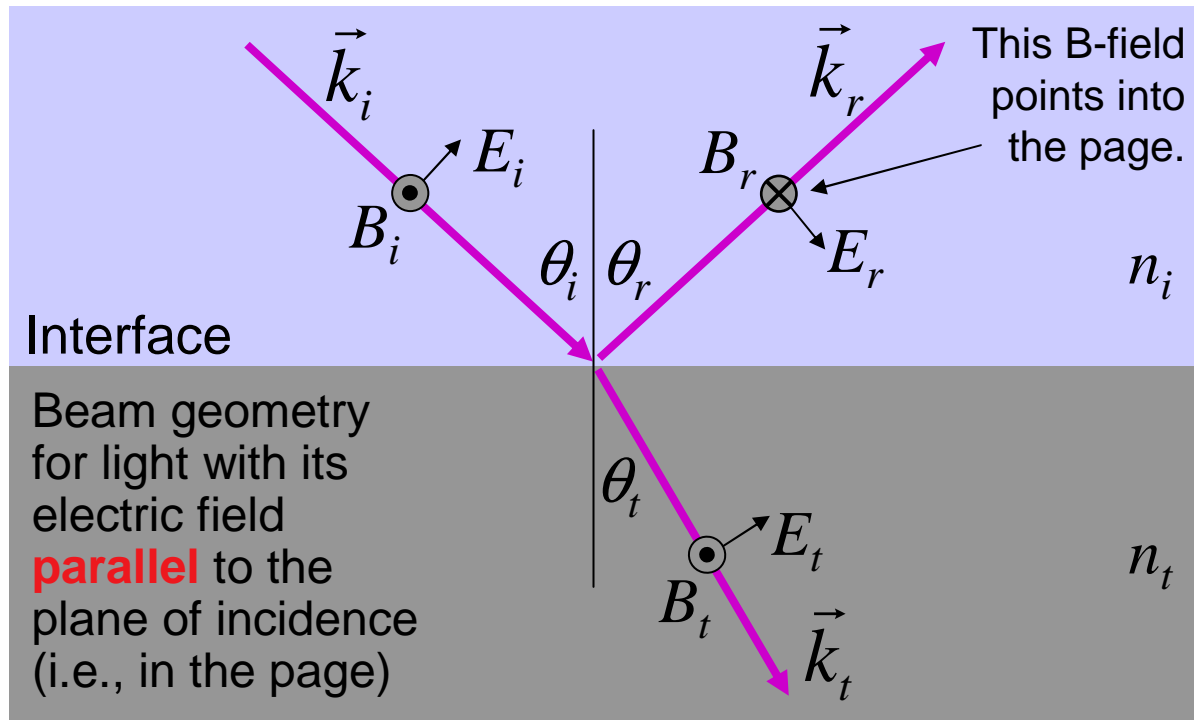
$$r_{\perp} = [n_i \cos(\theta_i) - n_t \cos(\theta_t)] / [n_i \cos(\theta_i) + n_t \cos(\theta_t)] = [1 - \rho m] / [1 + \rho m]$$

$$t_{\perp} = 2n_i \cos(\theta_i) / [n_i \cos(\theta_i) + n_t \cos(\theta_t)] = 2 / [1 + \rho m]$$

$$r_{\perp} = \frac{1 - \rho m}{1 + \rho m}$$

$$t_{\perp} = \frac{2}{1 + \rho m}$$

Fresnel Equations—Parallel electric field



Note that Hecht uses a different notation for the reflected field, which is confusing! Ours is better!

Note that the reflected magnetic field must point into the screen to achieve $\vec{E} \times \vec{B} \propto \vec{k}$. The x means "into the screen."

Reflection & Transmission Coefficients for Parallel (P) Polarized Light

For parallel polarized light, $B_{0i} - B_{0r} = B_{0t}$

and $E_{0i}\cos(\theta_i) + E_{0r}\cos(\theta_r) = E_{0t}\cos(\theta_t)$

Solving for E_{0r}/E_{0i} yields the reflection coefficient, r_{\parallel} :

$$r_{\parallel} = E_{0r} / E_{0i} = [n_i \cos(\theta_t) - n_t \cos(\theta_i)] / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]$$

Analogously, the transmission coefficient, $t_{\parallel} = E_{0t}/E_{0i}$, is

$$t_{\parallel} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]$$

These equations are called the Fresnel Equations for **parallel** polarized light.

Simpler expressions for r_{\parallel} and t_{\parallel}

$$r_{\parallel} = E_{0r} / E_{0i} = [n_i \cos(\theta_t) - n_t \cos(\theta_i)] / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]$$

$$t_{\parallel} = E_{0t} / E_{0i} = 2n_i \cos(\theta_i) / [n_i \cos(\theta_t) + n_t \cos(\theta_i)]$$

Again, use the magnification, m , and the refractive-index ratio, ρ .

And again dividing numerator and denominator of r and t by $n_i \cos(\theta_i)$:

$$r_{\square} = [m - \rho] / [m + \rho]$$

$$t_{\square} = 2 / [m + \rho]$$

$$r_{\square} = \frac{m - \rho}{m + \rho}$$

$$t_{\square} = \frac{2}{m + \rho}$$

Reflection Coefficients for an Air-to-Glass Interface

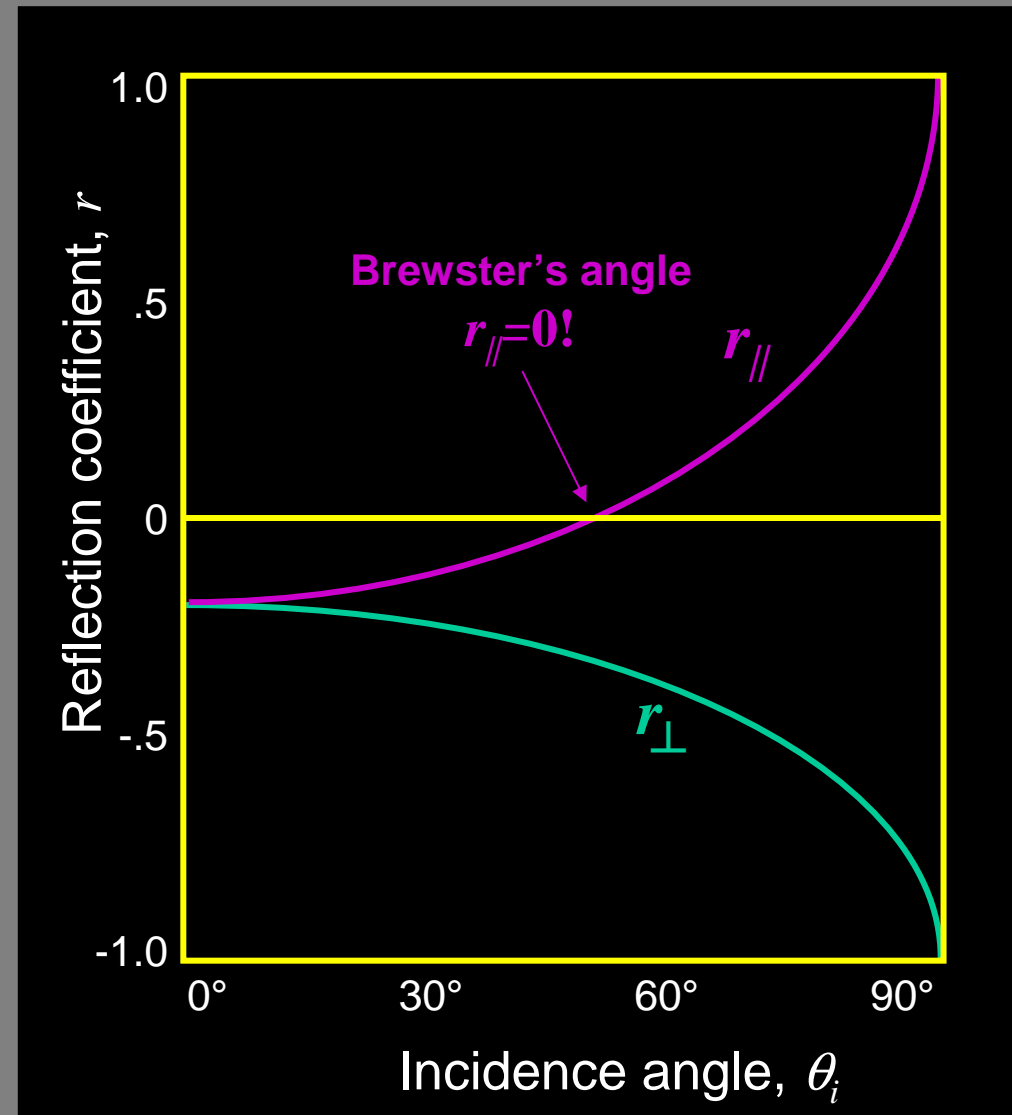
$$n_{air} \approx 1 < n_{glass} \approx 1.5$$

Note that:

Total reflection at $\theta = 90^\circ$
for both polarizations

Zero reflection for parallel
polarization at **Brewster's
angle** (56.3° for these
values of n_i and n_t).

(We'll delay a derivation of
a formula for Brewster's
angle until we do dipole
emission and polarization.)



