Ohm's Law and the Variation of Resistance with Temperature and Voltage
EECS 215-Circuits. Summary by Peter Diehr, PhD

Physical Origins of Electrical Conductivity and Resistivity

Some Definitions

Electrical current is a flow of electrically charged particles; current density is the rate of flow through a specified surface, $\vec{J} = \rho\vec{v}$, where the charge density $\rho = nq$, with $n = \frac{Number\ of\ Charges}{Unit\ of\ Volume}$ called the "number density"; $\vec{v}$ is the average velocity of the particles in the direction of the current.

Electric Field and Voltage

The potential difference ("voltage drop") is defined as $\varphi(x) = -\int_{Ref}^{x} \vec{E} \cdot d\vec{l}$, so that work performed moving a charge q from the reference point to location x is $w = \varphi(x)q$, which gives the potential difference relation $\varphi(x) = {dw}/{dq}$. Applying a voltage across the two ends of a wire produces an approximation to a uniform electric field, $E = -{V}/{L}$, with L the length of the wire.

Charged Particles in an Electric Field

Applying Newton's Second Law of Motion to a charged particle in an electric field gives $\vec{F} = \vec{E}q = m\vec{a}$, so the instantaneous acceleration is $\vec{a} = ({q}/{m})\vec{E}$. For a constant voltage the velocity due to the applied electric field is $\vec{v}_E = \vec{a}t$; if there was no "resistance" each electron would speed up as it moves along the wire, and the current would be stronger at one end than the other. This disagrees with Ohm's Law.

Statistical Mechanics: Mean Free Path and Random Scattering

Thermalization is the result of diffusion of excess energy; it is redistributed from the kinetic energy of organized motion to a collection of disorganized motions. If all of the motions were organized in a coherent fashion this motion would be called sound, and the material would have organized vibrations. If the mean free path (designated by λ) is the average distance travelled before the particle is scattered, then $\lambda = \frac{1}{2}at^2$, and the mean free time is $t_\lambda = \sqrt{{2\lambda}/{a}}$ which gives an average scattering velocity of $v_\lambda =$

$\frac{1}{2}at_\lambda = \sqrt{\lambda a/_2}$. But we previously found that $\vec{a} = \left({}^{q}/_{m}\right)\vec{E}$, which means that the mean velocity due to the presence of the electric field, and hence the current, is proportional to $\sqrt{E}$. This disagrees with Ohm's Law.

Statistical Mechanics of the Drude Model of an Electron Gas

The valence electrons of metals are loosely bound, there being no band gap between the valance band and the conduction band for metals. The good conductivity of metals is a consequence of this. The Drude model (Paul Drude, 1900) assumes that these "free" electrons form a gas. For a simple gas the kinetic energy of the particles is can be determined from the temperature: $\frac{1}{2}mv_{therm} = \frac{3}{2}kT$, assigning $\frac{1}{2}kT$of energy for each degree of freedom so that $v_{therm} = \sqrt{3\frac{kT}{m}}$; even at very low temperatures the ratio of the Boltzmann constant ($1.3806503 \times 10^{-23}$ m^2 kg $s^{-2}K^{-1}$) to the electron mass ($9.10938188 \times 10^{-31}$ kilograms) which is ${}^{k}/_{m} = 15{,}156{,}355.5\ \mathrm{m}^2\ \mathrm{s}^{-2}\mathrm{K}^{-1}$ is very large, so that for room temperature ($\sim 300\ K$) thermal velocity of $v_{therm} = 116{,}793\ {}^{\mathrm{m}}/_{\mathrm{s}}$, a random motion such that $\langle v_{therm}\rangle$, the average of many electrons over all directions is zero, just as it is for all thermal motion – otherwise hot things would move from place to place all on their own!

Ohm's Law is Not a Law of Nature

Ohm's Law is an empirical law which applies well to metals, but not to semi-conductors. We can arrive at Ohm's Law if we complete the analysis started above. Measured mean free paths are of the order $10^{-8}m = 10nm$, so the velocity due to an applied electric field is on the order of millimeters per second; thus $t_\lambda = {}^{\lambda}/_{(v_{therm} + v_E)} \approx {}^{\lambda}/_{v_{therm}}$ which then gives $v_{ave} = \frac{1}{2}at = {}^{a\lambda}/_{2v_{therm}}$.

Putting this together we have $\vec{a} = \left({}^{q}/_{m}\right)\vec{E}$, so the current density is

$\vec{J} = nq\vec{v}_{ave} = \left({}^{n\lambda q^2}/_{2mv_{therm}}\right)\vec{E}$, which gives a temperature and material dependent "constant" of proportionality between the current density and the applied electric field. This is Ohm's Law, which is summarized as $\vec{J} = \sigma\vec{E}$, with conductivity being the constant. When applied to a wire with cross section A, and length L, we get $I = JA = \sigma EA = \sigma A\, {}^{V}/_{L}$, or defining the resistance as $R = \frac{1}{\sigma}\frac{L}{A}$ we get the electrical engineer's version of Ohm's Law: $V = IR$.

Joule Heating and the Variation of Resistance with Voltage and Temperature

Looking closely at the detailed formula we see that the material parameters $\left[{}^{\lambda}/_{v_{therm}}\right]$ will control changes in the resistance; increases in temperature reduce the mean free path due to increased thermal motions of the metal atomic cores increases the resistance faster than the increased thermal velocity can reduce it. The scattering of the accelerated electrons adds to the thermal motion, increasing the temperature. This is the origin of the Joule heating law, $P = IV = I^2R$.

Not the Last Word

This presentation is based on classical ideas due to Drude and Lorentz. It was sound discovered that the Drude model worked for some things, but gave wrong answers for others – such as the electronic heat capacities of metals. A detailed analysis of the Drude model is often provided in the introductory chapters of books on condensed matter physics. Modern models which get all of the properties correct are based on a quantum mechanical analysis. But Ohm's Law still appears the same in the end.

References:

Drude Model, http://en.wikipedia.org/wiki/Drude_model

Introduction to Electrodynamics, David J. Griffiths, Chapter 7.