To prove: ψ is differentiable everywhere except at the surface.

Proof:

$$\psi = \int_{V'} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} dV' + \oint_{S'} \frac{\sigma}{|\mathbf{r} - \mathbf{r}'|} dS'$$

Let:

$$\psi^V = \int_{V'} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$\psi^S = \oint_{S'} \frac{\sigma}{|\mathbf{r} - \mathbf{r}'|} dS'$$

The proof consists of five parts.

Part I: Existence of partial derivatives of $\boldsymbol{\psi^V}$ at all points

$$\frac{\partial \psi^V}{\partial x} = \frac{\partial}{\partial x} \int\limits_{V'} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} dV'$$

$$= \lim_{\Delta x \to 0} \frac{\int_{V'} \frac{\rho}{|\mathbf{r}(x + \Delta x) - \mathbf{r}'|} dV' - \int_{V'} \frac{\rho}{|\mathbf{r}(x) - \mathbf{r}'|} dV'}{\Delta x}$$

$$= \int\limits_{V'} \rho \lim_{\Delta x \to 0} \left[\frac{\frac{1}{|\mathbf{r}(x + \Delta x) - \mathbf{r}'|} - \frac{1}{|\mathbf{r}(x) - \mathbf{r}'|}}{\Delta x} \right] dV'$$

$$= \int\limits_{V'} \rho \frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV'$$

$$= \int\limits_{V'} \rho \left(\frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} \right) dV'$$

Let us take the origin of the coordinate system at $\mathbf{r}$ and use spherical coordinate system.

$$\frac{\partial \psi^V}{\partial x} = \int\limits_{V'} \rho \frac{1}{r'^2} \frac{\xi}{r'} \; r'^2 \sin\theta' \; d\theta' \; d\phi' \; dr'$$

When $\mathbf{r} \notin V'$, the integrand is defined and continuous over domain V'. When $\mathbf{r} \in V'$, the integrand is defined and continuous over domain V' except at origin $\mathbf{r}$ where it is undefined. Therefore $\frac{\partial \psi^V}{\partial x}$ exists for any $\mathbf{r}$.

Similarly it can be shown that $\frac{\partial \psi^V}{\partial y}$ and $\frac{\partial \psi^V}{\partial z}$ exist for any $\mathbf{r}$.

Part II: Existence of partial derivatives of $\boldsymbol{\psi^S}$ at points not on $\boldsymbol{S'}$

$$\frac{\partial \psi^S}{\partial x} = \frac{\partial}{\partial x} \oint_{S'} \frac{\sigma}{|\mathbf{r} - \mathbf{r}'|} dS'$$

$$= \lim_{\Delta x \to 0} \frac{\oint_{S'} \frac{\sigma}{|\mathbf{r}(x + \Delta x) - \mathbf{r}'|} dS' - \oint_{S'} \frac{\sigma}{|\mathbf{r}(x) - \mathbf{r}'|} dS'}{\Delta x}$$

$$= \oint_{S'} \sigma \lim_{\Delta x \to 0} \left[\frac{\frac{1}{|\mathbf{r}(x + \Delta x) - \mathbf{r}'|} - \frac{1}{|\mathbf{r}(x) - \mathbf{r}'|}}{\Delta x} \right] dS'$$

$$= \oint_{S'} \sigma \frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) dS'$$

$$= \oint_{S'} \sigma \left(\frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3}\right) dS'$$

When $\mathbf{r} \notin S'$, the integrand is defined and continuous over domain S'. Therefore $\frac{\partial \psi^S}{\partial x}$ exists when $\mathbf{r} \notin S'$.

Similarly it can be shown that $\frac{\partial \psi^S}{\partial y}$ and $\frac{\partial \psi^S}{\partial z}$ exist when $\mathbf{r} \notin S'$.

From **Part I** and **Part II**, we conclude the following statement:

(i) Partial derivatives of ψ exist at points not in S'

Part III: Continuity of partial derivatives of $\boldsymbol{\psi}^{V}$ at points not in $\boldsymbol{V}'$

$\frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3}$ is continuous in space except at $\mathbf{r} = \mathbf{r}'$

Therefore $\rho \, \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, \Delta V'_i$ is continuous in space except at $\mathbf{r} = \mathbf{r}'$

Since superposition of continuous functions is a continuous function:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \rho \, \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, \Delta V'_i = \int_{V'} \rho \, \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, dV' = \frac{\partial \psi^V}{\partial x}$$

is continuous when $\mathbf{r} \notin V'$.

