**Definition.** A map $\varphi : V \to W$ is called a *morphism* (or *polynomial map* or *regular map*) of algebraic sets if there are polynomials $\varphi_1, \dots, \varphi_m \in k[x_1, x_2, \dots, x_n]$ such that

$$\varphi((a_1, \dots, a_n)) = (\varphi_1(a_1, \dots, a_n), \dots, \varphi_m(a_1, \dots, a_n))$$

for all $(a_1, \dots, a_n) \in V$. The map $\varphi : V \to W$ is an *isomorphism* of algebraic sets if there is a morphism $\psi : W \to V$ with $\varphi \circ \psi = 1_W$ and $\psi \circ \varphi = 1_V$.

Note that in general $\varphi_1, \varphi_2, \dots, \varphi_m$ are not uniquely defined. For example, both $f = x$ and $g = x + (xy - 1)$ in the example above define the same morphism from $V = \mathcal{Z}(xy - 1)$ to $W = \mathbb{A}^1$.

Suppose $F$ is a polynomial in $k[x_1, \dots, x_m]$. Then $F \circ \varphi = F(\varphi_1, \varphi_2, \dots, \varphi_m)$ is a polynomial in $k[x_1, \dots, x_n]$ since $\varphi_1, \varphi_2, \dots, \varphi_m$ are polynomials in $x_1, \dots, x_n$. If $F \in \mathcal{I}(W)$, then $F \circ \varphi((a_1, a_2, \dots, a_n)) = 0$ for every $(a_1, a_2, \dots, a_n) \in V$ since $\varphi((a_1, a_2, \dots, a_n)) \in W$. Thus $F \circ \varphi \in \mathcal{I}(V)$. It follows that $\varphi$ induces a well defined map from the quotient ring $k[x_1, \dots, x_m]/\mathcal{I}(W)$ to the quotient ring $k[x_1, \dots, x_n]/\mathcal{I}(V)$:

$$\begin{aligned} \widetilde{\varphi} : k[W] &\to k[V] \\ f &\mapsto f \circ \varphi \end{aligned}$$

where $f \circ \varphi$ is given by $F \circ \varphi + \mathcal{I}(V)$ for any polynomial $F = F(x_1, \dots, x_m)$ with $f = F + \mathcal{I}(W)$. It is easy to check that $\tilde{\varphi}$ is a $k$-algebra homomorphism (for example, $\widetilde{\varphi}(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \widetilde{\varphi}(f) + \widetilde{\varphi}(g)$ shows that $\widetilde{\varphi}$ is additive). Note also the contravariant nature of $\widetilde{\varphi}$: the morphism from $V$ to $W$ induces a $k$-algebra homomorphism from $k[W]$ to $k[V]$.

Suppose conversely that $\Phi$ is any $k$-algebra homomorphism from the coordinate ring $k[W] = k[x_1, \dots, x_m]/\mathcal{I}(W)$ to $k[V] = k[x_1, \dots, x_n]/\mathcal{I}(V)$. Let $F_i$ be a representative in $k[x_1, \dots, x_n]$ for the image under $\Phi$ of $\bar{x}_i \in k[W]$ (i.e., $\Phi(x_i \bmod \mathcal{I}(W))$ is $F_i \bmod \mathcal{I}(V)$). Then $\varphi = (F_1, \dots, F_m)$ defines a polynomial map from $\mathbb{A}^n$ to $\mathbb{A}^m$, and in fact $\varphi$ is a morphism from $V$ to $W$. To see this it suffices to check that $\varphi$ maps a point of $V$ to a point of $W$ since by definition $\varphi$ is already defined by polynomials. If $g \in \mathcal{I}(W) \subset k[x_1, \dots, x_m]$, then in $k[W]$ we have

$$g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W)) = g(x_1, \dots, x_m) + \mathcal{I}(W) = \mathcal{I}(W) = 0 \in k[W],$$

and so

$$\Phi(g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W))) = 0 \in k[V].$$

Since $\Phi$ is a $k$-algebra homomorphism, it follows that

$$g(\Phi(x_1 + \mathcal{I}(W)), \dots, \Phi(x_m + \mathcal{I}(W)) = 0 \in k[V].$$

By definition, $\Phi(x_i + \mathcal{I}(W)) = F_i \bmod \mathcal{I}(V)$, so

$$g(F_1 \bmod \mathcal{I}(V), \dots, F_m \bmod \mathcal{I}(V)) = 0 \in k[V],$$

i.e.,

$$g(F_1, \dots, F_m) \in \mathcal{I}(V).$$

It follows that $g(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) = 0$ for every $(a_1, \dots, a_n)$ in $V$. This shows that if $(a_1, \dots, a_n) \in V$, then every polynomial in $\mathcal{I}(W)$ vanishes