Reflection Coefficients for a Glass-to-Air Interface

$$n_{\text{glass}} \approx 1.5 > n_{\text{air}} \approx 1$$

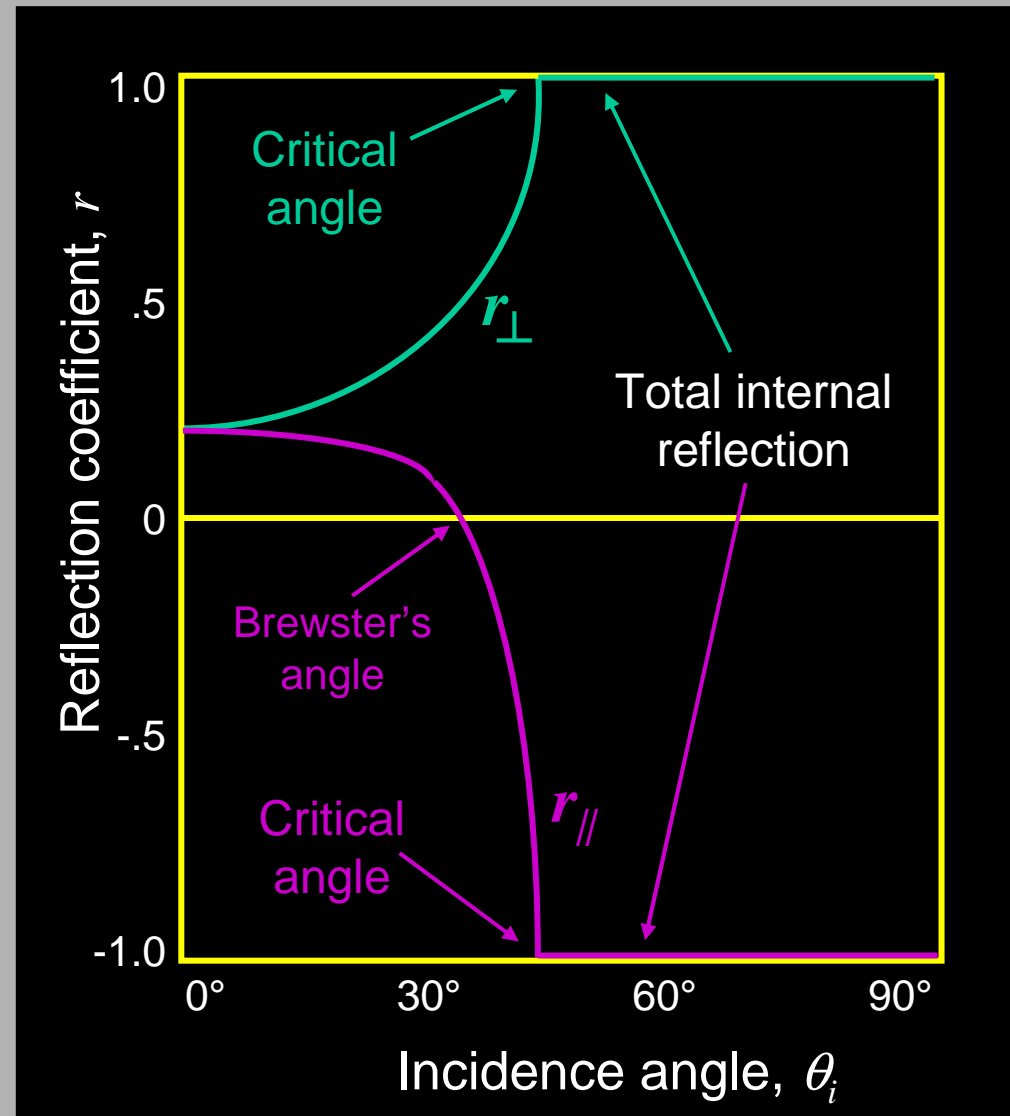
Note that:

Total internal reflection
above the **critical angle**

$$\theta_{\text{crit}} \equiv \arcsin(n_t/n_i)$$

(The sine in Snell's Law
can't be > 1):

$$\sin(\theta_{\text{crit}}) = n_t/n_i \sin(90^\circ)$$

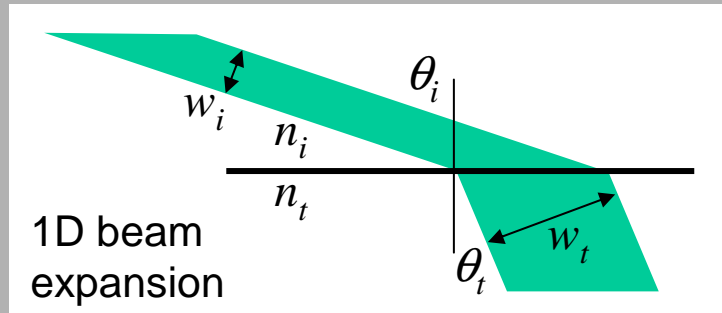


Transmittance (T)

$$T \equiv \text{Transmitted Power} / \text{Incident Power} = \frac{I_t A_t}{I_i A_i} \quad \leftarrow A = \text{Area}$$

$$I = \left(n \frac{\epsilon_0 c_0}{2} \right) |E_0|^2$$

Compute the ratio of the beam areas:



$$\frac{A_t}{A_i} = \frac{w_t}{w_i} = \frac{\cos(\theta_t)}{\cos(\theta_i)} = m$$

The beam expands in one dimension on refraction.

$$T = \frac{I_t A_t}{I_i A_i} = \frac{\left(n_t \frac{\epsilon_0 c_0}{2} \right) |E_{0t}|^2 \left[\frac{w_t}{w_i} \right]}{\left(n_i \frac{\epsilon_0 c_0}{2} \right) |E_{0i}|^2 w_i} = \frac{n_t |E_{0t}|^2 w_t}{n_i |E_{0i}|^2 w_i} = \frac{n_t}{n_i} t^2 \frac{\cos(\theta_t)}{\cos(\theta_i)}$$

$$\frac{|E_{0t}|^2}{|E_{0i}|^2} = t^2$$

\Rightarrow

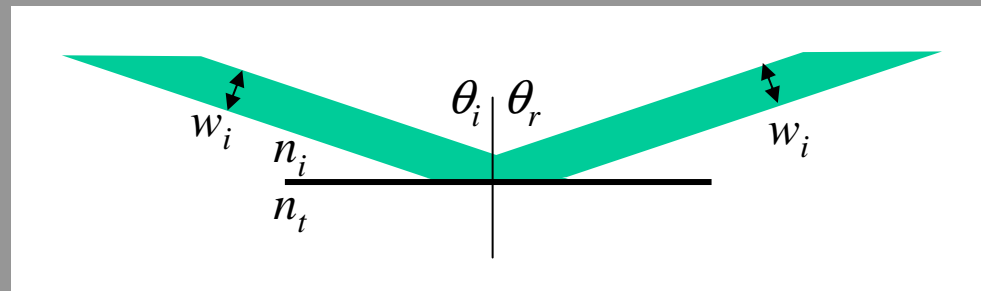
$$T = \left[\frac{(n_t \cos(\theta_t))}{(n_i \cos(\theta_i))} \right] t^2 = \rho m t^2$$

The Transmittance is also called the Transmissivity.

Reflectance (R)

$$R \equiv \text{Reflected Power} / \text{Incident Power} = \frac{I_r A_r}{I_i A_i}$$

$I = \left(n \frac{\epsilon_0 c_0}{2} \right) |E_0|^2$
 $A = \text{Area}$



Because the angle of incidence = the angle of reflection, the beam area doesn't change on reflection.

Also, n is the same for both incident and reflected beams.

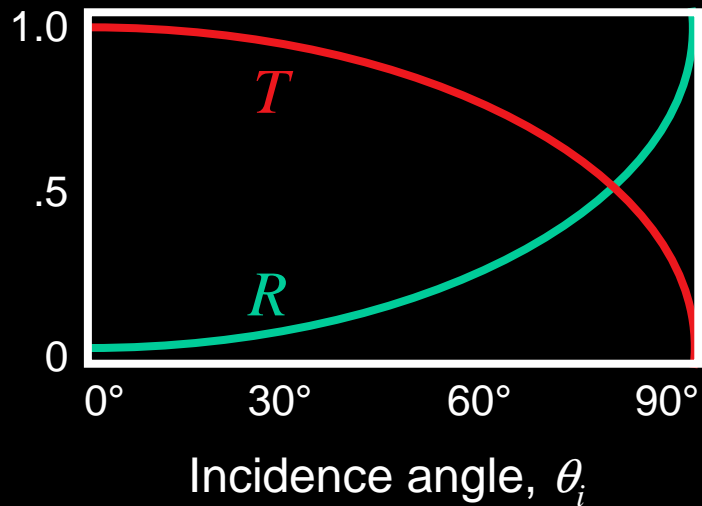
So:

$$R = r^2$$

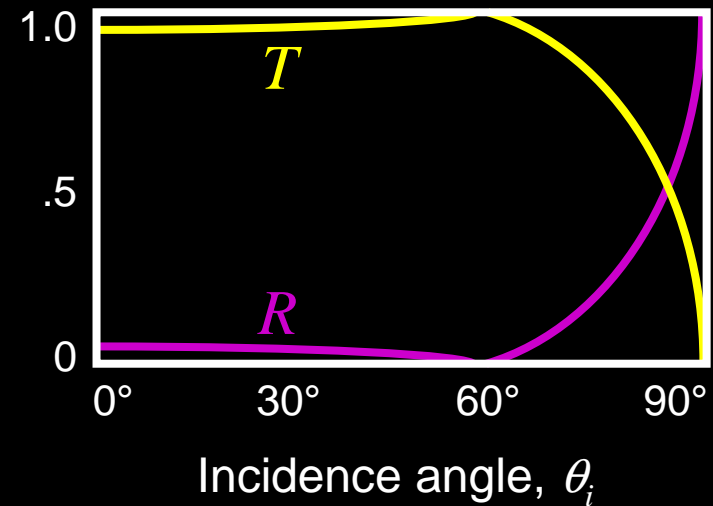
The Reflectance is also called the Reflectivity.

