in the original Lagrangian, this is equivalent to only a rescaling of masses, coupling constants and fields by (parameter dependent, divergent) constant numbers. Theories for which this can be done are called *renormalizable* theories.

We will not discuss the cut-off regularization procedure in any detail because it is not explicitly Lorentz invariant. But we will use the other two types of regularization. And we will also show that the physical results are independent of the regularization procedure.

12.4 Ward-Takahashi identity

We already identified three types of divergent amplitudes for QED. There is in fact a relation between them which makes it easier to tackle them. This can be guessed from the fact that the interaction between fermions and photons was introduced by the minimal substitution method summarized in Eq. (9.15), which means $\not{\partial} \to \not{\partial} + ieQ\not{A}$, using the gauge principle.

The relation can be explicitly stated now. The minimal substitution method implies that the vertex function introduced in §11.1 is given at the tree level by

$$\Gamma_\mu^{(0)}(p, p-q) = Q\gamma_\mu\,, \tag{12.11}$$

where the superscript on Γ denotes the number of loops, and we have indicated the momenta of the incoming and outgoing fermions in parenthesis. Thus,

$$\begin{aligned} q^\mu \Gamma_\mu^{(0)}(p, p-q) &= Q\not{q} = Q\Big[(\not{p} - m) - (\not{p} - \not{q} - m)\Big] \\ &= Q\Big[S_F^{-1}(p) - S_F^{-1}(p-q)\Big]^{(0)}. \end{aligned} \tag{12.12}$$

This relation turns out to be valid at all orders in perturbation theory. In other words, we can write

$$q^\mu \Gamma_\mu(p, p-q) = Q\Big[S_F^{-1}(p) - S_F^{-1}(p-q)\Big] \tag{12.13}$$

as an exact result. This equality is called the *Ward-Takahashi identity*.

On the left hand side we have Γ_μ, which is the general 3-point function involving two fermions and one photon, as we have seen in

Ch. 11. The usefulness of this identity can be seen by interpreting the right hand side as the difference between 2-point functions, i.e., Feynman amplitudes of diagrams with two external legs. Let us first draw the total 2-point amplitude as the sum of the tree-level and the rest.

(12.14)

The tree-level amplitude, which is the undecorated line on the right hand side of Eq. (12.14), can be read off from the Lagrangian by treating the quadratic term just like an interaction and using the procedure for writing Feynman rules described in §6.6. This will give $\not p - m$, which is just the inverse of the tree-level propagator. The Feynman amplitude for the remainder, which is drawn as the gray box, is conventionally denoted by $-\Sigma(p)$. Thus Eq. (12.14) can be written algebraically as

$$S_F^{-1}(p) = \not p - m - \Sigma(p)\,. \tag{12.15}$$

It is this full 2-point function that goes into the Ward-Takahashi identity of Eq. (12.13).

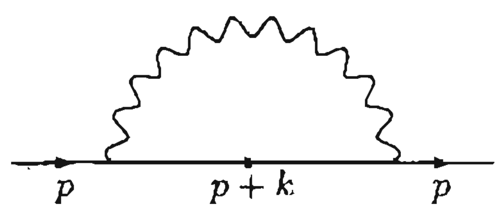

Figure 12.3: 1-loop diagram for fermion self-energy in QED.

To show that the Ward-Takahashi identity is at least valid for the 1-loop contributions, let us first evaluate the 1-loop corrections to the 2-point function for the fermions, i.e., the self-energy of the fermions. This comes from the diagram of Fig. 12.3. Using the Feynman rules of QED, we can write the 1-loop contribution to it as

$$-i\Sigma^{(1)}(p) = (-ieQ)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{i}{\not p + \not k - m} \gamma_\nu \, iD^{\mu\nu}(k)\,, \tag{12.16}$$

where $iD^{\mu\nu}(k)$ is the photon propagator.

As for the vertex function, we have already seen the 1-loop contribution in Eq. (11.37). Although that was written for on-shell electrons, the expression remains the same except $-e$ is now replaced by eQ:

$$-ie\Gamma_\mu^{(1)} = (-ieQ)^3 \int \frac{d^4k}{(2\pi)^4}\gamma_\lambda \frac{i}{\not{p}' + \not{k} - m}\gamma_\mu \frac{i}{\not{p} + \not{k} - m}\gamma_\rho \, iD^{\lambda\rho}(k), \tag{12.17}$$

where $p' = p - q$. Thus

$$q^\mu\Gamma_\mu^{(1)} = (-ieQ)^2 Q \int \frac{d^4k}{(2\pi)^4}\gamma_\lambda \frac{i}{\not{p}' + \not{k} - m}\not{q} \frac{i}{\not{p} + \not{k} - m}\gamma_\rho \, iD^{\lambda\rho}(k). \tag{12.18}$$

Since $p' = p - q$, we can write $\not{q}$ as $(\not{p} + \not{k} - m) - (\not{p}' + \not{k} - m)$. Thus,

$$\frac{i}{\not{p}' + \not{k} - m}\not{q}\frac{i}{\not{p} + \not{k} - m} = i\left[\frac{i}{\not{p}' + \not{k} - m} - \frac{i}{\not{p} + \not{k} - m}\right]. \tag{12.19}$$

Putting this back into Eq. (12.18) and comparing with the expression for the self-energy in Eq. (12.16), we obtain

$$q^\mu\Gamma_\mu^{(1)}(p,p') = Q\left[\Big(-\Sigma^{(1)}(p)\Big) - \Big(-\Sigma^{(1)}(p')\Big)\right]. \tag{12.20}$$

Adding this to the tree level relation gives Eq. (12.13) up to 1-loop contributions.

It is true that our proof takes only the 1-loop contribution into account and considers only the fermion-photon interaction. But one can construct a more general proof from current conservation. In fact the identity holds true to all orders, even if the fermion has other interactions which do not violate gauge invariance. Also, we know from Eq. (11.8) that $q^\mu\Gamma_\mu$ must vanish between the spinors. However, if we consider the effective electromagnetic vertex of a neutral fermion for which $Q = 0$, we see from Eq. (12.13) that for neutral fermions a more stringent relation, $q^\mu\Gamma_\mu = 0$, is satisfied irrespective of whether we consider it between spinors or not. These facts are very useful in considering electromagnetic interactions of particles in the presence of other interactions as well.