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PMath 345 – First Exam
Wednesday, October 5, 2005
1:30-2:20pm

THIS IS AN OPEN BOOK EXAM.

There are 6 pages in this exam – make sure you have them all, and that none of them are blank. Please remember to put your name on the front sheet of the exam; otherwise, I won't know who you are. Make sure you prove all of your answers unless otherwise specified. Good luck!

Note: Some of our definitions this term differ from those of the textbook. In particular, in PM345 we require that a ring have a unit element 1, and that a ring homomorphism ϕ satisfy $\phi(1) = 1$. You may further assume that all rings encountered on this exam are commutative.

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1. Answer the following yes/no questions. No need to justify your answers — just write yes or no. [2 points each]

$a+b\sqrt{2}$
$a,b\in\mathbb{Z}$.

(a) Consider the set S of real numbers of the form $a+b\sqrt{2}$, where a and b are integers. Is the set S a ring under the standard operations of addition and multiplication?

yes

$(a+b\cdot 2^{\frac{1}{3}})(c+d\cdot 2^{\frac{1}{3}})$
$= bd\,2^{2/3}$
$= bd\,2^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}$.

(b) Consider the set S of real numbers of the form $a+b\sqrt[3]{2}$, where a and b are integers. Is the set S a ring under the standard operations of addition and multiplication?

yes

$pq(x) = \frac{pq(1)+pq(2)}{2}$

(c) Let $\phi\colon \mathbb{C}[x]\to\mathbb{C}$ be the function defined by $\phi(p(x)) = (p(1)+p(2))/2$. Is ϕ a homomorphism?

$(p+q)(x)$

$\frac{(p+q)(1)+(p+q)(2)}{2}$

yes

(d) Let $\phi\colon \mathbb{Z}[x]\to\mathbb{C}$ be defined by $\phi(p(x)) = p(i)$, where $i^2=-1$. Is ϕ a homomorphism?

yes

$\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$

(e) Let I be the subset $\{0,3,6,9,12\}$ of $\mathbb{Z}/13\mathbb{Z}$. Is I an ideal of this ring?

No

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$\{(a,b) \mid a,b \in \mathbb{R}\}$

2. Recall that $\mathbb{R} \oplus \mathbb{R}$ is the ring of ordered pairs of real numbers, with coordinatewise addition and multiplication. Define $\phi: \mathbb{R}[x] \to \mathbb{R} \oplus \mathbb{R}$ by $\boxed{\phi(p(x)) = (p(0), p(1))}$. (You may assume that ϕ is a homomorphism.)

(a) Write down three nonzero elements of $\ker \phi$. [3 points]

$p \in \ker\phi$ if $\phi(p) = 0$, $p \neq 0$

So $p(0) = 0$ & $p(1) = 0$

$\underline{0} = (0,0)$

So 3 nonzero elts of $\ker\phi$ are:

$\longrightarrow x(x-1)$

$\longrightarrow x^2(x-1)^2$

$\longrightarrow x^3(x-1)^3$

(b) Is ϕ surjective? Injective? Bijective? Prove your answers. [7 points]

Injective: (1-1) ϕ is not injective because $\ker\phi \neq \{0\}$, by the above examples.

Bijective: (1-1 & onto) ϕ is not bijective because it is not injective.

Surjective: (onto) Let $(a,b) \in \mathbb{R} \oplus \mathbb{R}$ be an arbitrary element of the codomain of ϕ. Now, ϕ will be surjective iff we can find a $p(x) \in \mathbb{R}[x]$ s.t. $\phi(p(x)) = (a,b)$.

So, ~~choose~~ given (a,b), choose $p(x)$ s.t. $p(0) = a$ and $p(1) = b$. By Lagrange interpolation, such a polynomial may always be found.

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So we have found a $p(x) \in \mathbb{R}[x]$ satisfying these conditions

$\Longrightarrow \phi$ is surjective.

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3. Let $R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$, and let $P = (\sqrt{2})$ be the ideal generated by $\sqrt{2}$.

(a) Prove that P is a prime ideal. [5 points]

To do this, we can just show that R/I is a domain.

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So consider $R/P = \mathbb{Z}[\sqrt{2}]/(\sqrt{2})$. We see that in this ring, $\sqrt{2} \equiv 0 \pmod{I}$. Since each element of $\mathbb{Z}[\sqrt{2}]$ has the form $a + b\sqrt{2}$, $a, b \in \mathbb{Z}$, and $\sqrt{2} \equiv 0$, we have that each element in this quotient ring can be expressed as $a + b \cdot 0 = a$, $a \in \mathbb{Z}$. So $\mathbb{Z}[\sqrt{2}]/(\sqrt{2}) \cong \mathbb{Z}/2\mathbb{Z}$ and we know that $\mathbb{Z}$ is a domain (as it has no zero divisors). So R/P is a domain, and therefore $P = (\sqrt{2})$ is prime.

