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$$0 = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} \tag{1}$$

The geodesic equations in the new coordinates are

$$0 = \frac{\mathrm{d}^2 \bar{x}^\mu}{\mathrm{d}s^2} + \bar{\Gamma}^\mu_{\alpha\beta} \frac{\mathrm{d}\bar{x}^\alpha}{\mathrm{d}s} \frac{\mathrm{d}\bar{x}^\beta}{\mathrm{d}s} \tag{2}$$

The tangent vectors are related by

$$\frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}s} = \frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}x^\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \tag{3}$$

We can rearrange the geodesic equations in the $\bar{x}^\nu$ coordinates as follows

$$\begin{aligned}
0 &= \frac{d^2 \bar{x}^\nu}{ds^2} + \bar{\Gamma}^\nu_{\alpha\beta} \frac{d\bar{x}^\alpha}{ds} \frac{d\bar{x}^\beta}{ds} \\
&= \frac{d}{ds} \left(\frac{\partial \bar{x}^\nu}{\partial x^\tau} \frac{dx^\tau}{ds} \right) + \bar{\Gamma}^\nu_{\alpha\beta} \frac{dx^\sigma}{ds} \frac{dx^\theta}{ds} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\beta}{\partial x^\theta} \\
&= \frac{dx^\tau}{ds} \frac{dx^\rho}{ds} \frac{\partial^2 \bar{x}^\nu}{\partial x^\rho x^\tau} + \frac{\partial \bar{x}^\nu}{\partial x^\mu} \frac{d^2 x^\mu}{ds^2} + \bar{\Gamma}^\mu_{\alpha\beta} \frac{dx^\sigma}{ds} \frac{dx^\theta}{ds} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\theta}{\partial x^\beta} \\
\Rightarrow 0 &= \frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{dx^\tau}{ds} \frac{dx^\rho}{ds} \frac{\partial^2 \bar{x}^\nu}{\partial x^\rho x^\tau} + \frac{d^2 x^\mu}{ds^2} + \frac{\partial x^\mu}{\partial \bar{x}^\nu} \bar{\Gamma}^\nu_{\alpha\beta} \frac{dx^\sigma}{ds} \frac{dx^\theta}{ds} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\theta}{\partial x^\beta}
\end{aligned} \tag{4}$$

We also know that the geodesic equations hold in the old coordinates, so we can equate the right hand side of (1) with the right hand side of (4). This gives us

$$\begin{aligned}
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} &= \frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{dx^\tau}{ds} \frac{dx^\rho}{ds} \frac{\partial^2 \bar{x}^\nu}{\partial x^\rho x^\tau} + \frac{d^2 x^\mu}{ds^2} + \frac{\partial x^\mu}{\partial \bar{x}^\nu} \bar{\Gamma}^\nu_{\alpha\beta} \frac{dx^\sigma}{ds} \frac{dx^\theta}{ds} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\beta}{\partial x^\theta} \\
\Rightarrow \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} &= \frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{dx^\tau}{ds} \frac{dx^\rho}{ds} \frac{\partial^2 \bar{x}^\nu}{\partial x^\rho x^\tau} + \frac{\partial x^\mu}{\partial \bar{x}^\nu} \bar{\Gamma}^\nu_{\alpha\beta} \frac{dx^\sigma}{ds} \frac{dx^\theta}{ds} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\beta}{\partial x^\theta} \\
\Rightarrow \Gamma^\mu_{\sigma\theta} &= \frac{\partial x^\tau}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x^\theta} \frac{\partial^2 x^\mu}{\partial x^\rho x^\tau} + \bar{\Gamma}^\gamma_{\alpha\beta} \frac{\partial x^\mu}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\beta}{\partial x^\theta} \\
&= \frac{\partial x^\tau}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x^\theta} \frac{\partial^2 x^\mu}{\partial x^\rho x^\tau} + \bar{\Gamma}^\gamma_{\alpha\beta} \frac{\partial x^\mu}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^\sigma} \frac{\partial \bar{x}^\beta}{\partial x^\theta}
\end{aligned}$$