

Dear All.

I've read your discussion with interest and I think there is something that solves your conundrum with the missing quadratic term in the Least Action principle in the null world-line case.

Namely, for a null geodesic you should change the Lagrangian form in the Least Action principle, because it is then a constrained functional with the condition $\dot{x}.\dot{x} = 0$, and even more importantly, because this condition is not analytically compatible with the usual Lagrangian form $\sqrt{\dot{x}.\dot{x}}$ on account that the momentum $\frac{\dot{x}}{\sqrt{\dot{x}.\dot{x}}}$ is then ill defined. For a null case, the Lagrangian is simply $\lambda \dot{x}.\dot{x}$ with a Lagrange multiplier λ (which is an additional variable along with x 'es; λ must appropriately transform with re-parameterization in order the Lagrangian be of the first degree in the velocities, as required by the relativity theory). Then, in addition to the Euler-Lagrange equation for x (which assumes a geodesic-like form) we get another equation $\dot{x}.\dot{x} = 0$ by varying w.r.t. λ . Now, we may get rid off the term linear in \dot{x} in the geodesic-like form by introducing an appropriate parameterization (this is the affine parameterization) in which case we obtain the proper geodesic form and the condition $\dot{x}.\dot{x} = 0$ (the latter unaffected by this special choice of parameterization).

It is interesting to note, that the time-like and space-like geodesics can also be derived in a similar manner with the condition $\dot{x}.\dot{x} = \pm 1$, in which case the Lagrangian is:

A) $L = \frac{1}{2}\lambda \dot{x}.\dot{x} + \frac{1}{2}a\lambda^{-1}$ in general (with a dimensional constant a) if we want to have re-parameterization invariance, or

B) $L = \sqrt{\pm \dot{x}.\dot{x}} + \lambda(\dot{x}.\dot{x} \mp 1)$ if we want a special affine parameterization.

In the above cases, you obtain from A) the geodesic-like form (by varying w.r.t. x), while by varying w.r.t. λ you obtain a second equation: $\dot{x}.\dot{x} - \lambda^{-2}a = 0$ which gives $\lambda^{-1} = \sqrt{a^{-1} \dot{x}.\dot{x}}$ /we see that the sign of a determines the time- or space- like case/. By plugging this λ into A) you obtain $L = \sqrt{a \dot{x}.\dot{x}}$ – which reduces to the ordinary case without the λ variable (for $a > 0$ we have $a = \text{mass}^2$). In the B) case you get the condition $\dot{x}.\dot{x} = \pm 1$ which again reduces the geodesic-like form to one with affine parametrization without the linear term in \dot{x} .

The A) Lagrangian form is universal both for normalizable $\dot{x}.\dot{x}$ and for $\dot{x}.\dot{x} = 0$. In the latter case you require that the action is extremal for any λ , that is, λ is NOT a function of the velocities and this is possible only for $a = 0$ (massless case); in the former case when λ does is a function of the velocities, you get $a \neq 0$ and a condition for function λ which you may plug into the Lagrangian (which then attains the ordinary form). So, from this standpoint we see, that the geodesic-like forms are obtained in the null and the non-null case in the same general scheme, while the qualitative difference lies in that λ is or is not a function of \dot{x} .

In any case, the above reasoning shows that for null curves you also get a Lagrangian quadratic in \dot{x} .

Note also the qualitative difference, that for $a \neq 0$ the condition $\dot{x}.\dot{x} = \pm 1$ sets a parameterization, while for $a = 0$ the condition $\dot{x}.\dot{x} = 0$ does NOT set any parameterization and it is still re-parameterization invariant! This explains the presence of an additional degree of freedom in the null-like case represented by the independence of λ on \dot{x} . This also shows a fundamental difference which distinguishes null curves as such. In fact null curves are distinguished geometrically by the invariant null-cone structure of the empty spacetime – there is nothing similar in the Euclidean space!. Physically it is thus justified to consider always two Lagrangians in every mechanical case (one for timelike curves, the other for null curves): eg. for a structureless particle you consider $m\sqrt{\dot{x}.\dot{x}}$ for a massive particle or $\lambda \dot{x}.\dot{x}$ for a massless particle with independent additional intrinsic degree of freedom λ . L.B.