Reflectance and Transmittance for an Air-to-Glass Interface

Perpendicular polarization



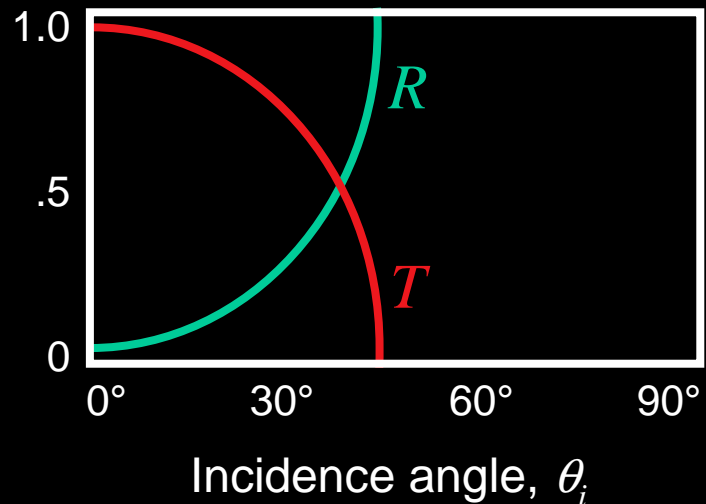
Parallel polarization



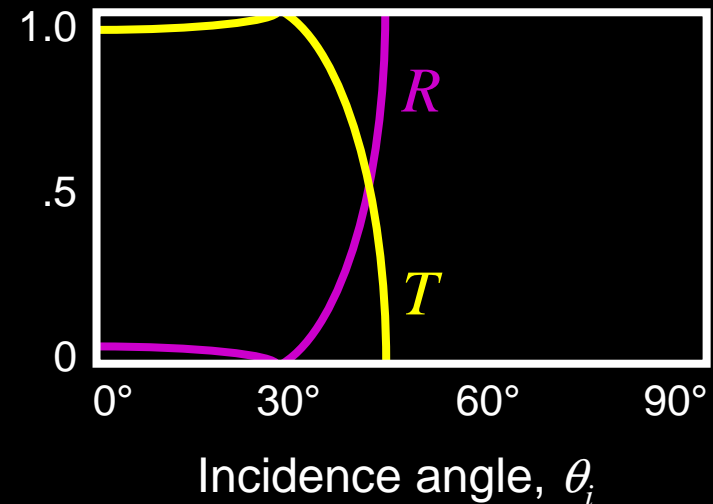
Note that $R + T = 1$

Reflectance and Transmittance for a Glass-to-Air Interface

Perpendicular polarization



Parallel polarization



Note that $R + T = 1$

Reflection at normal incidence

When $\theta_i = 0$,

$$R = \left(\frac{n_t - n_i}{n_t + n_i} \right)^2$$

and

$$T = \frac{4n_t n_i}{(n_t + n_i)^2}$$

For an air-glass interface ($n_i = 1$ and $n_t = 1.5$),

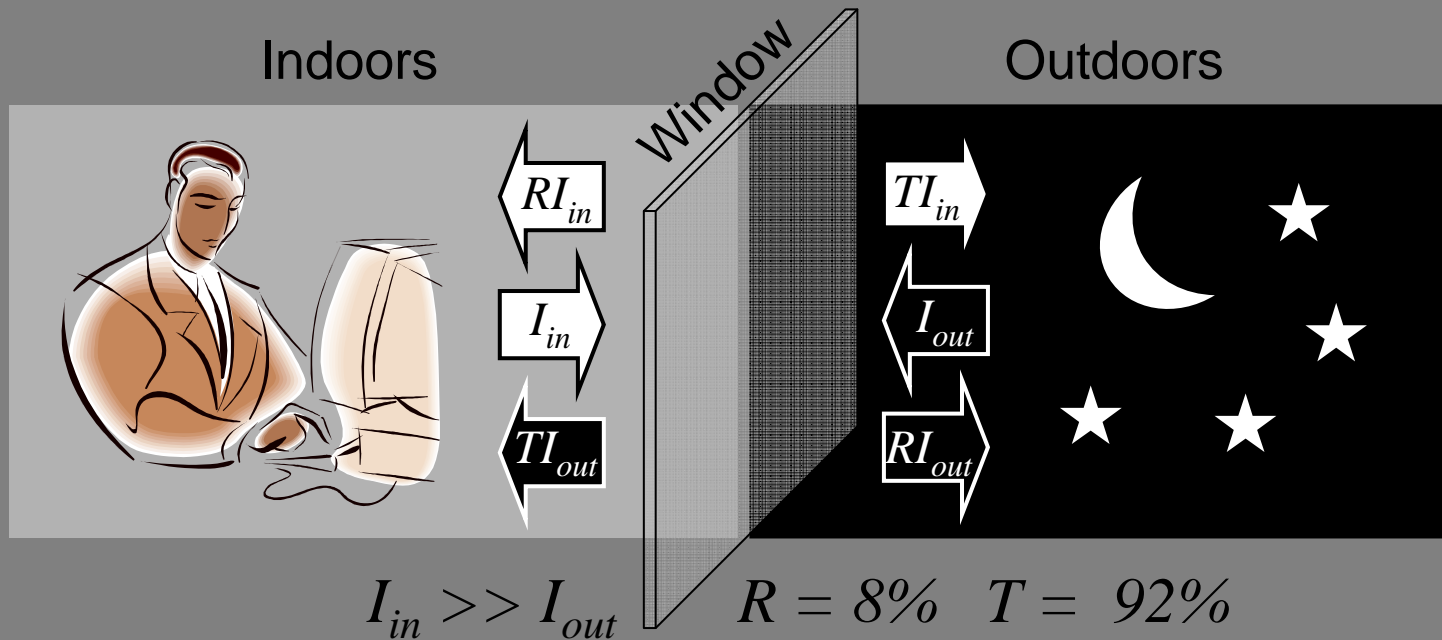
$$R = 4\% \text{ and } T = 96\%$$

The values are the same, whichever direction the light travels, from air to glass or from glass to air.

The 4% has big implications for photography lenses.

Practical Applications of Fresnel's Equations

Windows look like mirrors at night (when you're in a brightly lit room).

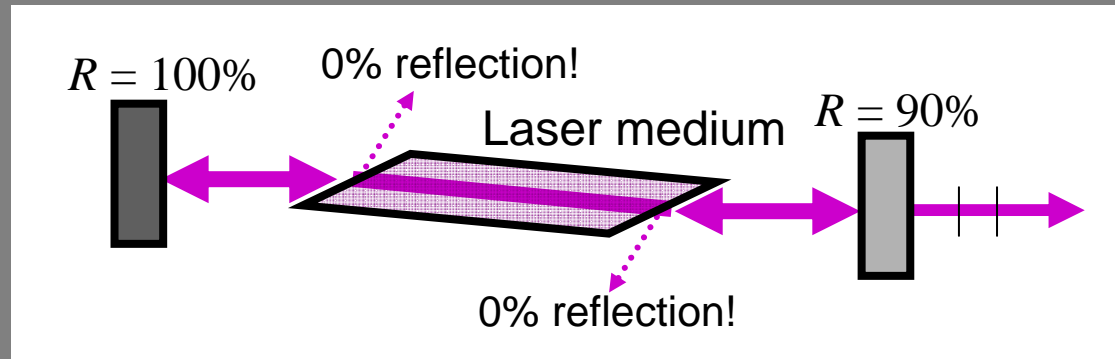


One-way mirrors (used by police to interrogate bad guys) are just partial reflectors (aluminum-coated), and you watch while in the dark.

Disneyland puts ghouls next to you in the haunted house using partial reflectors (also aluminum-coated).

Practical Applications of Fresnel's Equations

Lasers use Brewster's angle components to avoid reflective losses:



Optical fibers use total internal reflection.

