UNIVERSITY OF OTAGO EXAMINATIONS 2012

PHYSICS

PHSI 332

Electromagnetism and Condensed Matter

Semester Two

(TIME ALLOWED: 2 HOURS)

This examination paper comprises 6 pages

Candidates should answer questions as follows:

Answer THREE QUESTIONS

The following material is provided:

Use of calculators:

No restriction on the model of calculator to be used, but no device with communication capability shall be accepted as a calculator.
(Calculators are subject to inspection by examiners.)

Candidates are permitted copies of:

No additional material.

Other Instructions:

DO NOT USE RED INK OR PENCIL.
The values of PHYSICAL QUANTITIES which may be needed are given on page 2.
Symbols for physical quantities are given in italics.
Symbols for vector quantities are in bold.

UNIVERSITY OF OTAGO PHYSICS DEPARTMENT
PHYSICAL CONSTANTS
(for use in examinations)

Quantity	**Symbol**	**Value**			**Unit**
Speed of light (exact)	c	2.998	$\times$	10^{8}	m s^{-1}
Planck's constant	h	6.626	$\times$	10^{-34}	J s
Planck's constant (reduced)	$\hbar \equiv h/(2\pi)$	1.055	$\times$	10^{-34}	J s
		= 6.582	$\times$	10^{-22}	MeV s
|electron charge|	e	1.602	$\times$	10^{-19}	C
electron mass	m_e	9.109	$\times$	10^{-31}	kg
proton mass	m_p	1.673	$\times$	10^{-27}	kg
unified atomic mass unit	u	1.661	$\times$	10^{-27}	kg
		= 931.5			MeV c^{-2}
permittivity of free space (exact)	$\varepsilon_0 \equiv 1/(\mu_0 c^2)$	8.854	$\times$	10^{-12}	F m^{-1}
permeability of free space (exact)	μ_0	4π	$\times$	10^{-7}	N A^{-2}
Coulomb constant (exact)	$(4\pi\varepsilon_0)^{-1}$	8.988	$\times$	10^{9}	N m^2 C^{-2}
impedance of free space (exact)	$Z_0 \equiv c\mu_0$	376.7			Ω
(fine structure constant)$^{-1}$	α^{-1}	137.0			
classical electron radius	r_e	2.818	$\times$	10^{-15}	m
electron Compton wavelength/(2π)	$\lambda\!\!\!^{-}_e \equiv \lambda_e/(2\pi)$	3.862	$\times$	10^{-13}	m
Bohr radius ($m_{\text{nucleus}} = \infty$)	a_0	5.292	$\times$	10^{-11}	m
Rydberg energy	hcR_∞	13.61			eV
Bohr magneton	μ_B	9.274	$\times$	10^{-24}	J T^{-1}
		= 5.788	$\times$	10^{-5}	eV T^{-1}
nuclear magneton	μ_N	5.051	$\times$	10^{-27}	J T^{-1}
		= 3.152	$\times$	10^{-8}	eV T^{-1}
electron charge/mass	$-e/m_e$	-1.759	$\times$	10^{11}	C kg^{-1}
proton charge/mass	e/m_p	9.579	$\times$	10^{7}	C kg^{-1}
gravitational constant (Newton)	G	6.673	$\times$	10^{-11}	N m^2 kg^{-2}
standard g (sea level)	g	9.807			m s^{-2}
Avogadro number	N_A	6.022	$\times$	10^{23}	mol^{-1}
Boltzmann constant	k	1.381	$\times$	10^{-23}	J K^{-1}
		= 8.617	$\times$	10^{-5}	eV K^{-1}
molar gas constant	$R \equiv kN_A$	8.315			J mol^{-1} K^{-1}
molar volume (at STP)	V_m	2.241	$\times$	10^{-2}	m^3 mol^{-1}
Wien displacement constant	$b \equiv \lambda_{\text{max}} T$	2.898	$\times$	10^{-3}	m K
Stefan-Boltzmann constant	σ	5.671	$\times$	10^{-8}	W m^{-2} K^{-4}

All numerical values have been rounded to four significant figures, for use where necessary in this examination. Note that the actual situation is that except for G, k, R, N_A, and V_m all measured values are reliable to better than 1 part per million. Also μ_0, c, ε_0, and Z_0 are exact. μ_0 is given above, the exact value of the speed of light is $c = 299\ 792\ 458$ ms^{-1}, $\varepsilon_0 \equiv 1/(\mu_0 c^2)$, and $Z_0 \equiv c\mu_0 = \sqrt{\mu_0/\varepsilon_0}$.

