which reduces to

$$(x^2 - 1)u^{(\ell+2)} + 2xu^{(\ell+1)} - \ell(\ell+1)u^{(\ell)} = 0.$$

Changing the sign all through, we recover Legendre's equation (18.1) with $u^{(\ell)}$ as the dependent variable. Since, from (18.9), ℓ is an integer and $u^{(\ell)}$ is regular at $x = \pm 1$, we may make the identification

$$u^{(\ell)}(x) = c_\ell P_\ell(x), \tag{18.10}$$

for some constant c_ℓ that depends on ℓ. To establish the value of c_ℓ we note that the only term in the expression for the ℓth derivative of $(x^2-1)^\ell$ that does not contain a factor $x^2 - 1$, and therefore does not vanish at $x = 1$, is $(2x)^\ell \ell!(x^2-1)^0$. Putting $x = 1$ in (18.10) and recalling that $P_\ell(1) = 1$, therefore shows that $c_\ell = 2^\ell \ell!$, thus completing the proof of Rodrigues' formula (18.9).

►*Use Rodrigues' formula to show that*

$$I_\ell = \int_{-1}^{1} P_\ell(x)P_\ell(x)\,dx = \frac{2}{2\ell+1}. \tag{18.11}$$

The result is trivially obvious for $\ell = 0$ and so we assume $\ell \geq 1$. Then, by Rodrigues' formula,

$$I_\ell = \frac{1}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}\left[\frac{d^\ell(x^2-1)^\ell}{dx^\ell}\right]\left[\frac{d^\ell(x^2-1)^\ell}{dx^\ell}\right]dx.$$

Repeated integration by parts, with all boundary terms vanishing, reduces this to

$$\begin{aligned} I_\ell &= \frac{(-1)^\ell}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}(x^2-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell\,dx \\ &= \frac{(2\ell)!}{2^{2\ell}(\ell!)^2}\int_{-1}^{1}(1-x^2)^\ell\,dx. \end{aligned}$$

If we write

$$K_\ell = \int_{-1}^{1}(1-x^2)^\ell\,dx,$$

then integration by parts (taking a factor 1 as the second part) gives

$$K_\ell = \int_{-1}^{1} 2\ell x^2(1-x^2)^{\ell-1}\,dx.$$

Writing $2\ell x^2$ as $2\ell - 2\ell(1-x^2)$ we obtain

$$\begin{aligned} K_\ell &= 2\ell\int_{-1}^{1}(1-x^2)^{\ell-1}\,dx - 2\ell\int_{-1}^{1}(1-x^2)^\ell\,dx \\ &= 2\ell K_{\ell-1} - 2\ell K_\ell \end{aligned}$$

and hence the recurrence relation $(2\ell+1)K_\ell = 2\ell K_{\ell-1}$. We therefore find

$$K_\ell = \frac{2\ell}{2\ell+1}\,\frac{2\ell-2}{2\ell-1}\cdots\frac{2}{3}K_0 = 2^\ell \ell!\,\frac{2^\ell \ell!}{(2\ell+1)!}\,2 = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!},$$

which, when substituted into the expression for I_ℓ, establishes the required result. ◄