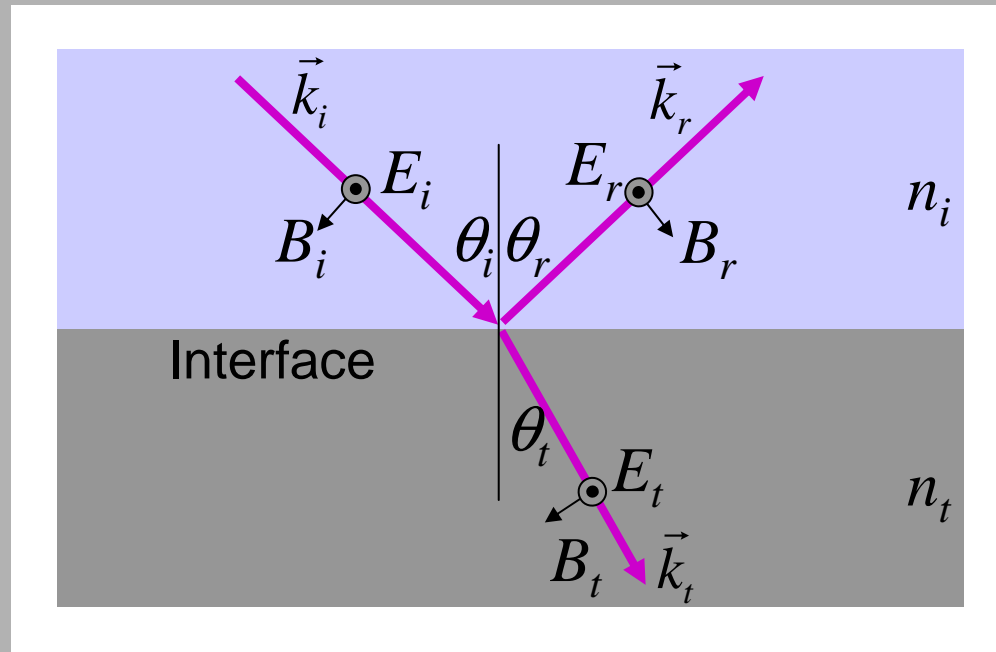
Hollow fibers use high-incidence-angle near-unity reflections.

Phase Shift in Reflection (for Perpendicularly Polarized Light)

$$r_{\perp} = E_{0r} / E_{0i} = \frac{[n_i \cos(\theta_i) - n_t \cos(\theta_t)]}{[n_i \cos(\theta_i) + n_t \cos(\theta_t)]}$$

$$\text{When } \theta_i = 0, \quad r = \frac{[n_i - n_t]}{[n_i + n_t]}$$

If $n_i < n_t$ (air to glass), $r < 0$



So there will be **destructive** interference between the incident and reflected beams just before the surface.

Analogously, if $n_i > n_t$ (glass to air), $r_{\perp} > 0$, and there will be **constructive** interference.

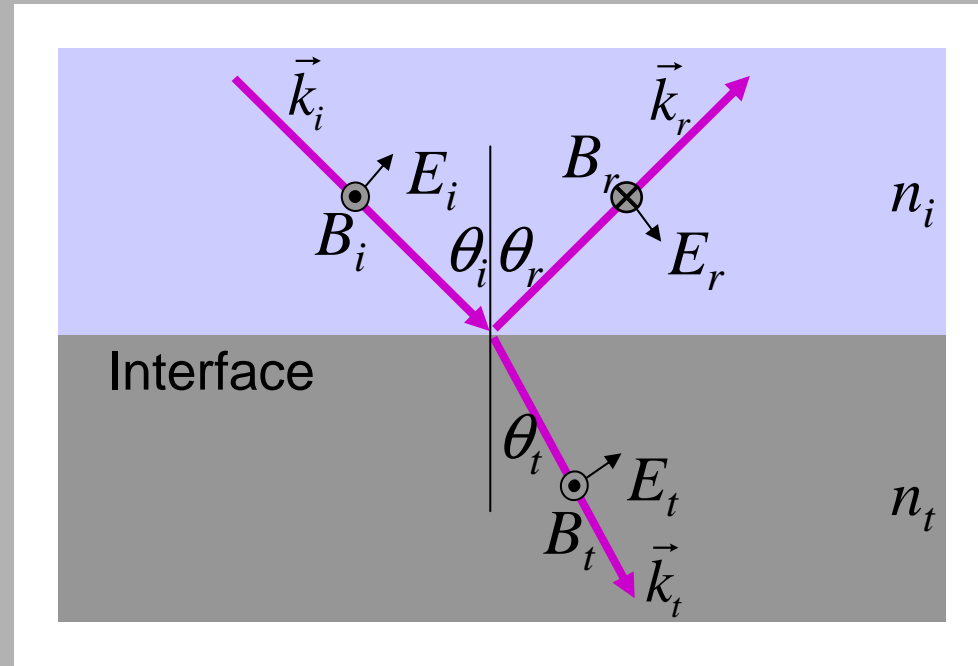
Phase Shift in Reflection (Parallel Polarized Light)

$$r_{\parallel} = E_{0r} / E_{0i} =$$

$$\frac{[n_i \cos(\theta_t) - n_t \cos(\theta_i)]}{[n_i \cos(\theta_t) + n_t \cos(\theta_i)]}$$

$$\text{When } \theta_i = 0, r_{\parallel} = \frac{[n_i - n_t]}{[n_i + n_t]}$$

If $n_i < n_t$ (air to glass), $r_{\parallel} < 0$



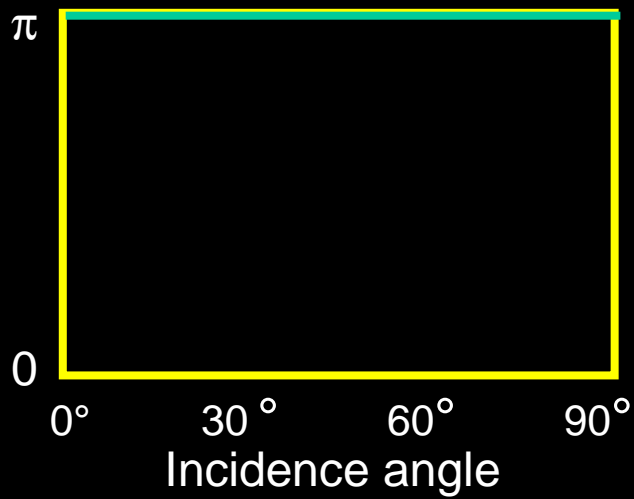
This also means **destructive** interference with incident beam.

Analogously, if $n_i > n_t$ (glass to air), $r_{\parallel} > 0$, and we have **constructive** interference just above the interface.

Good that we get the same result as for r_{\perp} ; it's the same problem when $\theta_i = 0$! Also, the phase is opposite above Brewster's angle.

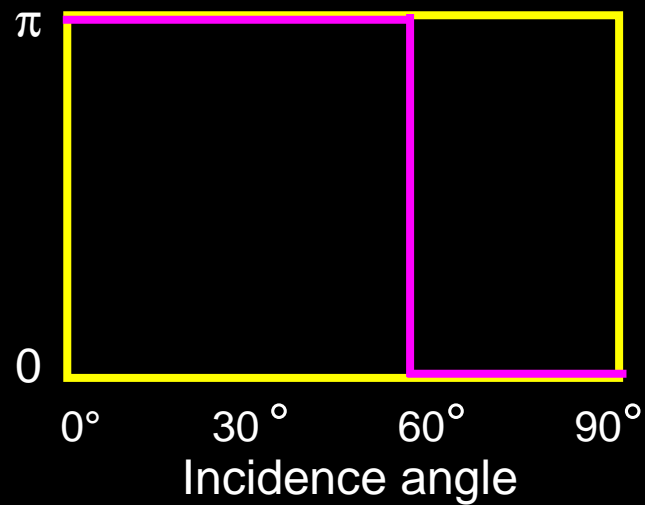
Phase shifts in reflection (air to glass)