Similarly it can be shown that $\frac{\partial \psi^V}{\partial y}$ and $\frac{\partial \psi^V}{\partial z}$ are continuous when $\mathbf{r} \notin V'$.

Part IV: Continuity of partial derivatives of $\boldsymbol{\psi}^{S}$ at points not in $\boldsymbol{S}'$

$\frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3}$ is continuous in space except at $\mathbf{r} = \mathbf{r}'$

Therefore $\sigma \, \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, \Delta S'_i$ is continuous in space except at $\mathbf{r} = \mathbf{r}'$

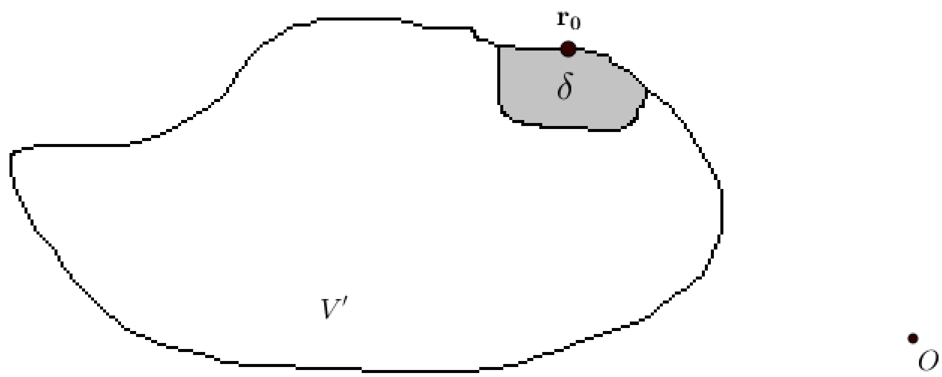
Since superposition of continuous functions is a continuous function:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \sigma \, \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, \Delta S'_i = \oint_{S'} \sigma \frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3} \, dS' = \frac{\partial \psi^S}{\partial x}$$

is continuous when $\mathbf{r} \notin S'$.

Similarly it can be shown that $\frac{\partial \psi^S}{\partial y}$ and $\frac{\partial \psi^S}{\partial z}$ are continuous when $\mathbf{r} \notin S'$.

Part V: Continuity of partial derivatives of ψ^V at points in V'



Let:

Origin O of our coordinate system lie outside V'

$\mathbf{r_0} = (x_0, y_0, z_0)$ be any point in V' where we wish to show continuity of ψ^V around which we remove a small volume $\delta \in V'$

B be the upper bound for $|\rho|$

A be the total volume in V'

δ be a volume of any shape with $\mathbf{r_0}$ in its interior (or boundary) and origin O to its exterior

For a particular $\mathbf{r}' \in (V' - \delta)$, $\frac{\xi}{|\mathbf{r}-\mathbf{r}'|^3}$ is continuous in space except at $\mathbf{r} = \mathbf{r}'$ and hence is continuous at $\mathbf{r_0}$.

Therefore for any $\frac{\varepsilon}{3kBA} > 0$, there exists a δ such that:

1. Whenever $\mathbf{r} \in \delta$ and $\mathbf{r}' \in (V' - \delta)$,

$$0 < \left| \frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\xi}{|\mathbf{r_0} - \mathbf{r}'|^3} \right| < \frac{\varepsilon}{3kBA} \qquad (1)$$

2. For all $\mathbf{r}$,

$$\left| k \int_{\delta} \rho \, \frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'_{sp} \right| < \frac{\varepsilon}{3} \qquad (2)$$

Let:

$$\frac{\partial \psi_1}{\partial x}(\mathbf{r}) = k \int_{\delta} \rho \, \frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'_{sp}$$

$$\frac{\partial \psi_2}{\partial x}(\mathbf{r}) = k \int_{V'-\delta} \rho \, \frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} \, dV'$$

By (2):

$$\left|\frac{\partial \psi_1}{\partial x}(\mathbf{r})\right| < \frac{\varepsilon}{3} \qquad \Rightarrow -\frac{\varepsilon}{3} < \frac{\partial \psi_1}{\partial x}(\mathbf{r}) < \frac{\varepsilon}{3} \tag{3}$$