(b) Let $D = R_P$ be the localization of R at the complement of the prime ideal $P = (\sqrt{2})$. Let P_P be the ideal of R_P generated by $(\sqrt{2})$ — that is, $P_P = \sqrt{2}R_P$. Is P_P a prime ideal of R_P? Prove your answer. [5 points]

$$D = R_P = R_{(\sqrt{2})} = \left\{ \frac{a}{b} \,\middle|\, a, b \in R,\ b \notin (\sqrt{2}) \right\}$$

and $(\sqrt{2})$ is a prime ideal

$P_P = (\sqrt{2})R_P$ = the ideal of R_P gen. by $(\sqrt{2})$.

Is P_P a prime ideal of R_P?

Sol

If P_P were to be a prime ideal of R_P, then $R_{(\sqrt{2})}/P_{(\sqrt{2})}$ would have to be a ~~field~~ domain.

Note that an element of this quotient ring will look like $\cancel{(\sqrt{2})\frac{a}{b}} = \frac{\sqrt{2}a}{b}$, $b \notin (\sqrt{2})$. So b does not contain a factor of $\sqrt{2}$, and so $\nexists q$ s.t. $\frac{\sqrt{2}a}{b} \cdot q = 1$ since $\sqrt{2}$ may not appear on the denominator. $\Rightarrow \frac{\sqrt{2}a}{b}$ is not a unit

$\Rightarrow$ this is not a field

$\Rightarrow$ P_P is not a prime ideal of R_P

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4. Recall that the set of Gaussian integers is defined by:

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

Find a homomorphism ϕ: $\mathbb{Z}[i]/(i-3) \to \mathbb{Z}/10\mathbb{Z}$. [10 points]

$$\left\{\begin{array}{c} 0+0\cdot 3 \\ 0+1\cdot 3 \\ \vdots \\ 1+1\cdot 3 \\ 2-2\cdot 3 \end{array}\right\} \xmapsto{\ \phi\ } \{0,1,2,3,4,5,6,7,8,9\}$$

$i-3 \equiv 0 \pmod{I}$

$\Rightarrow i \equiv 3 \pmod{I}$

Here, $\mathbb{Z}[i]/(i-3)$

$\cong \{a+b\cdot 3 \mid a,b \in \mathbb{Z}\}$

$\cong \mathbb{Z}$

So $\mathbb{Z}[i]/(i-3)$ describes the ring $\mathbb{Z}$. So we really need a hom $\phi: \mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$.

So let $\boxed{\cancel{\phi(x) = x \pmod{10}}}$ $\phi(a+bi) = a+3b \bmod 10$.

Show this is a hom:

(1) $\phi(1) = 1 \pmod{10}$

(2) $\phi(x+y) = (x+y) \bmod 10$
$= x \bmod 10 + y \bmod 10$
$= \phi(x) + \phi(y)$.

(3) $\phi(xy) = xy \text{ modulo } 5$
$= (x \bmod 10)(y \bmod 10)$
$= \phi(x)\phi(y)$.

So ϕ is a hom. which works.

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$\{(a,b) \mid a, b \in \mathbb{Z}\}$

5. Let $R = \mathbb{Z} \oplus \mathbb{Z}$, and let I be the ideal generated by $(4,9)$ and $(6,12)$. How many elements does the quotient ring R/I have? [10 points]

So we know that $(4,9) \equiv 0 \mod I$ ①

and $(6,12) \equiv 0 \mod I$ ②

$① \cdot ② \Rightarrow (4,9) \cdot (6,12) = (24,108) \equiv 0 \mod I$

$② - ① \Rightarrow (6,12) - (4,9) = (2,3) \equiv 0 \mod I$

So $(2,3)$ ~~appears~~ to be the smallest element we can obtain.

So $(2,3) \equiv 0 = (0,0) \pmod I$

* (?) So this gives that $(2,0) \equiv (0,0)$ and $(0,3) \equiv (0,0) \pmod I$,

not enough!

and that implies that we only have the following possible elements of R/I:

(a, b)

$\{\cancel{1}, 0, \text{ or } 2\}$ $\{-1, 0, \text{ or } 1\}$

So the ring R/I has $2 \cdot 3 = 6$ elements.