\perp

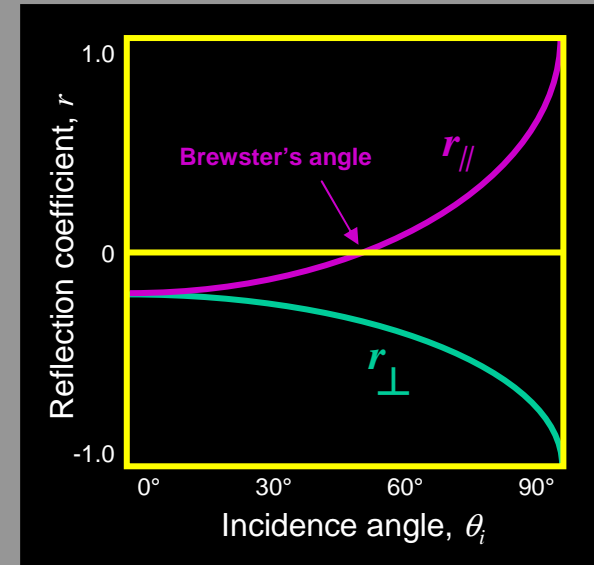


180° phase shift for all angles

\parallel

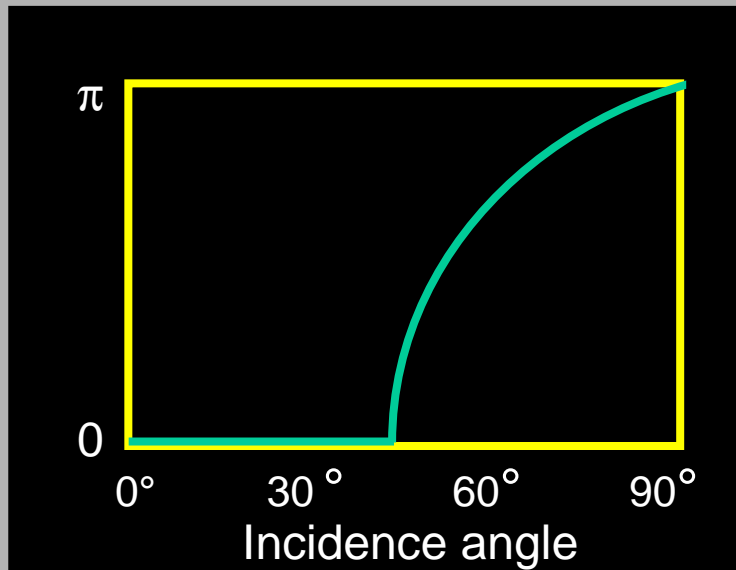


180° phase shift for angles below Brewster's angle; 0° for larger angles



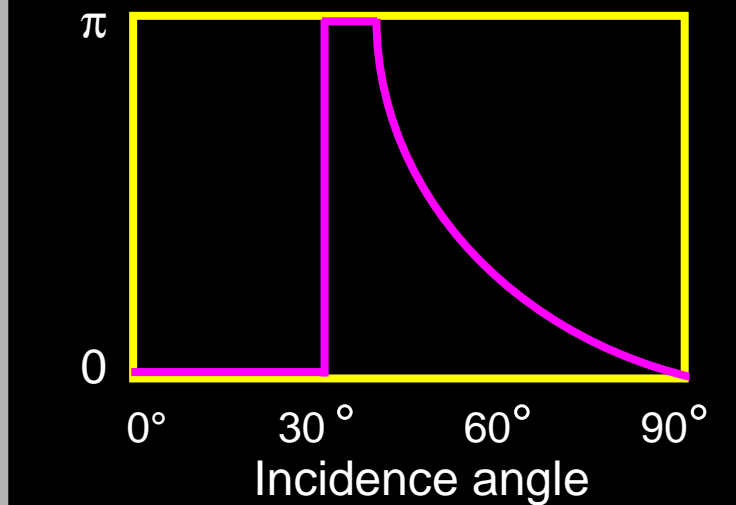
Phase shifts in reflection (glass to air)

⊥

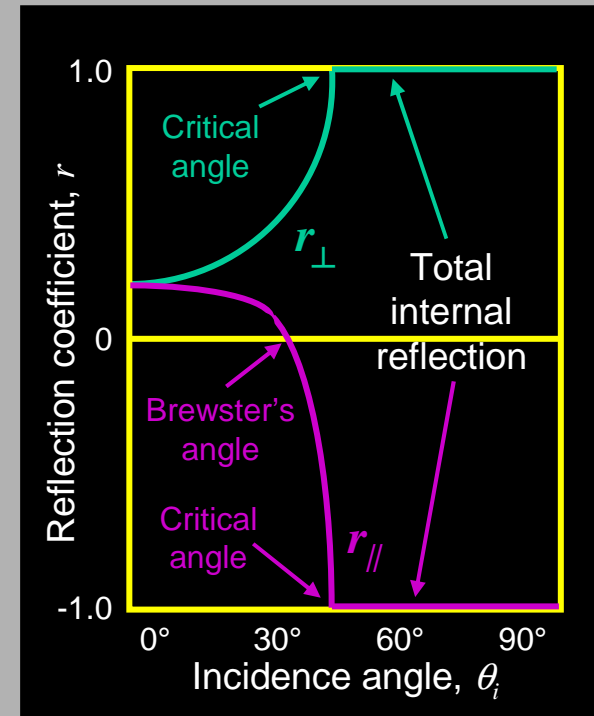


Interesting phase above the critical angle

||



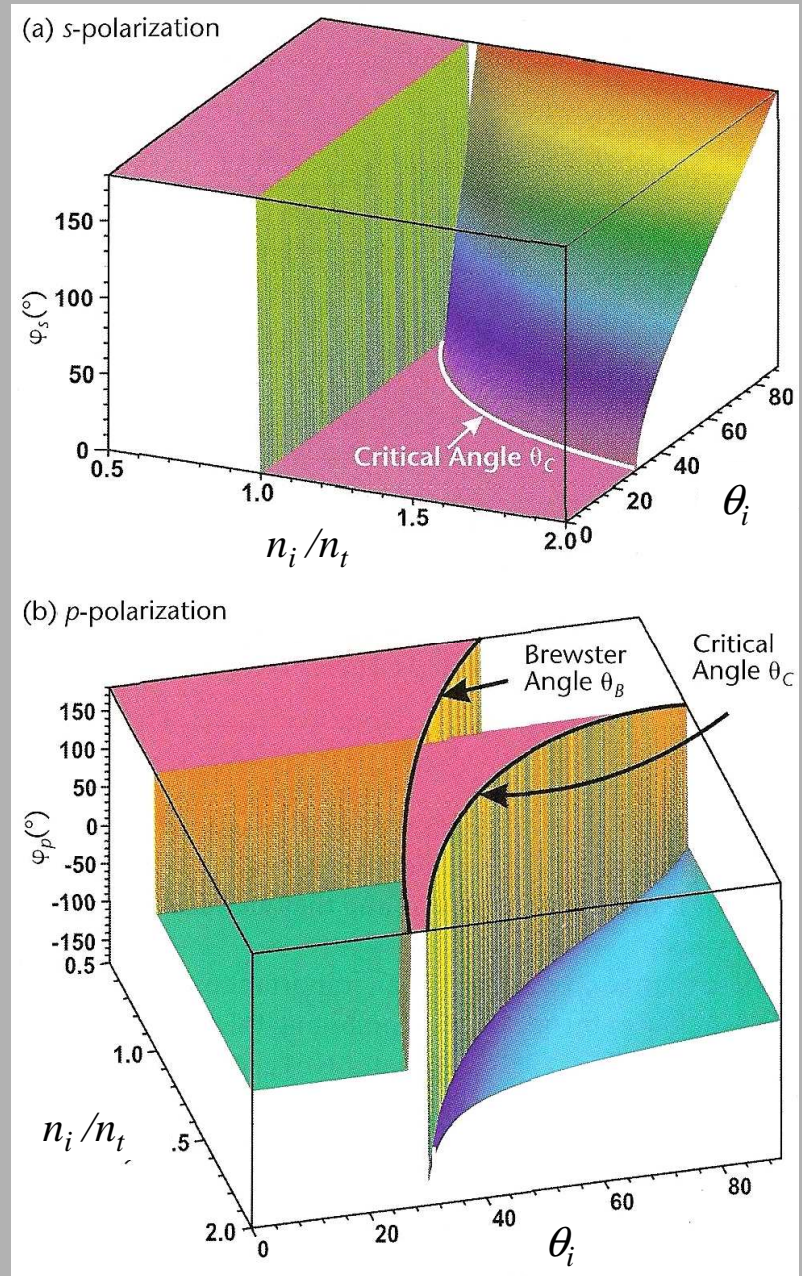
180° phase shift for angles below Brewster's angle; 0° for larger angles



Phase shifts vs. incidence angle and n_i/n_t

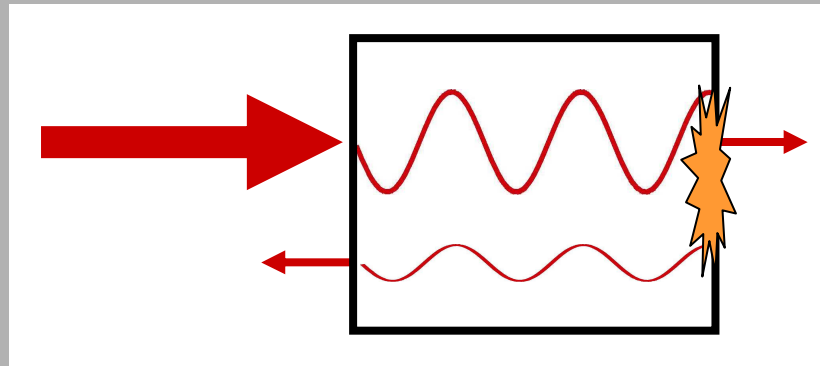
Note the general behavior
above and below the
various interesting
angles...

Li Li, OPN, vol. 14, #9,
pp. 24-30, Sept. 2003



If you slowly turn up a laser intensity incident on a piece of glass, where does damage happen first, the front or the back?

The obvious answer is the front of the object, which sees the higher intensity first.



But constructive interference happens at the back surface between the incident light and the reflected wave.

