# 1 Basic Equation

The basic differential operator in question is:

$$T = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} \tag{1}$$

Let $u = u(t, r, z)$ be a function, we apply the differential operator $T$ to this function to obtain the partial differential equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} = f(t, r, z) \tag{2}$$

With the boundary conditions:

$$u(0, r, z) = \partial_t u(0, r, z) = 0, \quad \partial_z u(t, r, 0) = g(t, r) \quad |u(t, r, z)| < \infty \tag{3}$$

The function $u(t, r, z)$ is supposed to represent the components of the vector potential and the function $f(t, r, z)$ is supposed to represent the current. So we want that the function is bounded and that at time $t = 0$ everything is zero and that at all time there is some form of energy coming in to the system (as the laser the beam is continuously being fired at the plasma). The function $u$ can be split up into two functions $u(t, r, z) = v(t, r, z) + w(t, r, z)$ where the function $v(t, r, z)$ satisfies:

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} + \alpha\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} = f(t, r, z) \tag{4}$$

With the boundary conditions:

$$v(0, r, z) = \partial_t v(0, r, z) = 0, \quad \partial_z v(t, r, 0) = 0 \quad |v(t, r, z)| < \infty \tag{5}$$

and the function $w(t, r, z)$ satisfies:

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} + \alpha\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} = 0 \tag{6}$$

With the boundary conditions:

$$w(0, r, z) = \partial_t w(0, r, z) = 0, \quad \partial_z w(t, r, 0) = g(t, r) \quad |w(t, r, z)| < \infty \tag{7}$$

Thus we have split up a PDE which is relatively complicated to two which are relatively straight forward to solve by transform/Green's function methods.

## 2 Perturbation Idea

The idea in this model is the laser has a finite radius $R$ which shines on the plasma which then creates the beam of electrons in the plasma which only them spreads out a little, this will be the basis of the perturbation model. Cylindrical symmetry will be assumed to keep the calculations simpler. Maxwell's equations with cylindrical symmetry are given by:

$$\begin{aligned}
\frac{\partial E_z}{\partial r} - \frac{\partial E_r}{\partial z} &= \frac{\partial B_\theta}{\partial t} &(8)\\
-\frac{\partial B_\theta}{\partial z} &= \mu_0 J_r + \mu_0 \epsilon_0 \frac{\partial E_r}{\partial t} &(9)\\
\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) &= \mu_0 J_z + \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t} &(10)\\
\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} &= \frac{\rho}{\epsilon_0} &(11)
\end{aligned}$$

Write $R = R + r'$, then:

$$\frac{\partial}{\partial r} = \frac{\partial r'}{\partial r}\frac{\partial}{\partial r'} = \frac{\partial}{\partial r'}, \tag{12}$$

then write:

$$r' = R\hat{r}' \Rightarrow \frac{\partial}{\partial r'} = \frac{1}{R}\frac{\partial}{\partial \hat{r}'}$$

Non-dimensionalise according to the following:

$$\mathbf{E} = \eta Nec\mathbf{E}', \mathbf{B} = \eta Ne\mathbf{B}', t = \eta\epsilon_0 t', z = c\eta\epsilon_0 z', \mathbf{J} = Nec\mathbf{J}', \rho = Ne\epsilon_0\rho' \tag{13}$$

Define the following quantity:

$$\varepsilon = \frac{R}{c\eta\epsilon_0} \tag{14}$$

The equations are then:

$$\begin{aligned}
\frac{\partial E_z'}{\partial \hat{r}'} - \varepsilon\frac{\partial E_r'}{\partial z'} &= \varepsilon\frac{\partial B_\theta'}{\partial t'} &(15)\\
\frac{\partial B_\theta'}{\partial z'} &= J_r + \frac{\partial E_r}{\partial t'} &(16)\\
\frac{1}{1+\hat{r}'}\frac{\partial}{\partial \hat{r}'}\left((1+\hat{r}')B_\theta'\right) &= \varepsilon J_z' + \varepsilon\frac{\partial E_z'}{\partial t'} &(17)\\
\frac{1}{1+\hat{r}'}\frac{\partial}{\partial \hat{r}'}\left((1+\hat{r}')E_r'\right) + \varepsilon\frac{\partial E_z'}{\partial z'} &= \rho' &(18)
\end{aligned}$$

Now as the beam is supposed to vary only a little, the co-ordinate $\hat{r}'$ will be small and so we set $\hat{r}' = \varepsilon\bar{r}$ and so:

$$\frac{\partial}{\partial \hat{r}'} = \frac{\partial \bar{r}}{\partial \hat{r}'}\frac{\partial}{\partial \bar{r}} = \frac{1}{\varepsilon}\frac{\partial}{\partial \bar{r}} \tag{19}$$

Upon dropping the bars and primes the equations are:

$$\begin{aligned}
\frac{\partial E_z}{\partial r} - \varepsilon^2 \frac{\partial E_r}{\partial z} &= \varepsilon^2 \frac{\partial B_\theta}{\partial t} && (20)\\
\frac{\partial B_\theta}{\partial z} &= J_r + \frac{\partial E_r}{\partial t} && (21)\\
\frac{1}{1+\varepsilon r}\frac{\partial}{\partial r}\left((1+\varepsilon r)B_\theta\right) &= \varepsilon^2 J_z + \varepsilon^2 \frac{\partial E_z}{\partial t} && (22)\\
\frac{1}{1+\varepsilon r}\frac{\partial}{\partial r}\left((1+\varepsilon r)E_r\right) + \varepsilon^2 \frac{\partial E_z}{\partial z} &= \varepsilon\rho && (23)
\end{aligned}$$