Since $\mathbf{r_0} \in \mathbf{r}$,

$$\left|\frac{\partial \psi_1}{\partial x}(\mathbf{r_0})\right| < \frac{\varepsilon}{3} \qquad \Rightarrow -\frac{\varepsilon}{3} < \frac{\partial \psi_1}{\partial x}(\mathbf{r_0}) < \frac{\varepsilon}{3} \tag{4}$$

By (3) and (4):

$$\left|\frac{\partial \psi_1}{\partial x}(\mathbf{r}) - \frac{\partial \psi_1}{\partial x}(\mathbf{r_0})\right| < \frac{2\varepsilon}{3} \tag{5}$$

By (1), whenever $\mathbf{r} \in \delta$ and $\mathbf{r}' \in (V' - \delta)$:

$$\left|\frac{\partial \psi_2}{\partial x}(\mathbf{r}) - \frac{\partial \psi_2}{\partial x}(\mathbf{r_0})\right| = \left| k \int_{V'-\delta} \rho \left(\frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\xi_0}{|\mathbf{r_0} - \mathbf{r}'|^3}\right) dV' \right|$$

$$\leq k \int_{V'-\delta} \left|\rho \left(\frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\xi_0}{|\mathbf{r_0} - \mathbf{r}'|^3}\right)\right| dV' = k \int_{V'-\delta} |\rho| \left|\frac{\xi}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\xi_0}{|\mathbf{r_0} - \mathbf{r}'|^3}\right| dV'$$

$$< k \int_{V'-\delta} B \frac{\varepsilon}{3kBA} \, dV' = \frac{\varepsilon}{3A} \int_{V'-\delta} dV' = \frac{\varepsilon \, (V' - \delta)}{3A} = \frac{\varepsilon (A - \delta)}{3A} = \frac{\varepsilon}{3} - \frac{\varepsilon \, \delta}{3A}$$

$$< \frac{\varepsilon}{3}$$

i.e. $$\left|\frac{\partial \psi_2}{\partial x}(\mathbf{r}) - \frac{\partial \psi_2}{\partial x}(\mathbf{r_0})\right| < \frac{\varepsilon}{3} \tag{6}$$

By (5) and (6), whenever $\mathbf{r} \in \delta$:

$$\left|\frac{\partial \psi^V}{\partial x}(\mathbf{r}) - \frac{\partial \psi^V}{\partial x}(\mathbf{r_0})\right| = \left|\left[\frac{\partial \psi_1}{\partial x}(\mathbf{r}) - \frac{\partial \psi_1}{\partial x}(\mathbf{r_0})\right] + \left[\frac{\partial \psi_2}{\partial x}(\mathbf{r}) - \frac{\partial \psi_2}{\partial x}(\mathbf{r_0})\right]\right|$$

$$\leq \left|\frac{\partial \psi_1}{\partial x}(\mathbf{r}) - \frac{\partial \psi_1}{\partial x}(\mathbf{r_0})\right| + \left|\frac{\partial \psi_2}{\partial x}(\mathbf{r}) - \frac{\partial \psi_2}{\partial x}(\mathbf{r_0})\right|$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$$\text{i.e.} \quad \left|\frac{\partial \psi^V}{\partial x}(\mathbf{r}) - \frac{\partial \psi^V}{\partial x}(\mathbf{r_0})\right| < \varepsilon$$

i.e. for any $\varepsilon > 0$, there exists a δ such that whenever $\mathbf{r} \in \delta$:

$$\left|\frac{\partial \psi^V}{\partial x}(\mathbf{r}) - \frac{\partial \psi^V}{\partial x}(\mathbf{r_0})\right| < \varepsilon$$

i.e. $\frac{\partial \psi^V}{\partial x}$ is continuous at $\mathbf{r_0}$

Since $\mathbf{r_0}$ is an arbitrary point in V', $\frac{\partial \psi^V}{\partial x}$ is continuous at every point in V'.

Similarly it can be shown that $\frac{\partial \psi^V}{\partial y}$ and $\frac{\partial \psi^V}{\partial z}$ are continuous at every point in V'.

From **Part III, Part IV** and **Part V**, we conclude the following statement:

(ii) Partial derivatives of ψ are continuous at points not in S'

By statements (i) and (ii), $\boldsymbol{\psi}$ is differentiable at points not in $\boldsymbol{S'}$