This yields an irradiance that is 44% higher just inside the back surface!

$$(1 + 0.2)^2 = 1.44$$

Phase shifts with coated optics

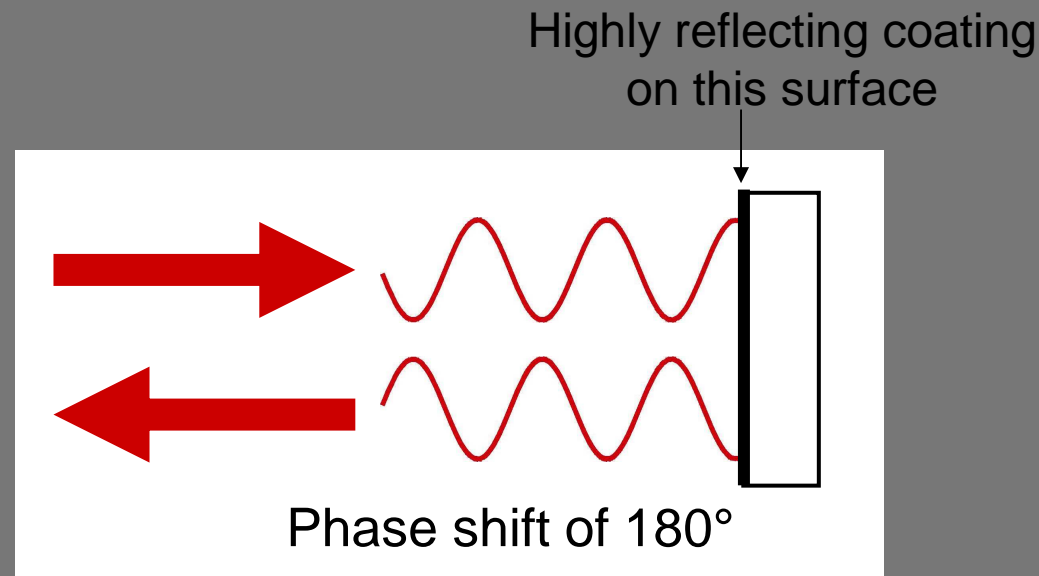
Reflections with different magnitudes can be generated using partial metallization or coatings. We'll see these later.

But the phase shifts on reflection are the same! For near-normal incidence:

180° if low-index-to-high and 0 if high-index-to-low.

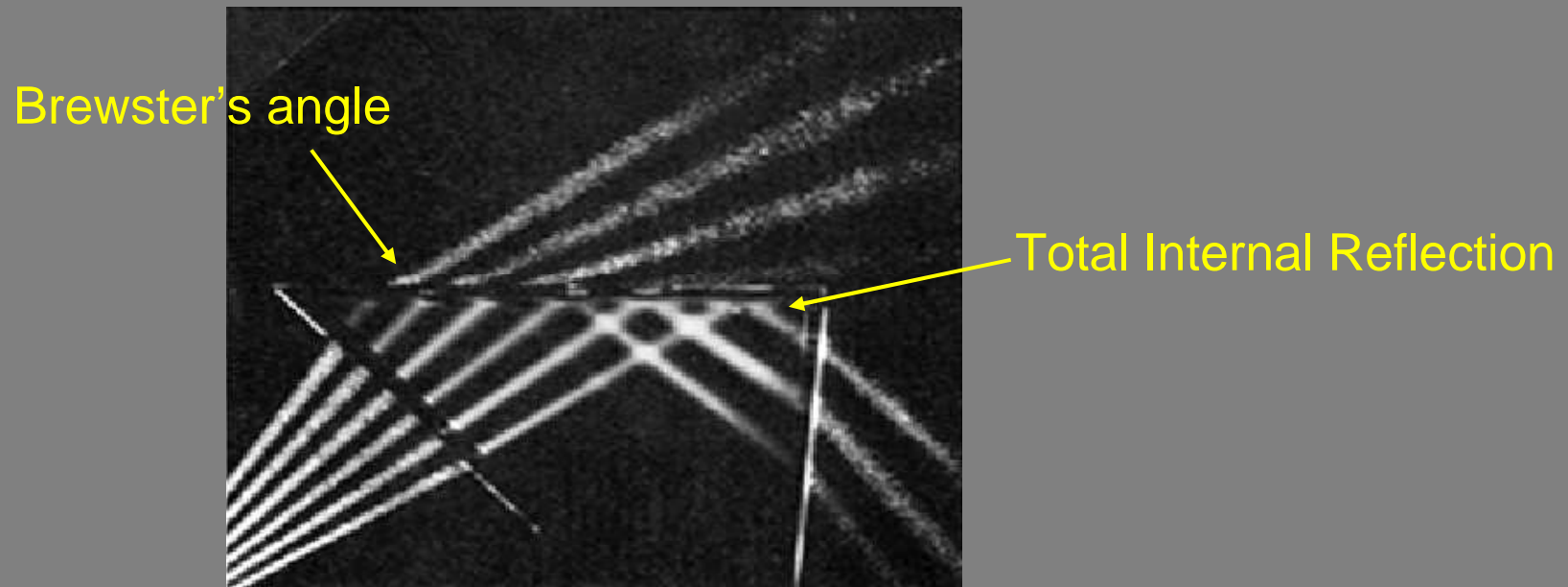
Example:

Laser Mirror



Total Internal Reflection occurs when $\sin(\theta_t) > 1$, and no transmitted beam can occur.

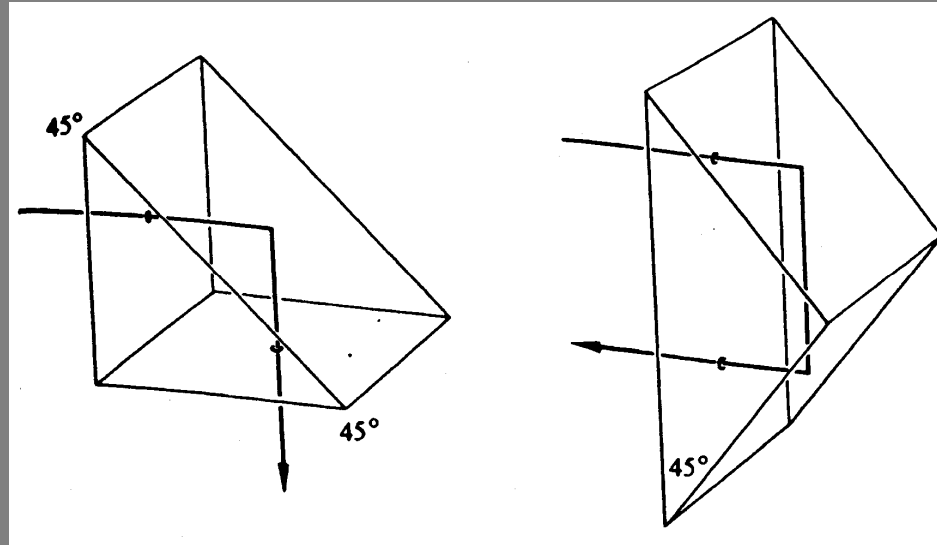
Note that the irradiance of the transmitted beam goes to zero (i.e., TIR occurs) as it grazes the surface.



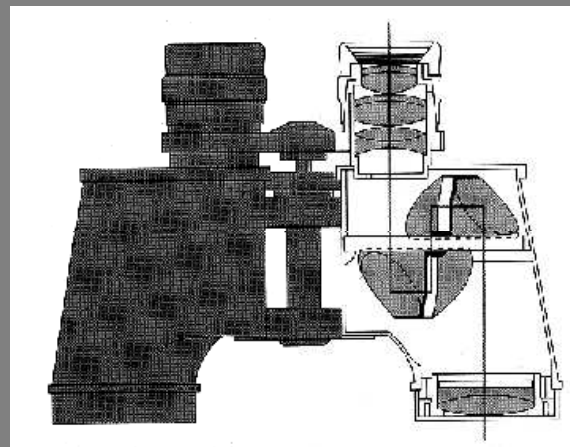
Total internal reflection is 100% efficient, that is, all the light is reflected.

Applications of Total Internal Reflection

Beam steerers

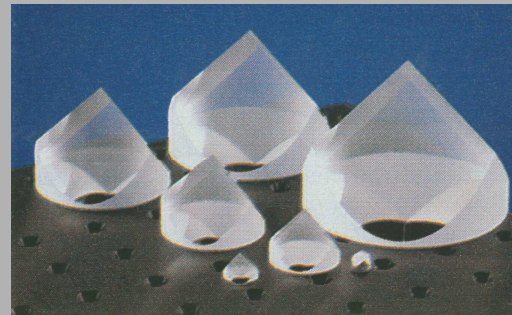
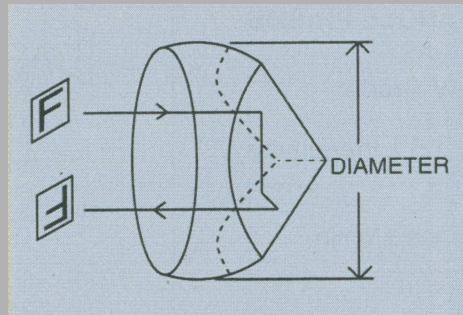


Beam steerers
used to compress
the path inside
binoculars



Three bounces: The Corner Cube

Corner cubes involve three reflections and also displace the return beam in space. Even better, they always yield a parallel return beam:

