Give a 3D point source (x_0, y_0, z_0) emitting radiation in uniform 3D random directions $t \sim U[-\pi, \pi)$ and $u \sim \frac{1}{2}\sin u \quad u \in [0, \pi]$, what is the density $p(y,z)$ of photons hitting a vertical plane positioned at $x = R$?

The governing equations of a photon traveling $\hat{r}$ distance in the direction of angles t, u to the detector surface

$$x = x_0 + \hat{r}\sin u \cos t = R$$
$$y = y_0 + \hat{r}\sin u \sin t$$
$$z = z_0 + \hat{r}\cos u$$

Let $r = \hat{r}\sin u$

then the equations are decoupled in terms of t and u

$$x = x_0 + r\cos t = R$$
$$y = y_0 + r\sin t$$
$$z = z_0 + r\cot u$$

Using Bayes' theorem, we can write

$$p(y,z) = p(z|y)\,p(y)$$

where $p(y)$ is the density from the line detector

That leaves $p(z|y)$ with

$$r = \sqrt{(R-x_0)^2 + (y-y_0)^2}$$

start with the distribution on u

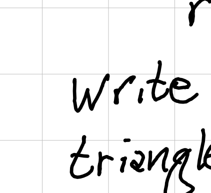
$$1 = \frac{1}{2}\int_0^{\pi} \sin u \, du$$

and make the substitution

$$z = z_0 + r\cot u \qquad \text{or} \qquad \cot u = \frac{z-z_0}{r}$$
$$dz = -r\csc^2 u \, du$$

$$\frac{-\sin^2 u}{r}\,dz = du$$

write $\sin u$ in terms of z by drawing a triangle reflecting the substitution



so $\sin u = \dfrac{r}{\sqrt{r^2 + (z-z_0)^2}}$

$$\frac{1}{2}\int_0^{\pi} \sin u \, du = \frac{-1}{2r}\int_{\infty}^{-\infty} \sin^3 u \, dz = \frac{1}{2r}\int_{-\infty}^{\infty} \frac{r^3}{\sqrt{r^2 + (z-z_0)^2}^3}\,dz$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{r^2}{\sqrt{r^2 + (z-z_0)^2}^3}\,dz$$

and so

$$p(z|y) = \frac{1}{2}\,\frac{r^2}{\sqrt{r^2 + (z-z_0)^2}^3}$$

where

$$r = \sqrt{(R-x_0)^2 + (y-y_0)^2}$$

Now write $p(y)$ in terms of r

$$p(y) = \frac{1}{\pi}\,\frac{|R-x_0|}{(R-x_0)^2 + (y-y_0)^2} = \frac{1}{\pi}\,\frac{|R-x_0|}{r^2}$$

Combining with Bayes' rule we get

$$p(y,z) = p(z|y)\,p(y) = \frac{1}{2\pi}\,\frac{\cancel{r^2}}{\sqrt{r^2 + (z-z_0)^2}^3}\,\frac{|R-x_0|}{\cancel{r^2}}$$

and expand r finally gives

$$p(y,z) = \frac{1}{2\pi}\,\frac{|R-x_0|}{\sqrt{(R-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}^3}$$

This is the bivariate Cauchy distribution

Much like the line detector and cylindrical detector, we can find a normalizer for $z \in [a, b]$ since the indefinite integral of $p(z|y)$ is

$$\int \frac{1}{2}\,\frac{r^2}{\left(\sqrt{r^2 + (z-z_0)^2}\right)^3}\,dz = \frac{1}{2}\,\frac{z-z_0}{\sqrt{r^2 + (z-z_0)^2}}$$

and so

$$\Phi(z) = \frac{1}{2}\int_{-\infty}^{z} \frac{r^2}{\left(\sqrt{r^2 + (s-z_0)^2}\right)^3}\,ds = \frac{1}{2}\,\frac{s-z_0}{\sqrt{r^2 + (s-z_0)^2}}\Bigg|_{-\infty}^{z}$$

$$= \frac{1}{2}\,\frac{z-z_0}{\sqrt{r^2 + (z-z_0)^2}} + \frac{1}{2}$$

And so, the normalized density

$$1 = \frac{1}{Z}\,\frac{1}{2}\int_a^b \frac{r^2}{\left(\sqrt{r^2 + (z-z_0)^2}\right)^3}\,dz$$

$$Z = \Phi(b) - \Phi(a)$$