The sum of the r equations yields

$$(1+r)^{p+2}-1=\sum_{n=0}^{p+1}\binom{p+2}{n}1^n+\sum_{n=0}^{p+1}\binom{p+2}{n}2^n+\dots+\sum_{n=0}^{p+1}\binom{p+2}{n}r^n$$

$$=\binom{p+2}{0}1^0+\dots+\binom{p+2}{p+1}1^{p+1}+\binom{p+2}{0}2^0+\dots+\binom{p+2}{p+1}2^{p+1}+\dots+\binom{p+2}{0}r^0+\dots+\binom{p+2}{p+1}r^{p+1}$$

$$=\binom{p+2}{0}1^0+\binom{p+2}{0}2^0+\dots+\binom{p+2}{0}r^0+\dots+\binom{p+2}{p+1}1^{p+1}+\binom{p+2}{p+1}2^{p+1}+\dots+\binom{p+2}{p+1}r^{p+1}$$

$$=\binom{p+2}{0}\left[1^0+2^0+\dots+r^0\right]+\dots+\binom{p+2}{p+1}\left[1^{p+1}+\dots+r^{p+1}\right]$$

$$=\binom{p+2}{0}\left[1^0+2^0+\dots+r^0\right]+\dots+\binom{p+2}{p}\left[1^p+\dots+r^p\right]+\binom{p+2}{p+1}\left[1^{p+1}+\dots+r^{p+1}\right]$$

$$=\binom{p+2}{0}\sum_{k=1}^{r}k^0+\dots+\binom{p+2}{p}\sum_{k=1}^{r}k^p+\binom{p+2}{p+1}\sum_{k=1}^{r}k^{p+1}$$

$$=\sum_{j=0}^{p}\sum_{k=1}^{r}\binom{p+2}{j}k^j+\binom{p+2}{p+1}\sum_{k=1}^{r}k^{p+1}$$

If the binomial theorem is used on the LHS, then we arrive at

$$\sum_{j=0}^{p+2}\binom{p+2}{j}r^j-1=\sum_{j=0}^{p}\sum_{k=1}^{r}\binom{p+2}{j}k^j+\binom{p+2}{p+1}\sum_{k=1}^{r}k^{p+1}$$

$$\Rightarrow\binom{p+2}{p+1}\sum_{k=1}^{r}k^{p+1}=\sum_{j=0}^{p+2}\binom{p+2}{j}r^j-1-\sum_{j=0}^{p}\sum_{k=1}^{r}\binom{p+2}{j}k^j$$