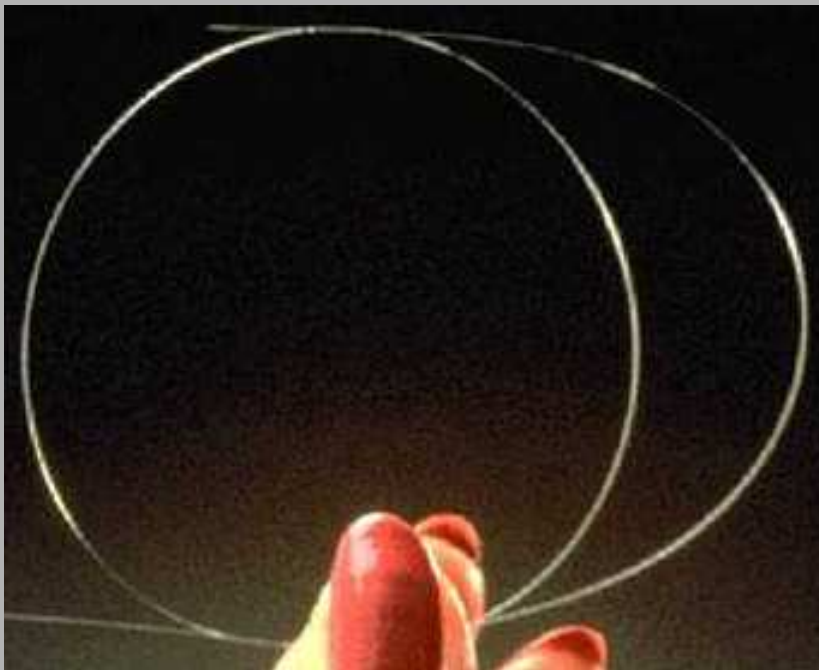


If the beam propagates in the z direction, it emerges in the $-z$ direction, with each point in the beam (x,y) reflected to the $(-x,-y)$ position.

Hollow corner cubes avoid propagation through glass and don't use TIR.

Fiber Optics

Optical fibers use TIR to transmit light long distances.



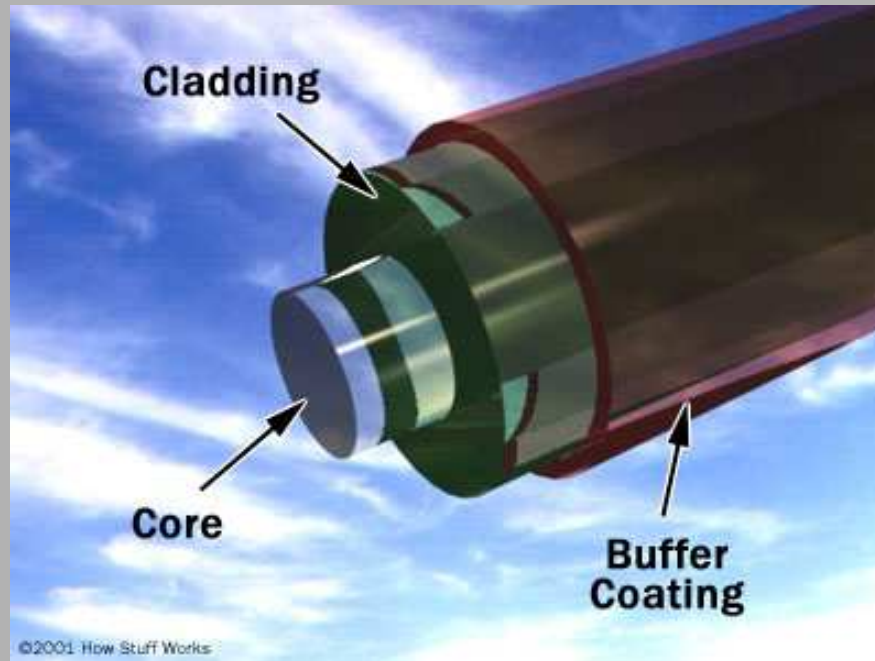
They play an ever-increasing role in our lives!

Design of optical fibers

Core: Thin glass center of the fiber that carries the light

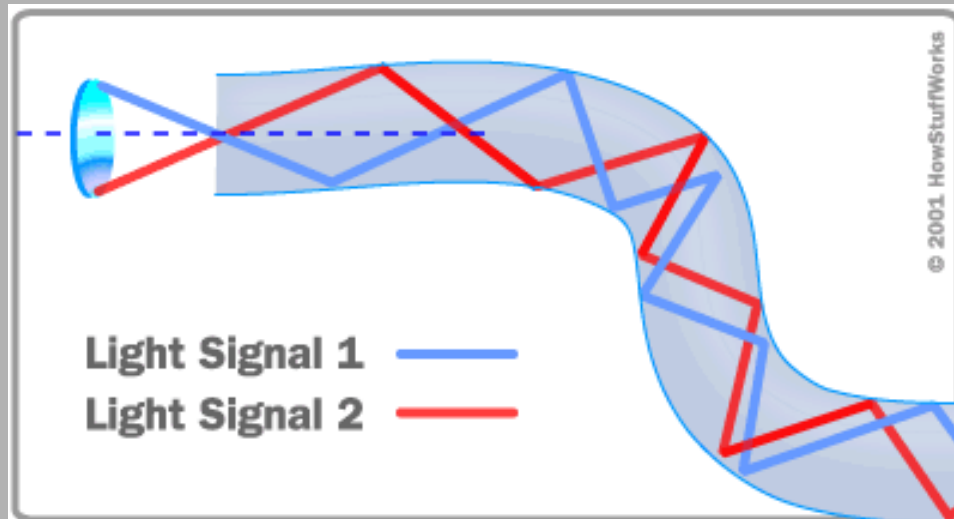
Cladding: Surrounds the core and reflects the light back into the core

Buffer coating: Plastic protective coating



$$n_{core} > n_{cladding}$$

Propagation of light in an optical fiber



Light travels through the core bouncing from the reflective walls. The walls absorb very little light from the core allowing the light wave to travel large distances.

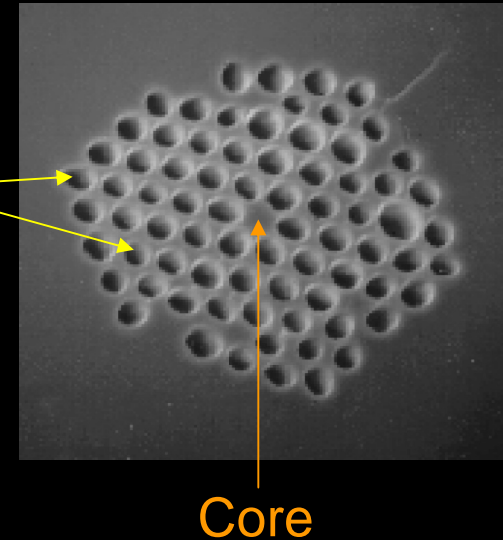
Some signal degradation occurs due to imperfectly constructed glass used in the cable. The best optical fibers show very little light loss -- less than 10%/km at 1.550 μm .

Maximum light loss occurs at the points of maximum curvature.

Microstructure fiber

In microstructure fiber, air holes act as the cladding surrounding a glass core. Such fibers have different dispersion properties.

Air holes



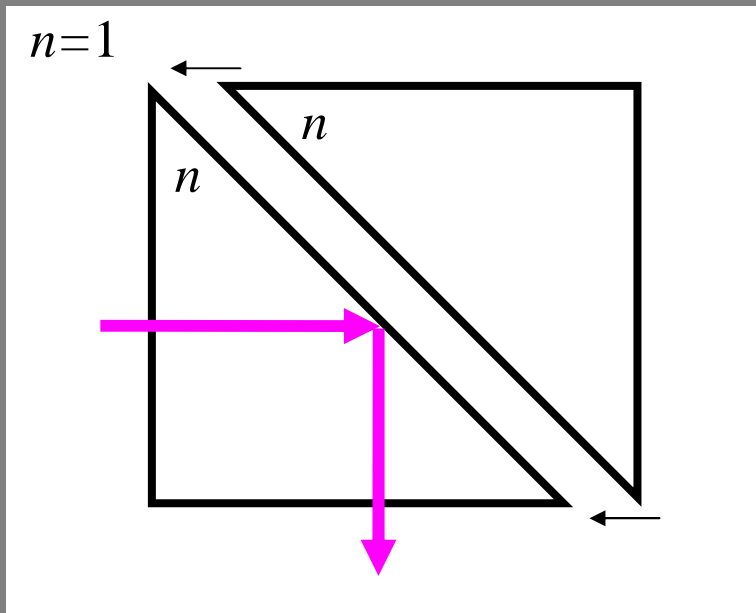
Such fiber has many applications, from medical imaging to optical clocks.