Definitions and equations:

$$\alpha \equiv e^2/(4\pi\varepsilon_0\hbar c), \qquad a_0 \equiv 4\pi\varepsilon_0\hbar^2/(m_e e^2) = r_e/\alpha^2,$$

$$u \equiv \text{mass}\,^{12}\text{C atom}/12 = (10^{-3}\ \text{kg})/N_A, \qquad r_e \equiv e^2/(4\pi\varepsilon_0 m_e c^2),$$

$$\lambda\!\!\!^{-}_e \equiv \lambda_e/(2\pi) \equiv \hbar/(m_e c) = r_e/\alpha, \qquad hcR_\infty \equiv m_e e^4/(2(4\pi\varepsilon_0)^2\hbar^2) = m_e c^2\alpha^2/2,$$

$$\mu_B \equiv e\hbar/(2m_e), \qquad \mu_N \equiv e\hbar/(2m_p).$$

TURN OVER

1. (a) (i) Briefly explain the method of images and give a justification of the method in terms of an appropriate uniqueness theorem for Laplace's equation (you do not need to prove this uniqueness theorem).

(ii) A single point charge is located between two infinitely large parallel conducting plates, closer to one plate than to the other. The plates are grounded, i.e., are held at zero potential. Draw a diagram representing this system and indicate on this diagram the position of any image charges needed to solve this problem.

(b) For a system with azimuthal symmetry, Laplace's equation in spherical coordinates is given by,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0\,.$$

The general solution to this equation is,

$$V\left(r,\theta\right)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l\left(\cos\theta\right)\,.$$

Suppose a spherical shell has a radius R and a surface charge density given by,

$$\sigma_0\left(\theta\right)=3\cos^2\theta+\cos\theta\,,$$

and the charge density inside and outside the spherical shell is zero.

(i) Write an expression for σ_0 as a linear combination of the first three Legendre polynomials.

(ii) Show that, for all values of l, $B_l=0$ inside the spherical shell, and $A_l=0$ outside the spherical shell.

(iii) Show that the coefficients A_l and B_l are related by the expression,

$$B_l=A_l R^{2l+1}$$

(iv) Using the boundary condition,

$$\left.\left(\frac{\partial V_{\text{out}}}{\partial r}-\frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R}=-\frac{1}{\epsilon_0}\sigma_0\left(\theta\right),$$

show that

$$A_l=\frac{1}{2\epsilon_0 R^{l-1}}\int_0^{\pi}\sigma_0\left(\theta\right)P_l\left(\cos\theta\right)\sin\theta\,d\theta.$$

(v) Derive an expression for the electrostatic potential inside and outside the sphere.

$$\boxed{\begin{aligned}P_0(x)&=1\\P_1(x)&=x\\P_2(x)&=\left(3x^2-1\right)/2\\P_3(x)&=\left(5x^3-3x\right)/2\end{aligned}}$$

$$\int_0^{\pi}P_l\left(\cos\theta\right)P_{l'}\left(\cos\theta\right)\sin\theta d\theta=\begin{cases}0, & \text{if } l'\neq l,\\ \dfrac{2}{2l+1}, & \text{if } l'=l\,.\end{cases}$$

2. (a) (i) From Maxwell's equations, derive a wave equation for the electric field component of an electromagnetic wave in a good conductor. Take care to define the terms you introduce and to explain any assumptions which you make. Note: $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

(ii) Consider monochromatic plane wave solutions to the electric and magnetic wave equations (derived above) with the form,

$$\begin{aligned} \tilde{\mathbf{E}}(z,t) &= \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)} \\ \tilde{\mathbf{B}}(z,t) &= \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}. \end{aligned}$$

(iii) Show that the magnitude of the wave vector, $\tilde{k}$, is given by the expression,

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

(iv) Given that the real and imaginary parts of $\tilde{k}$ are given by,

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2},$$

where $\tilde{k} = k + i\kappa$. Show that for a very good conductor, i.e., where $\sigma \gg \omega\epsilon$, the skin depth is given by

$$d = \frac{\lambda}{2\pi}$$

where λ is the wavelength of the plane wave in the conductor.

(b) An electromagnetic wave propagates through a vacuum and is incident at an angle θ_I on the plane boundary of a transparent, perfectly non-conducting material. The incident electric and magnetic fields are given by,

$$\begin{aligned} \tilde{\mathbf{E}}_I(\mathbf{r},t) &= \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)} \\ \tilde{\mathbf{B}}_I(\mathbf{r},t) &= \frac{1}{v_1}\left(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I\right). \end{aligned}$$

A portion of the incident electromagnetic wave is reflected at the boundary and a portion is transmitted into the transparent medium.

(i) How are the frequencies of the reflected and transmitted waves related to the frequency of the incident wave? Explain.

(ii) What condition must the phases of the incident, reflected and transmitted waves satisfy at the boundary?

(iii) Using the boundary condition on the phases given above derive the law of reflection and the law of refraction.

(iv) What boundary condition must the amplitudes, $\tilde{\mathbf{E}}_{0_I}$, $\tilde{\mathbf{E}}_{0_R}$, and $\tilde{\mathbf{E}}_{0_T}$ satisfy at the interface between the two media?

3. (a) Sketch a single unit cell of the body-centered orthorhombic crystal structure.

(b) Sketch a tetragonal unit cell, and within that cell indicate locations of the $\frac{1}{2}\,1\,\frac{1}{2}$ and $\frac{1}{4}\,\frac{1}{2}\,\frac{3}{4}$ point coordinates.

(c) Within a cubic unit cell, sketch the following directions:

(i) $[\bar{1}\,1\,0]$	(iii) $[0\,\bar{1}\,2]$	(v) $[\bar{1}\,\bar{1}\,1]$
(ii) $[\bar{1}\,\bar{2}\,1]$	(iv) $[1\,\bar{3}\,3]$	(vi) $[\bar{1}\,2\,2]$

(d) Consider the body-centered cubic (bbc) crystal structure. For this structure show that the unit cell edge length a and the atomic radius R are related by the expression,

$$a = \frac{4R}{\sqrt{3}}\,.$$

(e) For the hexagonal close-packed (hcp) crystal structure, show that the ratio of lattice constants c and a which gives the optimal packing is $\frac{c}{a} = 1.633$.

(f) (i) Give a brief definition of the *packing fraction* (also sometimes called the *atomic packing factor*) of a crystal structure.

(ii) Show that packing fraction for the body-centred cubic (bcc) crystal structure is $\frac{\sqrt{3}\pi}{8} = 0.68$ and for the hexagonal close-packed (hcp) crystal structure is $\frac{\sqrt{2}\pi}{6} = 0.74$.

4. (a) In a short paragraph describe the key elements of Bragg's analysis of X-ray diffraction.

   (b) In a short paragraph describe the key elements of von Laue's analysis of X-ray diffraction.

   (c) Monochromatic X-radiation having a wavelength of 0.0711 nm is used to investigate the crystal structure of an iron sample. You observe a first-order diffraction peak at a diffraction angle of 46.21°. Note that for a bcc crystal structure $a = \frac{4R}{\sqrt{3}}$, and $R = 0.1241$ nm for iron.

      (i) Calculate the interplanar spacing, d_{hkl}.

      (ii) What are the Miller indices of the planes which are responsible for this diffraction peak?

   (d) An iridium sample is investigated in an X-ray diffraction experiment using monochromatic X-radiation with a wavelength of 0.1542 nm. Iridium has an fcc crystal structure. The experiment shows that the (first-order reflection) diffraction angle for the (220) set of planes appears at an angle of 69.22°.

      (i) Calculate the interplanar spacing for this family of planes.

      (ii) Calculate the atomic radius of an iridium atom.

END