Photographs courtesy of
Jinendra Ranka, Lucent

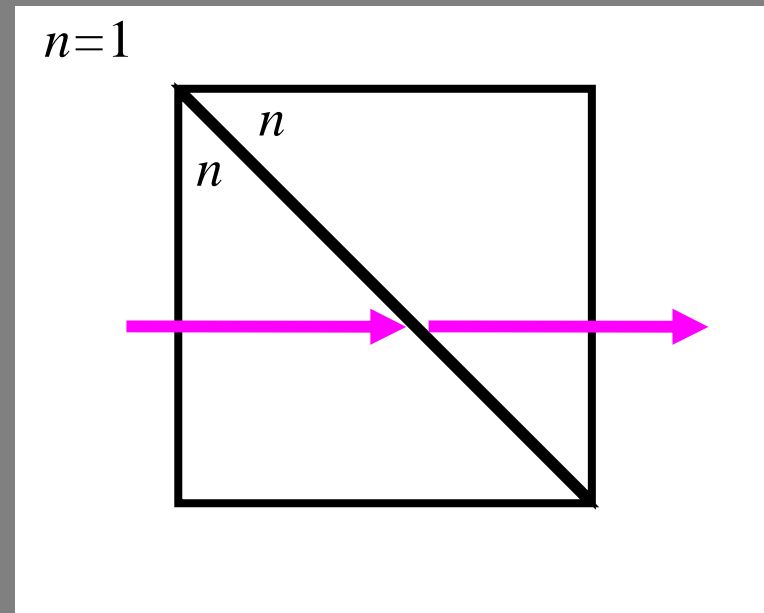
Frustrated Total Internal Reflection

By placing another surface in contact with a totally internally reflecting one, total internal reflection can be **frustrated**.

Total internal reflection



Frustrated total internal reflection

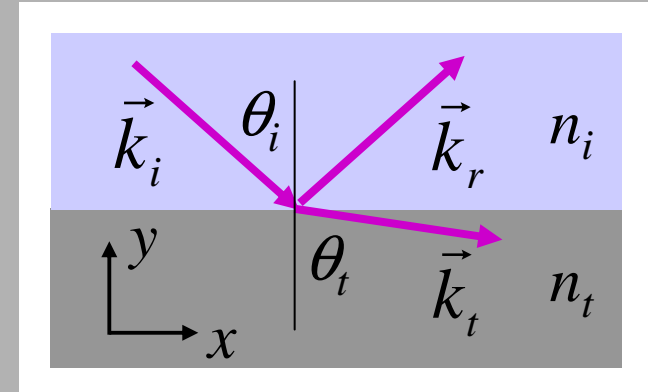


How close do the prisms have to be before TIR is frustrated?

This effect provides evidence for **evanescent fields**—fields that leak through the TIR surface—and is the basis for a variety of spectroscopic techniques.

The Evanescent Wave

The evanescent wave is the "transmitted wave" when total internal reflection occurs. A mystical quantity! So we'll do a mystical derivation:



$$r_{\perp} = \frac{E_{0r}}{E_{0i}} = \frac{[n_i \cos(\theta_i) - n_t \cos(\theta_t)]}{[n_i \cos(\theta_i) + n_t \cos(\theta_t)]}$$

Since $\sin(\theta_t) > 1$, θ_t doesn't exist, so computing r_{\perp} is impossible.

Let's check the reflectivity, R , anyway. Use Snell's Law to eliminate θ_t :

$$\cos(\theta_t) = \sqrt{1 - \sin^2(\theta_t)} = \sqrt{1 - \left(\frac{n_i}{n_t}\right)^2 \sin^2(\theta_i)} = \sqrt{\text{Neg. Number}}$$

Substituting this expression into the above one for r_{\perp} and

redefining R yields:

$$R \equiv r_{\perp} r_{\perp}^* = \left(\frac{a - bi}{a + bi} \right) \left(\frac{a + bi}{a - bi} \right) = 1$$

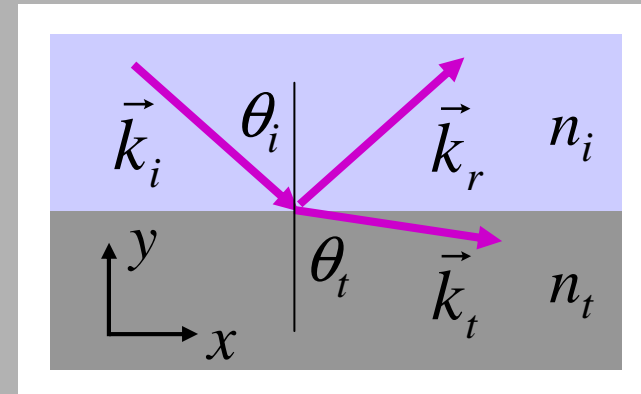
So all power is reflected; the evanescent wave contains no power.

The Evanescent-Wave k-vector

The evanescent wave k-vector must have x and y components:

Along surface: $k_{tx} = k_t \sin(\theta_t)$

Perpendicular to it: $k_{ty} = k_t \cos(\theta_t)$



Using Snell's Law, $\sin(\theta_t) = (n_i/n_t) \sin(\theta_i)$, so k_{tx} is meaningful.

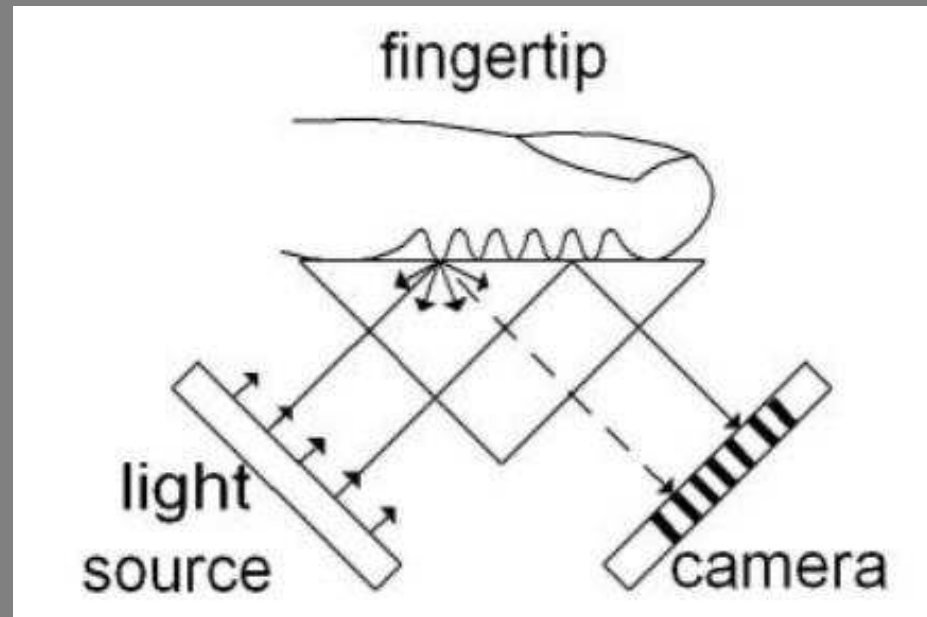
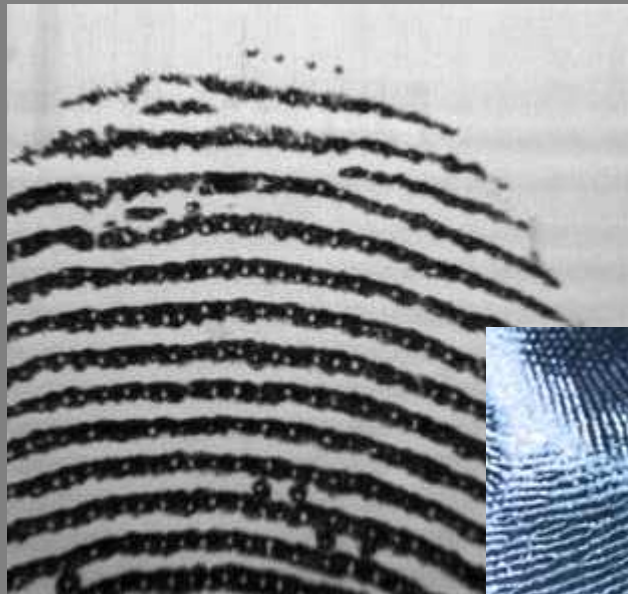
$$\begin{aligned} \text{And again: } \cos(\theta_t) &= [1 - \sin^2(\theta_t)]^{1/2} = [1 - (n_i/n_t)^2 \sin^2(\theta_i)]^{1/2} \\ &= \pm i\beta \end{aligned}$$

Neglecting the unphysical $-i\beta$ solution, we have:

$$E_t(x,y,t) = E_0 \exp[-k\beta y] \exp i [k (n_i/n_t) \sin(\theta_i) x - \omega t]$$

The evanescent wave decays exponentially in the transverse direction.

FTIR, the evanescent wave, and fingerprinting



See TIR from a fingerprint valley and FTIR from a ridge. This works because the ridges are higher than the evanescent wave penetration.

Complex refractive indices: optical properties of metals

A simple model of a metal is a gas of free electrons (the Drude model).

These free electrons and their accompanying positive nuclei can undergo "plasma oscillations" at frequency, ω_p .

where:
$$\omega_p^2 = \frac{N e^2}{(\epsilon_0 m_e)}$$

The refractive index for a metal is:
$$n^2 = 1 - \left(\frac{\omega_p}{\omega} \right)^2$$

When $n^2 < 0$, n is imaginary, and absorption is strong.

So for $\omega < \omega_p$ metals absorb strongly. For $\omega > \omega_p$ metals are transparent.

Reflection from metals

At normal incidence in air:

$$R = \frac{(n-1)^2}{(n+1)^2}$$

Generalizing to complex refractive indices:

$$R = \frac{(n-1)(n^*-1)}{(n+1)(n^*+1)}$$

